Mathematical Modelling and Analysis
Volume 15 Number 1, 2010, pages 69–82
Doi:10.3846/1392-6292.2010.15.69-82

ISSN 1392-6292 print
ISSN 1648-3510 online
www.vgtu.lt/mma/

On Fully Discrete Collocation Methods for Solving Weakly Singular Integro-Differential Equations*

R. Kangro and E. Tamme

Faculty of Mathematics and Computer Science, University of Tartu
Liivi 2, 50409 Tartu, Estonia
E-mail(*corresp.*): `raul.kangro@ut.ee`
E-mail: `enn.tamme@ut.ee`

Received June 19, 2009; revised November 24, 2009; published online February 15, 2010

Abstract. In order to find approximate solutions of Volterra and Fredholm integro-differential equations by collocation methods it is necessary to compute certain integrals that determine the required algebraic systems. Those integrals usually can not be computed exactly and if the kernels of the integral operators are not smooth, simple quadrature formula approximations of the integrals do not preserve the convergence rate of the collocation method. In the present paper fully discrete analogs of collocation methods where non-smooth integrals are replaced by appropriate quadrature formulas approximations, are considered and corresponding error estimates are derived. Presented numerical examples display that theoretical results are in a good accordance with the actual convergence rates of the proposed algorithms.

Keywords: weakly singular integro-differential equation; collocation method; fully discrete collocation method; graded grid.

AMS Subject Classification: 45J05; 65R20.

1 Introduction

In the present paper we study the convergence behaviour of fully discrete analogs of collocation methods for the numerical solution of initial or boundary value problems of the form

$$u^{(n)}(t) = \sum_{i=0}^{n-1} a_i(t)u^{(i)}(t) + \sum_{i=0}^{n} \int_0^b K_i(t,s)u^{(i)}(s)\,ds + f(t), \quad 0 \le t \le b, \tag{1.1}$$

$$\sum_{j=1}^{n} \left[\alpha_{ij}u^{(j-1)}(0) + \beta_{ij}u^{(j-1)}(b)\right] = 0, \quad i = 1,\dots,n, \tag{1.2}$$

* The work of E. Tamme was supported by Estonian Science Foundation Grant No. 7353. The work of R. Kangro was supported by Estonian Science Foundation Grant N0. 7044.

where $n \in \mathbb{N} = \{1, 2, \ldots\}$, $\alpha_{ij}, \beta_{ij} \in \mathbb{R} = (-\infty, \infty)$ $(i, j = 1, \ldots, n)$ and $a_i, f : [0, b] \to \mathbb{R}$ $(i = 0, 1, \ldots, n-1)$ are some continuous functions. The kernels K_i $(i = 0, \ldots, n)$ are assumed to be m times continuously differentiable with respect to s on the set

$$\Delta = \{(t, s) : \, 0 \le t \le b, \ 0 \le s \le b, \ t \ne s\} \tag{1.3}$$

and they satisfy on this set for $j = 0, 1, \ldots, m$ the estimates

$$\left| \frac{\partial^j K_i(t, s)}{\partial s^j} \right| \le c \begin{cases} 1 & \text{if} \quad j < -\nu\,, \\ 1 + \big| \log |t - s| \big| & \text{if} \quad j = -\nu\,, \\ |t - s|^{-\nu - j} & \text{if} \quad j > -\nu\,, \end{cases} \tag{1.4}$$

where c is a positive constant, $-\infty < \nu < 1$ and $m \in \mathbb{N}$. For example, the kernels in the form

$$K_i(t, s) = K_{i,1}(t, s)|t - s|^{-\alpha_i} + K_{i,2}(t, s), \quad i = 0, \ldots, n,$$

satisfy this conditions if $\alpha_i \le \nu$ and $K_{i,p}$ $(i = 0, \ldots, n, \ p = 1, 2)$ are some m times continuously differentiable with respect to s functions on the square $[0, b] \times [0, b]$. If all kernels are identically equal to 0 above the diagonal $t = s$ (i.e. $K_i(t, s) \equiv 0$ for $s > t$ and for all $i \in \{0, 1, \ldots, n\}$), then we have a Volterra integro-differential equation, otherwise we have a Fredholm integro-differential equation. In case of a Volterra integro-differential equation it is sufficient to require that the functions $K_{i,p}$ $(i = 0, \ldots, n, p = 1, 2)$ are m times continuously differentiable with respect to s on the triangle $\{(t, s) : 0 \le s \le t \le b\}$. The problems of the form (1.1), (1.2) arise in many applications (see, e.g., [1, 3] and references therein).

We use a reformulation of the problem (1.1), (1.2) and introduce a new unknown function $v = u^{(n)}$. We assume that from all solutions of the linear homogeneous differential equation $u^{(n)} = 0$ only $u = 0$ satisfies the conditions (1.2), which is equivalent to the invertibility of the matrix Z with the elements

$$z_{ij} = (j-1)!\alpha_{ij} + \sum_{k=1}^{j} \frac{(j-1)!}{(j-k)!}\, b^{j-k} \beta_{ik}, \quad i, j = 1, \ldots, n.$$

Then the nonhomogeneous equation

$$u^{(n)}(t) = v(t), \quad t \in [0, b], \ \ v \in L^\infty(0, b),$$

with boundary conditions (1.2), has a unique solution $u(t) = (Jv)(t)$, $t \in [0, b]$, where the operator J has a representation

$$(Jv)(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s)\, ds + \sum_{i=1}^{n} c_i t^{i-1}.$$

Here the vector $\mathbf{c} = (c_1, \ldots, c_n)'$ is defined by $\mathbf{c} = -Z^{-1}\Psi\mathbf{d}$, where $\Psi = (\beta_{i,j})$ and the components of the vector $\mathbf{d} = (d_1, \ldots, d_n)'$ are given by

$$d_j = \int_0^b \frac{(b-s)^{n-j}}{(n-j)!} v(s)\, ds, \quad j = 1, \ldots, n.$$

We also define the following operators:

$$(J_i v)(t) = (Jv)^{(i)}(t), \quad t \in [0, b],\ i = 0, \dots, n\,. \tag{1.5}$$

We see that $J_n v = v$ and J_i ($i = 0, \dots, n-1$) are bounded linear operators from $L^\infty(0, b)$ into $C[0, b]$ and that for piecewise polynomial functions v the functions $J_i v$ can be computed exactly.

Using $u^{(n)} = v$ and (1.5), the problem (1.1), (1.2) may be rewritten as a linear operator equation of the second kind with respect to v:

$$v = Tv + f,$$

where

$$T = A + \sum_{i=0}^{n} B_i J_i\,, \tag{1.6}$$

$$(Av)(t) = \sum_{i=0}^{n-1} a_i(t)(J_i v)(t),\ (B_i w)(t) = \int_0^b K_i(t, s) w(s)\, ds, \quad t \in [0, b]. \tag{1.7}$$

A popular class of methods for solution of weakly singular integro-differential equations is the class of piecewise polynomial collocation methods using nonuniform grids (see, e.g., [1, 2, 3, 7, 8, 12]). In order to apply these methods it is necessary to compute certain integrals that determine the linear systems to be solved. Unfortunately those integrals usually cannot be computed exactly and even when analytic formulas exist, their straightforward application may be numerically unstable in the case of highly nonuniform grids (see [5]). Therefore it is of great practical and theoretical interest to consider methods (so called fully discrete methods), where the integrals are computed by quadrature formulas. If the kernels of the integral operators are not smooth, then it is not easy to define a quadrature approximation to the system integrals so that the order of convergence of the original collocation method is preserved. One way of coping with this difficulty is to use for approximating the integral operators additional graded grids that take into account the singularities of the kernels. Such methods for solution of weakly singular integral equations are proposed and investigated in [4, 6, 11]. In the present paper we propose fully discrete analogs of collocation methods to solve integro-differential equations with weakly singular kernels. Here we use results from [6].

In Section 2 of the present paper we introduce an algorithm of a collocation method for solving (1.1), (1.2). In Section 3 we construct a fully discrete analog of this algorithm and in Section 4 derive estimates for the difference of approximations obtained by the exact collocation method and by the fully discrete collocation method (Theorem 1). Theorem 1 together with the convergence rate estimates for the exact collocation methods enable us to estimate the errors of the approximations computed by the fully discrete collocation method. In the last section the obtained theoretical results are verified by some numerical experiments.

2 Collocation Method

For a given $N \in \mathbb{N}$ let $\Pi_N = \{t_0, \dots, t_N : 0 = t_0 < t_1 < \dots < t_N = b\}$ be a partition (a grid) of the interval $[0, b]$ (for the ease of notation we suppress the index N of $t_j = t_j^{(N)}$ indicating the dependence of grid points on N).

For given integers $m \geq 0$ and $-1 \leq d \leq m-1$, let $S_m^{(d)}(\Pi_N)$ be the spline space of piecewise polynomial functions on the grid Π_N:

$$S_m^{(d)}(\Pi_N) = \left\{v \in C^d[0,b] : v\big|_{[t_{j-1},t_j]} \in \pi_m,\ j = 1,\dots,N\right\},\ 0 \leq d \leq m-1,$$
$$S_m^{(-1)}(\Pi_N) = \left\{v : v\big|_{(t_{j-1},t_j)} \in \pi_m,\ j = 1,\dots,N\right\}.$$

Here π_m denotes the set of polynomials of degree not exceeding m and $v\big|_{(t_{j-1},t_j)}$ is the restriction of $v : [0,b] \to \mathbb{R}$ onto the subinterval (t_{j-1}, t_j). Note that the elements of $S_m^{(-1)}(\Pi_N)$ may have jump discontinuities at the interior points $t_1, \dots, t_{N-1}$ of the grid Π_N.

We define $m \geq 1$ collocation points in every subinterval $[t_{j-1}, t_j]$ $(j = 1, \dots, N)$ by

$$t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1}), \quad k = 1, \dots, m, \tag{2.1}$$

where $\eta_1 \dots, \eta_m$ are some fixed parameters which do not depend on j and N and satisfy $0 \leq \eta_1 < \eta_2 < \dots < \eta_m \leq 1$.

We look for an approximate solution u_N of the problem (1.1), (1.2) in the form $u_N(t) = (J_0 v_N)(t)$, where $v_N \in S_{m-1}^{(-1)}(\Pi_N)$ $(m, N \in \mathbb{N})$ is determined by the following collocation conditions:

$$v_N(t_{jk}) = (Tv_N)(t_{jk}) + f(t_{jk}), \quad k = 1, \dots, m,\ j = 1, \dots, N. \tag{2.2}$$

Here J_0 and T are defined by (1.5) and (1.6), respectively. If $\eta_1 = 0$, then by $v_N(t_{j1})$ we denote the right limit $\lim_{t \to t_{j-1},\, t > t_{j-1}} v_N(t)$, if $\eta_m = 1$, then $v_N(t_{jm})$ denotes the left limit $\lim_{t \to t_j,\, t < t_j} v_N(t)$. The convergence of such collocation method for solving (1.1), (1.2) is investigated in [8]. In order to obtain a high-order convergence a special graded grid reflecting the possible singular behavior of the solution is used.

3 A Fully Discrete Collocation Method

For determining v_N from (2.2) it is necessary to compute integrals which usually cannot be computed exactly. In order to discretize the integrals in (2.2) we introduce a set of points $\Sigma_M = \{s_j : j = -M, \dots, M\}$ where

$$s_j = b\left(\frac{j}{M}\right)^{r_1},\ j = 0, 1, \dots, M\,, \quad s_j = -s_{-j}\,,\ j = -M, \dots, -1\,, \tag{3.1}$$

$M > 1$ is a natural number and $r_1 \geq 1$ is a real number that determines the nonuniformity of Σ_M at zero. Basically, this grid, shifted to be centered at the current value of t, is the grid on which we can approximate our integrals with nonsmooth kernels well. But since our integrands may have discontinuities at the points in Π_N, we refine the integration grid with respect to those points. More precisely, for given $t \in [0, b]$ we divide the interval $[0, b]$ into subintervals with the points

$$X_t = \big(\Pi_N \cup (t + \Sigma_M)\big) \cap [0, b],$$

where $t + \Sigma_M = \{t + s : s \in \Sigma_M\}$. Let us number the points in X_t in the increasing order, i.e., $X_t = \{x_p : p = 0, 1, \ldots, p_1\}$, where

$$0 = x_0 < x_1 < \cdots < x_{p_1} = b.$$

Additionally, we choose a quadrature formula

$$\int_0^1 g(x)\,dx \approx \sum_{q=1}^{m_1} \omega_q\, g(\xi_q) \tag{3.2}$$

with knots $0 \le \xi_1 < \cdots < \xi_{m_1} \le 1$ and weights $\omega_1, \ldots, \omega_{m_1}$. We denote

$$x_{pq} = x_{p-1} + \xi_q(x_p - x_{p-1}), \quad q = 1, \ldots, m_1\,, \;\; p = 1, \ldots, p_1,$$

and approximate the integrals $(B_i w)(t)$ (see (1.7)) for $t \in [0, b]$ by

$$(\tilde{B}_i w)(t) = \begin{cases} \sum\limits_{p=1}^{p_1} (\tilde{B}_{ip} w)(t) & \text{if } \nu < 0\,, \\ \sum\limits_{p:0<x_p\le t-s_1} (\tilde{B}_{ip} w)(t) + \sum\limits_{p:t+s_1<x_p\le b} (\tilde{B}_{ip} w)(t) & \text{if } \nu \ge 0, \end{cases} \tag{3.3}$$

where $\nu < 1$ as (1.4) and

$$(\tilde{B}_{ip} w)(t) = (x_p - x_{p-1}) \sum_{q=1}^{m_1} \omega_q\, K_i(t, x_{pq}) w(x_{pq}), \quad p=1, \ldots, p_1,\;\; i=0, \ldots, n. \tag{3.4}$$

Our fully discrete collocation method is as follows: we look for an approximate solution $\tilde{u}_N$ of (1.1), (1.2) in the form $\tilde{u}_N = J_0 \tilde{v}_N$ where $\tilde{v}_N \in S_{m-1}^{(-1)}(\Pi_N)$ is determined by the conditions

$$\tilde{v}_N(t_{jk}) = (A\tilde{v}_N)(t_{jk}) + \sum_{i=0}^{n} (\tilde{B}_i J_i \tilde{v}_N)(t_{jk}) + f(t_{jk}), \quad k=1, \ldots, m,\;\; j=1, \ldots, N. \tag{3.5}$$

Here J_i, A, $\tilde{B}_i$ and t_{jk} are defined by (1.5), (1.7), (3.4) and (2.1), respectively. Recall that for piecewise polynomial functions v_N the values of the functions $J_i v_N$ can be computed exactly.

4 Convergence of the Fully Discrete Collocation Method

We obtain the following estimates for the errors of the quadrature approximations $\tilde{B}_i$ of the integral operators B_i.

Lemma 1. *Assume that for some $k \in \mathbb{N}$ and for a fixed $t \in [0,b]$ we have*

$$w \in S^{(-1)}_{k-1}(X_t) = \{ w : w|_{[x_{p-1},x_p]} \in \pi_{k-1},\ p = 1,\ldots,p_1 \},\ t \in [0,b].$$

Assume also that the quadrature formula (3.2) *is exact for all polynomials of degree μ where $\mu \geq k-1$ and that the kernels K_i, $(i = 0,\ldots,n)$ are $\mu - k + 2$ times continuously differentiable with respect to s on the set Δ (see* (1.3)*) and satisfy on this set for $j = 0,\ldots,\mu - k + 2$ the estimates* (1.4) *with $\nu < 1$.*

Then, for $r_1 \geq 1$ as (3.1) *and for $i = 0,\ldots,n$, we have*

$$|(B_i w)(t) - (\tilde{B}_i w)(t)| \leq c\,\|w\|_\infty \Omega_M(\mu - k + 2, \nu, r_1),\ t \in [0,b], \tag{4.1}$$

where c does not depend on t, M, N and the form of Π_N, the function Ω_M is defined by

$$\Omega_M(\beta,\nu,r) = \begin{cases} M^{-r(1-\nu)} & \text{if} \quad 1 \leq r < \frac{\beta}{1-\nu},\ -\nu \notin \mathbb{N}_0, \\ M^{-r(1-\nu)}(1 + \log M) & \text{if} \quad r < \frac{\beta}{1-\nu},\ -\nu \in \mathbb{N}_0, \\ M^{-\beta}(1 + \log M) & \text{if} \quad r = \frac{\beta}{1-\nu}, \\ M^{-\beta} & \text{if} \quad r > \frac{\beta}{1-\nu}, \end{cases} \tag{4.2}$$

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and B_i and $\tilde{B}_i$ are defined by* (1.7) *and* (3.3)*, respectively.*

Proof. Let us examine the case $0 \leq \nu < 1$. Then we have for fixed $i \in \{0,\ldots,n\}$ and $t \in [0,b]$

$$|(B_i w)(t) - (\tilde{B}_i w)(t)| \leq \Delta_1(t) + \Delta_2(t) + \|w\|_\infty \int\limits_{\max\{0,t-s_1\}}^{\min\{b,t+s_1\}} |K_i(t,s)|\,ds, \tag{4.3}$$

where

$$\Delta_1(t) = \sum_{p:0<x_p\leq t-s_1} |\delta_p(t)|, \quad \Delta_2(t) = \sum_{p:t+s_1<x_p\leq b} |\delta_p(t)|, \tag{4.4}$$

$$\delta_p(t) = \int_{x_{p-1}}^{x_p} K_i(t,s) w(s)\,ds - \tilde{B}_{ip}(t)$$

and $(\tilde{B}_{ip})(t)$ is defined by (3.4). Since the quadrature formula (3.2) is exact for all polynomials of degree μ and w is on each interval $[x_{p-1},x_p]$, $p = 1,\ldots,p_1$, a polynomial of degree $k-1$, we have for all polynomials ϕ of degree $\mu - k + 1$

$$\begin{aligned} \delta_p(t) = &\int_{x_{p-1}}^{x_p} \big[K_i(t,s) - \phi(s)\big] w(s)\,ds \\ &- (x_p - x_{p-1}) \sum_{q=1}^{m_1} w_q \big[K_i(t,x_{pq}) - \phi(x_{pq})\big] w(x_{pq})\,ds,\ t \in [0,b]. \end{aligned}$$

Thus for any such polynomial we get an estimate

$$|\delta_p(t)| \leq c\|w\|_\infty (x_p - x_{p-1}) \sup_{s\in(x_{p-1},x_p)} |K_i(t,s) - \phi(s)|, \quad t \in [0,b], \tag{4.5}$$

where c is a positive constant not depending on t and p. For values p such that $t+s_1 < x_p \le b$ we use

$$\phi(s) = \sum_{j=0}^{\mu-k+1} \frac{1}{j!} \frac{\partial^j K_i(t,s)}{\partial s^j}\Big|_{s=x_p} (s-x_p)^j, \quad s \in [x_{p-1}, x_p],$$

which, together with the well-known estimate for Taylor expansion

$$|K_i(t,s) - \phi(s)| \le \frac{|s-x_p|^{\mu-k+2}}{(\mu-k+2)!} \sup_{x_{p-1}<\sigma<x_p} \left| \frac{\partial^{\mu-k+2} K_i(t,s)}{\partial s^{\mu-k+2}}\Big|_{s=\sigma} \right|$$

and the estimates (4.5) and (1.4) with $j = \mu-k+2 \ge 1 > -\nu$ gives us

$$|\delta_p(t)| \le c_1 \|w\|_\infty (x_p - x_{p-1})^{\mu-k+3} \sup_{x_{p-1}<\sigma<x_p} |t-\sigma|^{-\nu-\mu+k-2},$$

where c_1 does not depend on p and $t \in [0,b]$.

For $\Delta_2(t)$ defined by (4.4) we obtain for $t \in [0,b]$

$$\Delta_2(t) = \sum_{j=1}^{M-1} \sum_{\substack{p\,:\, t+s_j<x_p \\ \le t+s_{j+1}}} |\delta_p(t)| \le c_1 \|w\|_\infty \sum_{j=1}^{M-1} s_j^{-\nu-\mu+k-2} (s_{j+1}-s_j)^{\mu-k+3}.$$

Since

$$s_j = b\left(\frac{j}{M}\right)^{r_1}, \quad 0 < s_{j+1} - s_j \le \frac{br_1}{M} \left(\frac{j+1}{M}\right)^{r_1-1}, \quad j = 1, \ldots, M,$$

we get for $t \in [0,b]$

$$\begin{aligned}
\Delta_2(t) &\le c_1 \|w\|_\infty \sum_{j=1}^{M-1} \left[\left(\frac{j}{M}\right)^{r_1}\right]^{-\nu-\mu+k-2} \left[\frac{br_1}{M} \left(\frac{j+1}{M}\right)^{r_1-1}\right]^{\mu-k+3} \\
&\le c_2 \|w\|_\infty M^{-r_1(1-\nu)} \sum_{j=1}^{M-1} j^{r_1(1-\nu)-\mu+k-3} \\
&\le c_3 \|w\|_\infty \begin{cases} M^{-r_1(1-\nu)} & \text{if} \quad r_1 < \dfrac{\mu-k+2}{1-\nu}, \\ M^{-\mu+k-2}(1+\log M) & \text{if} \quad r_1 = \dfrac{\mu-k+2}{1-\nu}, \\ M^{-\mu+k-2} & \text{if} \quad r_1 > \dfrac{\mu-k+2}{1-\nu}, \end{cases}
\end{aligned}$$

where c_3 does not depend on M and t.

In a similar way we get the same estimates for $\Delta_1(t)$ (see [6]). Using (3.1) and (1.4) we obtain (see [6])

$$\int_{\max\{0,t-s_1\}}^{\min\{b,t+s_1\}} |K_i(t,s)|\, ds \le c \begin{cases} M^{-r_1(1-\nu)} & \text{if} \quad 0<\nu<1, \\ M^{-r_1(1-\nu)}(1+\log M) & \text{if} \quad \nu = 0, \end{cases}$$

and so from (4.3) the estimates (4.1) for $0 \le \nu < 1$ follow. Repeating the steps of the corresponding part of the proof of Theorem 1 in [6] we can prove the estimates (4.1) also for $\nu < 0$. □

We derive sharper estimates for the errors of the approximations $\tilde{B}_i J_i v$ of the integrals $B_i J_i v$ for $i = 0, \dots, n$.

Lemma 2. *Assume that the quadrature formula* (3.2) *is exact for all polynomials of degree* μ *where* $\mu \ge m+n-i-1$, $m \in \mathbb{N}$ *and* $i \in \{0, \dots, n\}$. *Assume also, that* $v \in S_{m-1}^{(-1)}(\Pi_N)$ *and the kernel* K_i *is on the set* Δ $(\mu - m + 2)$ *times continuously differentiable with respect to* s *and satisfies on this set the estimates* (1.4) *with* $\nu < 1$ *for* $j = 0, \dots, \mu - m + 2$. *Then we have*

$$\|B_i J_i v - \tilde{B}_i J_i v\|_\infty \le c\,\|v\|_\infty \Omega_M(\mu - m + 2, \nu, r_1), \tag{4.6}$$

where c *does not depend on* M, N *and the form of* Π_N *and* J_i, B_i, $\tilde{B}_i$ *and* Ω_M *are defined by* (1.5), (1.7), (3.3) *and* (4.2), *respectively.*

Proof. Note that since $\Pi_N \subset X_t$ and $v \in S_{m-1}^{(-1)}(\Pi_N)$, we have

$$J_i v \in S_{m+n-i-1}^{(n-i-1)}(\Pi_N) \subset S_{m+n-i-1}^{(-1)}(X_t)$$

for any $t \in [0, b]$ (see [8]).

If $i = n$ then $J_i v = v \in S_{m-1}^{(-1)}(\Pi_N)$ and the estimate (4.6) follows from Lemma 1 immediately. Consider the case $0 \le i < n$. We use the equality

$$B_i J_i v - \tilde{B}_i J_i v = (B_i - \tilde{B}_i)(J_i v - \phi) + (B_i - \tilde{B}_i)\phi, \tag{4.7}$$

where $\phi \in S_{n-i-1}^{(-1)}(X_t)$ is defined by

$$\phi(s) = \sum_{j=0}^{n-i-1} \frac{1}{j!}(J_i v)^{(j)}(x_p)(s - x_p)^j,\ \ s \in [x_{p-1}, x_p],\ \ p = 1, \dots, p_1.$$

By the well-known estimate for the error of Taylor expansion, we obtain

$$|(J_i v)(s) - \phi(s)| \le \frac{|s - x_p|^{n-i}}{(n-i)!} \sup_{x_{p-1} < \sigma < x_p} \left|(J_i v)^{(n-i)}(\sigma)\right|,\ \ s \in [x_{p-1}, x_p].$$

Since $(J_i v)^{(n-i)} = v$ and $0 < x_p - x_{p-1} \le r_1 b/M$ $(p = 1, \dots, p_1)$ we get

$$\|J_i v - \phi\|_\infty \le c\, M^{-n+i} \|v\|_\infty. \tag{4.8}$$

Since $J_i v - \phi \in S_{m+n-i-1}^{(-1)}(X_t)$ we estimate by (4.1) and (4.8)

$$\begin{aligned} \|(B_i - \tilde{B}_i)(J_i - \phi)\|_\infty &\le c\,\|J_q v - \phi\|_\infty \Omega_M(\mu - m - n + i + 2, \nu, r_1) \\ &\le c_1\,\|v\|_\infty M^{-n+i} \Omega_M(\mu - m - n + i + 2, \nu, r_1) \\ &\le c_1\,\|v\|_\infty \Omega_M(\mu - m + 2, \nu, r_1). \end{aligned} \tag{4.9}$$

In order to estimate the norm of $(B_i - \tilde{B}_i)\phi$, we write $\phi = \sum_{j=0}^{n-i-1} \phi_j$, where

$$\phi_j(s)=\frac{1}{j!}(J_i v)^{(j)}(x_p)(s-x_p)^j, \ s \in [x_{p-1}, x_p], \ p=1,\dots,p_1, \ j=0,\dots,n-i-1.$$

Since $\phi_j \in S_j^{(-1)}(X_t) \subset S_{m-1}^{(-1)}(X_t)$ for $j \le m-1$, we have according to (4.1) the estimate

$$\|(B_i - \tilde{B}_i)\phi_j\|_\infty \le c\,\|\phi_j\|_\infty \varOmega_M(\mu - m + 2, \nu, r_1), \ \ 0 \le j \le m-1.$$

As $\|\phi_j\|_\infty \le c_1\,\|v\|_\infty M^{-j}, \ j = 0,\dots,n-j-1$, and in case of $n-j > m$ we have

$$\begin{aligned}\|(B_i - \tilde{B}_i)\phi_j\|_\infty &\le c\,\|\phi_j\|_\infty \varOmega_M(\mu - j + 1, \nu, r_1)\\ &\le c_2\,\|v\|_\infty M^{-j} \varOmega_M(\mu - j + 1, \nu, r_1)\\ &\le c_2\,\|v\|_\infty \varOmega_M(\mu - m + 2, \nu, r_1), \ j = m,\dots,n-i-1,\end{aligned}$$

then we obtain

$$\|(B_i - \tilde{B}_i)\phi\|_\infty \le c\,\|v\|_\infty \varOmega_M(\mu - m + 2, \nu, r_1). \tag{4.10}$$

From (4.7), (4.9) and (4.10) the estimate (4.6) follows. □

Lemma 2 enables us to estimate easily the difference of solutions of the exact collocation method (2.2) and the fully discrete collocation method (3.5).

Theorem 1. *Assume that the following conditions are fulfilled:*

1) *problem* (1.1), (1.2) *is uniquely solvable in* $C^n[0,b]$ *and of all solutions of the equation* $u^{(n)} = 0$ *only* $u = 0$ *satisfies* (1.2)*;*

2) *quadrature formula* (3.2) *is exact for all polynomials of degree* μ *where* $\mu \ge m + n - 1$ *and* $m \in \mathbb{N}$*;*

3) *in equation* (1.1) f *and* a_i $(i = 0,\dots,n-1)$ *are continuous functions on* $[0,b]$ *and the kernels* K_i $(i = 0,\dots,n)$ *are on the set* Δ $(\mu - m + 2)$ *times continuously differentiable with respect to s and satisfy on this set the estimates* (1.4) *with* $\nu < 1$ *for* $j = 0,\dots,\mu - m + 2$*;*

4) *the sequence of grids* $\varPi_N$, $N \in \mathbb{N}$, *is such that*

$$\max_{1 \le j \le N} |\,t_j - t_{j-1}| \to 0 \quad \text{as} \quad N \to \infty.$$

Then there exist integers N_0 *and* M_0 *such that for all* $N \ge N_0$ *and* $M \ge M_0$ *both conditions* (2.2) *and* (3.5) *uniquely determine* $v_N \in S_{m-1}^{(-1)}(\varPi_N)$ *and* $\tilde{v}_N \in S_{m-1}^{(-1)}(\varPi_N)$*, respectively. For their difference the following estimates hold:*

$$\|J_i v_N - J_i \tilde{v}_N\|_\infty \le c\,\varOmega_M(\mu - m + 2, \nu, r_1), \quad i = 0,1,\dots,n, \tag{4.11}$$

$$\|\hat{v}_N - \hat{\tilde{v}}_N\|_\infty \le c\,\varOmega_M(\mu - m + 2, \nu, r_1), \tag{4.12}$$

where c is a constant not depending on M and N and the form of Π_N, $-\infty < \nu < 1$, $r_1 \geq 1$,

$$\hat{v}_N = Tv_N + f, \quad \hat{\tilde{v}}_N = \big(A + \sum_{i=0}^{n} \tilde{B}_i J_i\big)\tilde{v}_N + f, \tag{4.13}$$

and J_i, T, A, $\tilde{B}_i$ *and* Ω_M *are defined by* (1.5), (1.6), (1.7), (3.3) *and* (4.2), *respectively.*

Proof. Define a piecewise polynomial interpolation operator $\mathcal{P}_N : C[0,b] \to S_{m-1}^{(-1)}(\Pi_N) \subset L^\infty(0,b)$ by conditions

$$(\mathcal{P}_N v)(t_{jk}) = v(t_{jk}), \quad k = 1,\dots,m,\ j = 1,\dots,N,$$

where t_{jk} $(k = 1,\dots,m,\ j = 1,\dots,N)$ are defined by (2.1). Then we can represent conditions (2.2) and (3.5) as operator equations

$$v_N = \mathcal{P}_N T v_N + \mathcal{P}_N f \tag{4.14}$$

$$\tilde{v}_N = \mathcal{P}_N \tilde{T} \tilde{v}_N + \mathcal{P}_N f \quad \text{with} \quad \tilde{T} = A + \sum_{i=0}^{n} \tilde{B}_i J_i, \tag{4.15}$$

respectively. It is well known that $\mathcal{P}_N$ is a sequence of uniformly bounded operators satisfying

$$\lim_{N\to\infty} \|\mathcal{P}_N v - v\|_\infty = 0 \quad \text{for all} \quad v \in C[0,b].$$

In the same way as in the proof of Theorem 4.1 of [8] we obtain that there exists an integer N_0 such that for every $N \geq N_0$ equation (4.14) possesses a unique solution $v_N \in S_{m-1}^{(-1)}(\Pi_N)$ and

$$\|(I - \mathcal{P}_N T)^{-1}\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \leq c \tag{4.16}$$

where c does not depend on N. Let's denote the space of piecewise polynomial functions $S_{m-1}^{(-1)}(\Pi_N)$ equipped with $L^\infty(0,b)$ norm by E_N and consider (4.14) and (4.15) as equations in E_N. It follows from (4.6) that

$$\begin{aligned}\|\mathcal{P}_N T - \mathcal{P}_N \tilde{T}\|_{\mathcal{L}(E_N,E_N)} &= \big\|\mathcal{P}_N \sum_{i=0}^{n} (B_i - \tilde{B}_i) J_i\big\|_{\mathcal{L}(E_N,E_N)} \\ &\leq c\Omega_M(\mu - m + 2, \nu, r_1) \to 0 \quad \text{as} \quad M \to \infty.\end{aligned}$$

Using equality

$$I - \mathcal{P}_N \tilde{T} = (I - \mathcal{P}_N T)[I - (I - \mathcal{P}_N T)^{-1}\mathcal{P}_N(\tilde{T} - T)]$$

and inequality (4.16) we get that there exists $M_0 \in \mathbb{N}$ such that for every $M \geq M_0$ and $N \geq N_0$ operator $(I - \mathcal{P}_N \tilde{T})$ is invertible in E_N, equation (4.15) possesses a unique solution $\tilde{v}_N \in E_N$ and

$$\|\tilde{v}_N\|_\infty = \|(I - P_N \tilde{T})^{-1}\mathcal{P}_N f\|_\infty \leq c, \tag{4.17}$$

where c does not depend on M and N. It follows from (4.14) and (4.15) that

$$(I - \mathcal{P}_N T)(v_N - \tilde{v}_N) = \mathcal{P}_N \sum_{i=0}^{n} (B_i - \tilde{B}_i) J_i \tilde{v}_N.$$

On the basis of (4.6), (4.16) and (4.17) we obtain from this that for $M \geq M_0$ and $N \geq N_0$

$$\|v_N - \tilde{v}_N\|_\infty \leq c \Big\| \mathcal{P}_N \sum_{i=0}^{n} (B_i - \tilde{B}_i) J_i \Big\|_{\mathcal{L}(E_N, E_N)} \leq c_1 \, \Omega_M(\mu - m + 2, \nu, r_1),$$

where c and c_1 do not depend on M and N. This together with boundedness of J_i $(i = 0, \ldots, n)$ yields the estimate (4.11). The estimate (4.12) follows from the equalities

$$\hat{v}_N - \hat{\tilde{v}}_N = T v_N - \tilde{T} \tilde{v}_N = T(v_N - \tilde{v}_N) + \sum_{i=0}^{n} (B_i - \tilde{B}_i) J_i \tilde{v}_N.$$

□

Theorem 1 tells us that for any collocation method we can choose appropriate M depending on N, an appropriate quadrature formula (3.2) and a suitable value for the nonuniformity parameter r_1 so that the convergence rate of the fully discrete collocation method is of the same order as the convergence rate of the collocation method.

Remark 1. If in the integral equation (1.1) the kernels $K_i(t, s) \equiv 0$ for $i = 0, \ldots, p$ where $p < n$, then the assertions of Theorem 1 are valid when we replace the assumption 2) with the assumption that quadrature formula (3.2) is exact for all polynomials of degree $\mu \geq m + n - p - 2$.

5 Numerical Experiments

Let us consider the following boundary value problem:

$$u''(t) = \sqrt{t}\, u(t) + \int_0^1 |t - s|^{-1/2} u(s)\, ds + f(t), \quad t \in [0, 1], \tag{5.1}$$

$$u(0) = u'(1) = 0. \tag{5.2}$$

The forcing function f is selected such that

$$u(t) = t^{5/2} + (1 - t)^{5/2} - 1 - \frac{5}{2} t$$

is the exact solution. Actually, this is a problem of the form (1.1), (1.2) where $n = 2,\ b = 1,\ a_0(t) = \sqrt{t},\ a_1 = 0,\ K_0(t, s) = |t - s|^{-1/2},\ K_1 = K_2 = 0$. It is easy to check that the kernel K_0 satisfies the conditions (1.4) with $\nu = 1/2$ and arbitrary $j \in \mathbb{N}_0$.

Problem (5.1), (5.2) is solved numerically by collocation method (2.2) and by fully discrete collocation method (3.5). At that grid points

$$t_j = \frac{1}{2}\Big(\frac{2j}{N}\Big)^r,\ j = 0, 1, \ldots, \frac{N}{2}, \quad t_{N/2+j} = 1 - t_{N/2-j},\ j = 1, \ldots, \frac{N}{2}, \qquad (5.3)$$

with even N, $r = 2$ and collocation points (2.1) with $m = 2$ and Gaussian parameters $\eta_1 = (3 - \sqrt{3})/6$, $\eta_2 = 1 - \eta_1$ are used. Then from the results of [8, 9, 10] we get for sufficiently large N the estimates

$$\begin{aligned} &\|u - J_0 v_N\|_\infty \le c\, N^{-3}, \quad \|u' - J_1 v_N\|_\infty \le c\, N^{-3}(1 + \log N), \\ &\|u'' - \hat{v}_N\|_\infty \le c\, N^{-3}, \end{aligned} \qquad (5.4)$$

where $v_N \in S_1^{(-1)}(\varPi_N)$ is evaluated by (2.2), $\hat{v}_N$ by (4.13) and $J_i v_N$ $(i = 0, 1)$ are computed by formula (1.5) with $b = 1$ and

$$(Jv)(t) = (J_0 v)(t) = \int_0^t (t - s) v(s)\, ds - t \int_0^1 v(s)\, ds.$$

In order to evaluate $\tilde{v}_N \in S_1^{(-1)}(\varPi_N)$ by (3.5) we use in addition to (5.3) another graded grid (3.1) and quadrature formula (3.2) with two Gaussian knots $\xi_1 = (3-\sqrt{3})/6$, $\xi_2 = 1-\xi_1$ and weights $w_1 = w_2 = 1/2$. This formula is exact for all polynomials of order $\mu = 3$. The differences of the approximations evaluated by the exact and the fully discrete collocation methods are, according to Theorem 1, for sufficiently large M and N bounded by

$$c\, \varOmega_M(3, 1/2, r_1) = c \begin{cases} M^{-r_1/2} & \text{if} \quad 1 \le r_1 < 6, \\ M^{-3}(1 + \log M) & \text{if} \quad r_1 = 6, \\ M^{-3} & \text{if} \quad r_1 > 6. \end{cases} \qquad (5.5)$$

In Tab. 5.1 some results for $r = 2$, $r_1 = 6$ and different values of the parameters N and M are presented. The quantities $\varepsilon_N^{(i)}$, $\tilde{\varepsilon}_N^{(i)}$ $(i = 0, 1)$, $\varepsilon_N^{(2)}$ and $\tilde{\varepsilon}_N^{(2)}$ are the approximate values of the norms $\|u^{(i)} - J_i v_N\|_\infty$, $\|u^{(i)} - J_i \tilde{v}_N\|_\infty$ $(i = 0, 1)$, $\|u'' - \hat{v}_N\|_\infty$ and $\|u'' - \hat{\tilde{v}}_N\|_\infty$, respectively, calculated as follows:

$$\varepsilon_N^{(i)} = \max_{j=1,\ldots,N}\ \max_{k=0,\ldots,10} \left|u^{(i)}(\tau_{jk}) - (J_i v_N)(\tau_{jk})\right|, \quad i = 0, 1,$$

$$\tilde{\varepsilon}_N^{(i)} = \max_{j=1,\ldots,N}\ \max_{k=0,\ldots,10} \left|u^{(i)}(\tau_{jk}) - (J_i \tilde{v}_N)(\tau_{jk})\right|, \quad i = 0, 1,$$

$$\varepsilon_N^{(2)} = \max_{j=1,\ldots,N}\ \max_{k=0,\ldots,10} \left|u''(\tau_{jk}) - \hat{v}_N(\tau_{jk})\right|,$$

$$\tilde{\varepsilon}_N^{(2)} = \max_{j=1,\ldots,N}\ \max_{k=0,\ldots,10} \left|u''(\tau_{jk}) - \hat{\tilde{v}}_N(\tau_{jk})\right|,$$

where

$$\tau_{jk} = t_{j-1} + \frac{k}{10}(t_j - t_{j-1}), \quad k = 0, \ldots, 10,\ j = 1, \ldots, N,$$

with the grid points $\{t_j\}$, defined by the formula (5.3). In Tab. 5.1 we also present the ratios

$$\varrho_N^{(i)} = \frac{\varepsilon_{N/2}^{(i)}}{\varepsilon_N^{(i)}}, \quad \tilde{\varrho}_N^{(i)} = \frac{\tilde{\varepsilon}_{N/2}^{(i)}}{\tilde{\varepsilon}_N^{(i)}}, \quad i = 0, 1, 2,$$

Table 5.1. Results for $r = 2$, $r_1 = 6$, $m = m_1 = 2$, $\eta_1 = \xi_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = \xi_2 = 1 - \eta_1$, $w_1 = w_2 = \frac{1}{2}$.

			$M = N$		$M = 2N$		$M = 4N$		$M = 8N$	
N	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\tilde{\varepsilon}_N^{(0)}$	$\tilde{\varrho}_N^{(0)}$	$\tilde{\varepsilon}_N^{(0)}$	$\tilde{\varrho}_N^{(0)}$	$\tilde{\varepsilon}_N^{(0)}$	$\tilde{\varrho}_N^{(0)}$	$\tilde{\varepsilon}_N^{(0)}$	$\tilde{\varrho}_N^{(0)}$
8	1.4E−4	7.2	9.9E−3	5.6	1.6E−3	6.8	3.4E−4	7.4	1.7E−4	7.3
16	1.8E−5	7.7	1.5E−3	6.7	2.2E−4	7.4	4.4E−5	7.7	2.1E−5	8.0
32	2.3E−6	7.9	2.0E−4	7.4	2.8E−5	7.7	4.9E−6	9.0	2.6E−6	7.9
64	2.9E−7	8.0	2.6E−5	7.7	2.9E−6	9.7	6.2E−7	8.0	3.1E−7	8.4
N	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\tilde{\varepsilon}_N^{(1)}$	$\tilde{\varrho}_N^{(1)}$	$\tilde{\varepsilon}_N^{(1)}$	$\tilde{\varrho}_N^{(1)}$	$\tilde{\varepsilon}_N^{(1)}$	$\tilde{\varrho}_N^{(1)}$	$\tilde{\varepsilon}_N^{(1)}$	$\tilde{\varrho}_N^{(1)}$
8	6.7E−4	7.1	1.6E−2	5.6	2.6E−3	6.6	8.3E−4	6.6	6.9E−4	7.0
16	9.0E−5	7.4	2.3E−3	6.8	3.6E−4	7.4	1.2E−4	7.2	9.2E−5	7.5
32	1.2E−5	7.6	3.1E−4	7.4	4.6E−5	7.7	1.4E−5	8.0	1.2E−5	7.6
64	1.5E−6	7.8	4.0E−5	7.7	5.0E−6	9.2	1.9E−6	7.7	1.5E−6	7.9
N	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\tilde{\varepsilon}_N^{(2)}$	$\tilde{\varrho}_N^{(2)}$	$\tilde{\varepsilon}_N^{(2)}$	$\tilde{\varrho}_N^{(2)}$	$\tilde{\varepsilon}_N^{(2)}$	$\tilde{\varrho}_N^{(2)}$	$\tilde{\varepsilon}_N^{(2)}$	$\tilde{\varrho}_N^{(2)}$
8	3.2E−4	6.9	2.5E−2	6.4	3.6E−3	6.8	2.7E−4	8.8	2.8E−4	7.0
16	4.2E−5	7.6	3.8E−3	6.6	4.9E−4	7.3	3.2E−5	8.3	3.7E−5	7.5
32	5.3E−6	7.8	5.2E−4	7.3	6.6E−5	7.4	4.3E−6	7.6	4.6E−6	8.2
64	6.7E−7	7.9	7.0E−5	7.4	6.4E−6	10.3	8.1E−7	5.3	7.4E−7	6.2

characterizing the observed convergence rate. From (5.4), (4.11), (4.12) and (5.5) we obtain for $r = 2$, $r_1 = 6$ and for sufficiently large N and $M = \gamma N$ with a positive constant γ the estimates

$$\tilde{\varepsilon}_N^{(i)} \le c\,N^{-3}(1 + \log N), \quad i = 0, 1, 2.$$

The corresponding ratios should be equal $\hat{\varrho}_N^{(i)} \approx 8$ ($i = 0, 1, 2$).

The results in Tab. 5.1 are calculated using the second expression of $\tilde{B}_0 w$ in formula (3.3). Although the kernel K_0 in this example satisfies estimates (1.4) with $\nu = 1/2 > 0$ and it is not bounded at $t = s$, we can also use the first expression of $\tilde{B}_0 w$ in (3.3) (since 0 and 1 are not among the knots of the quadrature formula). This version of the method converges nearly with the same rate although the absolute values of the errors are somewhat smaller.

The numerical results presented in Tab. 5.1 show that in general the derived theoretical error estimates express the actual convergence rate of the collocation method and the fully discrete collocation method well enough. We note, that a good choice of M is $M = 2N$.

References

[1] H. Brunner. *Collocation Methods for Volterra Integral and Related Functional Equations.* Cambridge Monogr. Appl. and Comput. Math., 15. Cambridge University Press, Cambridge, UK, 2004.

[2] H. Brunner, A. Pedas and G. Vainikko. Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels. *SIAM J. Numer. Anal.*, **39**(3):957–982, 2001.

[3] M. Ganesh and I. H. Sloan. Optimal order spline methods for nonlinear differential and integro-differential equations. *Appl. Numer.Math.*, **29**(4):445–478, 1999. Doi:10.1016/S0168-9274(98)00067-1.

[4] H. Kaneko and Y. Xu. Gauss-type quadratures for weakly singular integrals and their application to Fredholm integral equations of second kind. *Math. Comp.*, **62**(206):739–753, 1994. Doi:10.2307/2153534.

[5] R. Kangro and I. Kangro. On the stability of piecewise polynomial collocation methods for solving weakly singular integral equations of the second kind. *Math. Model. Anal.*, **13**(1):29–36, 2008. Doi:10.3846/1392-6292.2008.13.29-36.

[6] R. Kangro and I. Kangro. On fully discrete collocation methods for solving weakly singular integral equations. *Math. Model. Anal.*, **14**(1):69–78, 2009. Doi:10.3846/1392-6292.2009.14.69-78.

[7] I. Parts, A. Pedas and E. Tamme. Piecewise polynomial collocation for Fredholm integro-differential equations with weakly singular kernels. *SIAM J. Numer. Anal.*, **43**(5):1897–1911, 2005. Doi:10.1137/040612452.

[8] A. Pedas and E. Tamme. Spline collocation method for integro-differential equations with weakly singular kernels. *J. Comput. Appl. Math.*, **197**(1):253–269, 2006. Doi:10.1016/j.cam.2005.07.035.

[9] A. Pedas and E. Tamme. Discrete Galerkin method for Fredholm integro-differential equations with weakly singular kernels. *J. Comput. Appl. Math.*, **213**(1):111–126, 2008. Doi:10.1016/j.cam.2006.12.024.

[10] A. Pedas and E. Tamme. Fully discrete Galerkin method for Fredholm integro-differential equations with weakly singular kernels. *Comput. Methods Appl. Math.*, **8**(3):294–308, 2008.

[11] E. Tamme. Fully discrete collocation method for weakly singular integral equations. *Proc. Estonian Acad. Sci. Phys. Math.*, **50**(3):133–144, 2001.

[12] W. Volk. The numerical solution of linear integrodifferential equations by projection methods. *J. Integral Equations*, **9**(1):171–190, 1985.