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Almost Periodic Solutions of Impulsive Integro-Differential Neural Networks*

Gani Tr. Stamov[1] **and Jehad O. Alzabut**[2]

[1] *Department of Mathematics, Technical University–Sofia*
8800 Sliven, Bulgaria
E-mail: gstamov@abv.bg

[2] *Department of Mathematics and Physical Sciences, Prince Sultan University*
P. O. Box 66833, Riyadh 11586, Saudi Arabia
E-mail(*corresp.*): jalzabut@psu.edu.sa

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Abstract. In this paper, sufficient conditions are established for the existence of almost periodic solutions for system of impulsive integro–differential neural networks. Our approach is based on the estimation of the Cauchy matrix of linear impulsive differential equations. We shall employ the contraction mapping principle as well as Gronwall–Bellman's inequality to prove our main result.

Keywords: almost periodic solution; neural networks; contraction mapping principle; Gronwall–Bellman's inequality.

AMS Subject Classification: 34K14; 34K45.

1 Introduction

Due to their immense potentials of application perspective in different areas such as pattern recognition, optimization, signal and image processing, robotics and psychophysics, models of neural networks have been extensively studied in the recent years. Indeed, they have been the object of intensive analysis by many authors who established good results concerning the qualitative properties of their solutions. These models have been mainly investigated in form of delayed neural networks or impulsive delayed neural networks, see the papers [7, 8, 12, 17, 22, 24, 19, 20, 25, 32, 33, 34, 35, 38].

The reader can easily realize, nevertheless, that most of equations of models considered in the above mentioned papers are subject to periodic assumptions and the authors, in particular, studied the existence of their periodic

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solutions, see also [11, 13, 18, 37, 42]. On the other hand, upon considering long–term dynamical behaviors it is possible for the various components of the model to be periodic with rationally independent periods, and therefore it is more reasonable to consider the various parameters of models to be changing almost–periodically rather than periodically with a common period. Thus, the investigation of almost periodic behavior of solutions is considered to be more accordant with reality. Although it has widespread applications in real life, the generalization to the notion of almost periodicity is not as developed as that of periodic solutions. To the best of authors' knowledge, there are a few recent published papers considering the notion of almost periodicity of differential equations with or without impulses, see the papers [1, 2, 3, 14, 15, 16, 23, 28, 29, 30, 31, 36, 39, 40, 41].

Motivated by this, the aim of this paper is to establish sufficient conditions for the existence and exponential stability of almost periodic solutions of general model of neural networks. However, it is known that many real world phenomena often behave in a piecewise continuous frame interlaced with abrupt changes. Thus, the choice of system of neural networks accompanied with impulsive conditions would be more appropriate. For more details on impulsive differential equations and their applications, we refer the readers to [4, 5, 6, 9, 10, 21, 26, 27].

In this paper, we shall employ the contraction mapping principle as well as the Gronwall-Bellman's inequality to prove the existence of almost periodic exponential stable solutions for system of impulsive integro-differential neural networks. Our approach is based on the estimation of the Cauchy matrix of linear impulsive differential equations. The impulsive integro-differential neural networks are natural generalizations of Hopfield neural networks and may be used for applying certain mathematical simulations and describing some real life phenomena which are subject to short-term perturbations during their evolutions.

2 Preliminary Notes

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with elements $x=(x_1, x_2, .., x_n)^T$ and the norm $|x| = \max_i |x_i|$. Let Ω be a domain in $\mathbb{R}^n$ such that $\Omega \neq \emptyset$. By $\mathcal{B} = \big\{\{\tau_k\} : \tau_k \in \mathbb{R}, \tau_k < \tau_{k+1}, k \in \mathbb{Z}\big\}$, we denote the set of all unbounded and strictly increasing sequences. We shall investigate the problem of existence of exponentially stable almost periodic solution for system of impulsive integro–differential neural networks of the form

$$\begin{cases} \dfrac{dx_i(t)}{dt} = \sum\limits_{j=1}^{n} a_{ij}(t)x_j(t) + \sum\limits_{j=1}^{n} \int\limits_{t_0}^{t} k_{ij}(t,s)x_j(s)\,\mathrm{d}s \\ \qquad\qquad + \sum\limits_{j=1}^{n} \alpha_{ij}(t)f_j(x_j(t)) + \gamma_i(t), \quad t \neq \tau_k, \quad i = 1, 2, \dots, n, \\ \Delta x(t) = A_k x(t) + I_k(x(t)) + \gamma_k, \quad t = \tau_k, \ k \in \mathbb{Z}, \end{cases} \tag{2.1}$$

where

(i) $t_0, t \in \mathbb{R},\ a_{ij}(t),\ \alpha_{ij}(t),\ \gamma_i(t) \in C(\mathbb{R}, \mathbb{R}),\ k_{ij}(t,s) \in C(\mathbb{R}^2, \mathbb{R}),$ $f_j(t) \in C(\mathbb{R}, \mathbb{R}),\ i, j = 1, 2, \dots, n;$

(ii) $A_k \in \mathbb{R}^{n\times n},\ I_k(x) \in C(\Omega, \mathbb{R}^n),\ \gamma_k \in \mathbb{R}^n,\ k \in \mathbb{Z};$

(iii) $\Delta x(t) = x(t+0) - x(t-0),\ \{\tau_k\} \in \mathcal{B}.$

Let $J \subset \mathbb{R}$. By $PLC(J, \mathbb{R}^n)$, we denote the space of all piecewise continuous functions $x : J \to \mathbb{R}^n$ with points of discontinuity of the first kind at τ_k such that x is left continuous, i.e. the following relations hold

$$x(\tau_k - 0) = x(\tau_k),\ x(\tau_k + 0) = x(\tau_k) + \Delta x(\tau_k),\ k \in \mathbb{Z}.$$

It follows that the solution $x(t)$ of (2.1) is from the space $PLC(J, \mathbb{R}^n)$ and thus one may adopt some definitions and facts concerning the concept of almost periodicity for piecewise continuous functions.

Definition 1. [27] The set of sequences $\{\tau_{k+j} - \tau_k\}$ $k, j \in \mathbb{Z}$ is said to be *uniformly almost periodic* if for arbitrary $\varepsilon > 0$ there exists a relatively dense set of $\varepsilon-$almost periods common for any sequences.

Definition 2. [27] The function $g \in PLC(\mathbb{R}, \mathbb{R}^n)$ is said to be *almost periodic* if

(a1) the set of sequences $\{\tau_{k+j} - \tau_k\},\ k, j \in \mathbb{Z}$, $\{\tau_k\} \in \mathcal{B}$ is uniformly almost periodic.

(a2) for any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $g(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|g(t') - g(t'')| < \varepsilon$.

(a3) for any $\varepsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $\Big|g(t+\tau) - g(t)\Big| < \varepsilon$ for all $t \in \mathbb{R}$ satisfying the condition $|t - \tau_k| > \varepsilon$, $k \in \mathbb{Z}$.

Together with (2.1), we consider the linear integro-differential system

$$\begin{cases} \dfrac{\partial R(t,s)}{\partial t} = A(t)R(t,s) + \int\limits_{t_0}^{t} K(t,v)R(v,s)\,\mathrm{d}v,\ s \neq \tau_k,\ t \neq \tau_k, \\ R(\tau_k + 0, s) = (E + A_k)R(\tau_k, s),\ k = \pm 1, \pm 2, \ldots, \\ R(s,s) = E, \end{cases} \tag{2.2}$$

where $R(t,s) \in \mathbb{R}^{n\times n}$, $t_0, t \in \mathbb{R}$, $A(t) = (a_{ij}(t))$, $K(t,v) = (k_{ij}(t,v))$, $i,j = 1, 2, \ldots, n$, $E \in \mathbb{R}^{n\times n}$ and $\{\tau_k\} \in \mathcal{B}$.

Introduce the following conditions:

(H1) There exists a matrix function $R(t,s) \in \mathbb{R}^{n\times n}$ that satisfies (2.2);

(H2) $\det(E + A_k) \neq 0,\ k \in \mathbb{Z};$

(H3) $\mu[A(t) - R(t,t)] \leq -\alpha$, where $\alpha > 0$ and $\mu[\cdot]$ denotes the logarithmic norm.

Lemma 1. [26] *Let conditions* H1–H3 *be fulfilled. Then*

$$|R(t,s)| \le Ke^{-\alpha(t-s)}, \quad t > s, \tag{2.3}$$

where $K > 0$.

Introduce the following conditions:

(H4) The matrix $A(t) = (a_{ij}(t)),\ i,j = 1,2,\dots,n$, is almost periodic in the sense of Bohr;

(H5) The sequence $\{A_k\},\ k \in \mathbb{Z}$, is almost periodic;

(H6) The set of sequences $\{\tau_{k+j} - \tau_k\},\ k,j \in \mathbb{Z}$, $\{\tau_k\} \in \mathcal{B}$ is uniformly almost periodic and $\inf_k\{\tau_{k+1} - \tau_k\} = \theta > 0$;

(H7) The matrix $K(t,s) = (k_{ij}(t,s)),\ i,j = 1,2,\dots,n$, is almost periodic along diagonal line, i.e. for any $\varepsilon > 0$, the set $T(K,\varepsilon)$ composed from ε-almost period τ such that for $\tau \in T(K,\varepsilon)$, satisfies the inequality

$$\big|K(t+\tau, s+\tau) - K(t,s)\big| \le \varepsilon e^{-\frac{\alpha}{2}(t-s)},\ t > s$$

and $T(K,\varepsilon)$ is a relatively dense in $\mathbb{R}$;

(H8) The functions $\alpha_{ij}(t)$ are almost periodic in the sense of Bohr such that

$$0 < \sup_{t\in\mathbb{R}} |\alpha_{ij}(t)| = \overline{\alpha}_{ij} < \infty,\ \ i,j = 1,2,\dots,n;$$

(H9) The functions $\gamma_i(t),\ i = 1,2,\dots,n$, are almost periodic in the sense of Bohr, $\{\gamma_k\},\ k \in \mathbb{Z}$, is almost periodic sequence and there exists $C_0 > 0$ such that

$$\max\big\{\max_i |\gamma_i(t)|, \max_k |\gamma_k|\big\} \le C_0;$$

(H10) The functions $f_j(t)$ are almost periodic in the sense of Bohr such that

$$0 < \sup_{t\in\mathbb{R}} |f_j(t)| = \overline{f}_j < \infty,\ f_j(0) = 0$$

and there exists $L_1 > 0$ such that for $t,s \in \mathbb{R}$

$$\max_j \big|f_j(t) - f_j(s)\big| < L_1|t-s|,\ j = 1,2,\dots,n;$$

(H11) The sequence of functions $I_k(x)$ is almost periodic uniformly with respect to $x \in \Omega$ such that

$$0 < \sup_{x\in\Omega} |I_k(x)| = \overline{I}_k < \infty,\ I_k(0) = 0$$

and there exists $L_2 > 0$ for which $\big|I_k(x) - I_k(y)\big| \le L_2|x-y|$, when $k \in \mathbb{Z},\ x,y \in \Omega$.

The following lemmas are essential in proving our main results.

Lemma 2. [27] *Let conditions* H1–H6, H8–H10 *be fulfilled. Then for each* $\varepsilon > 0$ *there exist* ε_1, $0 < \varepsilon_1 < \varepsilon$ *and relatively dense sets* T *of real numbers and* Q *of whole numbers such that the following relations hold:*

(a) $|A(t+\tau) - A(t)| < \varepsilon,\ t \in \mathbb{R},\ \tau \in T;$

(b) $|\alpha_{ij}(t+\tau) - \alpha_{ij}(t)| < \varepsilon,\ t \in \mathbb{R},\ \tau \in T,\ k \in \mathbb{Z},\ i,j = 1,2,\ldots,n;$

(c) $|f_j(t+\tau) - f_j(t)| < \varepsilon,\ t \in \mathbb{R},\ \tau \in T,\ k \in \mathbb{Z},\ j = 1,2,\ldots,n;$

(d) $|A_{k+q} - A_k| < \varepsilon,\ q \in Q,\ k \in \mathbb{Z};$

(e) $|\gamma_j(t+\tau) - \gamma_j(t)| < \varepsilon,\ t \in \mathbb{R},\ \tau \in T,\ k \in \mathbb{Z},\ j = 1,2,\ldots,n;$

(f) $|\gamma_{k+q} - \gamma_k| < \varepsilon,\ q \in Q,\ k \in \mathbb{Z};$

(g) $|\tau_{k+q} - \tau| < \varepsilon_1,\ q \in Q,\ \tau \in T,\ k \in \mathbb{Z}.$

Lemma 3. [27] *Let the set of sequences* $\{\tau_{k+j} - \tau_k\},\ k,j \in \mathbb{Z},\ \{\tau_k\} \in \mathcal{B}$ *be uniformly almost periodic. Then for each* $p > 0$ *there exists a positive integer* N *such that on each interval of length* p *there is no more than* N *elements of the sequence* $\{\tau_k\}$*, i.e.,* $i(s,t) \le N(t-s) + N$*, where* $i(s,t)$ *is the number of points* τ_k *in the interval* (s,t)*.*

3 The Main results

Lemma 4. *Let conditions* H1–H7 *be fulfilled. Then the matrix function* $R(t,s)$ *is almost periodic along diagonal line and*

$$\big|R(t+\tau, s+\tau) - R(t,s)\big| \le \varepsilon \Gamma e^{-\frac{\alpha}{2}(t-s)},\ t > s, \tag{3.1}$$

where $\Gamma > 0,\ \varepsilon > 0$ *and* τ *is almost period.*

Proof. Let $\varepsilon > 0$ be given and τ is common ε-almost period of $A(t)$ and $K(t,s)$. Then

$$\begin{aligned}\frac{\partial R(t+\tau,s+\tau)}{\partial t} &= A(t)R(t+\tau,s+\tau) + \big(A(t+\tau) - A(t)\big)R(t+\tau,s+\tau)\\ &+ \int_s^t \big(K(t+\tau,v+\tau) - K(t,v)\big)R(v+\tau,s+\tau)\,\mathrm{d}v\\ &+ \int_s^t K(t,v)R(v+\tau,s+\tau)\,\mathrm{d}v, \quad s \ne \tau_k',\ t \ne \tau_k',\end{aligned}$$

for

$$R(\tau_k' + \tau, s+\tau) = (E + A_k)R(\tau_k' + \tau, s+\tau) + (A_{k+q} - A_k)R(t_k + \tau, s+\tau),$$

where $\tau'_k = \tau_k - \tau$ and $\tau,\ q$ are the numbers defined in Lemma 2. Hence from (2.2) we obtain

$$R(t+\tau, s+\tau) - R(t,s) = \int_s^t R(t,u)\big(A(u+\tau) - A(u)\big)R(u+\tau, s+\tau)\,\mathrm{d}u$$
$$+ \int_s^t R(t,u)\Big(\int_s^u \big(K(u+\tau, v+\tau) - K(u,v)\big)R(v+\tau, s+\tau)\,\mathrm{d}v\Big)\,\mathrm{d}u$$
$$+ \sum_{s \le \tau'_v < t} R(t, \tau'_v + 0)\big(A_{v+q} - A_v\big)R(\tau'_v + \tau, s+\tau). \tag{3.2}$$

In view of Lemma 2, it follows that if $|t - \tau'_k| > \varepsilon,\ t \in \mathbb{R}$ then $\tau'_{k+q} < t + \tau < \tau'_{k+q+1}$. Then from (2.3), (3.2) and Lemma 2, we have

$$\big|R(t+\tau, s+\tau) - R(t,s)\big| \le K^2\varepsilon\Big(e^{-\alpha(t-s)}(t-s) + \frac{4}{\alpha^2}e^{-\frac{\alpha}{2}(t-s)}$$
$$+ i(s,t)e^{-\alpha(t-s)}\Big) \le \varepsilon\Gamma e^{-\frac{\alpha}{2}(t-s)},\ \ t > s,$$

where $\Gamma = K^2\frac{2}{\alpha}\big(1 + \frac{2}{\alpha} + N + \frac{N\alpha}{2}\big)$. Thus, proof of Lemma 4 is complete. □

Now we are in a position to state and prove the main theorem of our paper.

Theorem 1. *Let the following assumptions be satisfied:*

(i) *Conditions* H1–H11 *be fulfilled;*

(ii) *The number* $r = K\big\{\max_i L_1\alpha^{-1}\sum_{j=1}^n \overline{\alpha}_{ij} + 2NL_2/(1 - e^{-\alpha})\big\} < 1.$

Then

(1) *There exists a unique almost periodic solution* $x(t)$ *of* (2.1).

(2) *If the following inequalities*

$$1 + KL_2 < e \ \text{ and } \ \alpha - KL_1\max_i \sum_{j=1}^n \overline{\alpha}_{ij} - N\ln(1 + KL_2) > 0$$

hold, then the solution $x(t)$ *of* (2.1) *is exponentially stable.*

Proof of Assertion 1. By $D,\ D \subset PLC(\mathbb{R}, \mathbb{R}^n)$, we denote the set of all almost periodic functions $\varphi(t)$ satisfying the inequality $\|\varphi\| < \overline{K}$, where

$$\|\varphi\| = \sup_{t\in\mathbb{R}}|\varphi(t)|, \quad \overline{K} = KC_0\Big(\frac{1}{\alpha} + \frac{2N}{1 - e^{-\alpha}}\Big).$$

Set $\gamma(t) = col(\gamma_1(t), \gamma_2(t), .., \gamma_n(t)),\ F(t,x) = col\big\{F_1(t,x), F_2(t,x), .., F_n(t,x)\big\}$, where

$$F_i(t,x) = \sum_{j=1}^n \alpha_{ij}(t)f_j(x_j), \ \ i = 1, 2, \dots, n.$$

In virtue of [27], we know that the solution $x(t)$ of (2.1) has the form

$$x(t) = R(t,t_0)x(t_0) + \int_{t_0}^{t} R(t,s)\big[F(s,x(s)) + \gamma(s)\big]\,\mathrm{d}s + \sum_{t_0<\tau_k<t} R(t,\tau_k)\big[I_k(x(\tau_k)) + \gamma_k\big].$$

Define an operator S in D as follows

$$S\varphi = \int_{-\infty}^{t} R(t,s)\big[F\big(s,\varphi(s)\big) + \gamma(s)\big]\,\mathrm{d}s + \sum_{\tau_k<t} R(t,\tau_k)\big[I_k\big(\varphi(\tau_k)\big) + \gamma_k\big]. \tag{3.3}$$

Let the subset D^*, $D^* \subset D$, be defined as follows

$$D^* = \big\{\varphi \in D : \|\varphi - \varphi_0\| \le r\overline{K}/(1-r)\big\},$$

where

$$\varphi_0 = \int_{-\infty}^{t} R(t,s)\gamma(s)\,\mathrm{d}s + \sum_{t_k<t} R(t,\tau_k)\gamma_k.$$

Taking the norm, we obtain

$$\begin{aligned}
\|\varphi_0\| &= \sup_{t\in\mathbb{R}}\Big\{\max_i\Big(\int_{-\infty}^{t} |R(t,s)||\gamma_i(s)|\Big)\,\mathrm{d}s + \sum_{\tau_k<t}|R(t,\tau_k)||\gamma_k|\Big\}\\
&\le \sup_{t\in\mathbb{R}}\Big\{\max_i\Big(\int_{-\infty}^{t} Ke^{-\alpha(t-s)}|\gamma_i(s)|\,\mathrm{d}s\Big) + \sum_{\tau_k<t} Ke^{-\alpha(t-\tau_k)}|\gamma_k|\Big\}\\
&\le K\Big(\frac{C_0}{\alpha} + \frac{2NC_0}{1-e^{-\alpha}}\Big) = \overline{K}.
\end{aligned} \tag{3.4}$$

Then for arbitrary $\varphi \in D^*$, it follows from (3.3) and (3.4) that

$$\|\varphi\| \le \|\varphi-\varphi_0\| + \|\varphi_0\| \le \frac{r\overline{K}}{1-r} + \overline{K} = \frac{\overline{K}}{1-r}.$$

Now we prove that S is self–mapping from D^* to D^*. For arbitrary $\varphi \in D^*$, it follows that

$$\begin{aligned}
&\|S\varphi - \varphi_0\| \le \sup_{t\in\mathbb{R}}\Big\{\max_i\Big(\int_{-\infty}^{t} |R(t,s)|\sum_{j=1}^{n}|\alpha_{ij}(s)||f_j(\varphi_j(s))|\,\mathrm{d}s\Big)\\
&\quad + \sum_{\tau_k<t}|R(t,\tau_k)||I_k(\varphi(\tau_k))|\Big\} \le \Big\{\max_i\Big(\int_{-\infty}^{t} Ke^{-\alpha(t-s)}\sum_{j=1}^{n}\overline{\alpha}_{ij}L_1\,\mathrm{d}s\Big)\\
&\quad + \sum_{\tau_k<t} Ke^{-\alpha(t-\tau_k)}L_2\Big\}\|\varphi\| \le K\Big\{\max_i \alpha^{-1}\sum_{j=1}^{n}\overline{\alpha}_{ij}L_1 + \frac{2NL_2}{1-e^{-\alpha}}\Big\}\|\varphi\|\\
&\quad = r\|\varphi\| \le \frac{r\overline{K}}{1-r}.
\end{aligned} \tag{3.5}$$

Let $\tau \in T,\ q \in Q$ where the sets T and Q are defined as in Lemma 2. Then

$$\begin{aligned}
&\|S\varphi(t+\tau) - S\varphi(t)\| \le \sup_{t\in\mathbb{R}} \Big\{ \max_i \Big[\int_{-\infty}^{t} \big|R(t+\tau, s+\tau) - R(t,s)\big| \\
&\Big|\sum_{j=1}^{n} \alpha_{ij}(s+\tau) f_j\big(\varphi_j(s+\tau)\big)\Big|\, \mathrm{d}s + \int_{-\infty}^{t} |R(t,s)| \Big|\sum_{j=1}^{n} \alpha_{ij}(s+\tau) f_j\big(\varphi_j(s+\tau)\big) \\
&- \sum_{j=1}^{n} \alpha_{ij}(s) f_j\big(\varphi_j(s)\big)\Big|\, \mathrm{d}s\Big] + \sum_{\tau_k<t} \big|R(t+\tau, \tau_{k+q}) - R(t,\tau_k)\big| \big|I_{k+q}\big(\varphi(\tau_{k+q})\big)\big| \\
&+ \sum_{\tau_k<t} |R(t,\tau_k)| \big|I_{k+q}\big(\varphi(\tau_{k+q})\big) - I_k\big(\varphi(\tau_k)\big)\big| \Big\} \le \varepsilon C_1,
\end{aligned} \tag{3.6}$$

where

$$C_1 = \frac{2\Gamma}{\alpha} \max_i \sum_{j=1}^{n} \Big((1+L_1)\overline{\alpha}_{ij} + \overline{f}_j\Big) + 2N\Big(\frac{\Gamma}{1-e^{-\frac{\alpha}{2}}} \overline{I}_k + \frac{L_2+1}{1-e^{-\alpha}}\Big).$$

From (3.5) and (3.6), we deduce that $S\varphi \in D^*$. Let $\varphi \in D^*$, $\psi \in D^*$. We get

$$\begin{aligned}
\|S\varphi - S\psi\| &\le \sup_{t\in\mathbb{R}} \Big\{ \max_i \Big[\int_{-\infty}^{t} |R(t,s)| \sum_{j=1}^{n} |\alpha_{ij}(s)| \big|f_j(\varphi_j(s)) - f_j(\psi_j(s))\big|\, \mathrm{d}s\Big] \\
&\quad + \sum_{\tau_k<t} |R(t,\tau_k)| \big|I_k(\varphi(\tau_k)) - I_k(\psi(\tau_k)\big| \Big\} \\
&\le K\Big\{ \max_i L_1\alpha^{-1} \sum_{j=1}^{n} \overline{\alpha}_{ij} + \frac{2NL_2}{1-e^{-\alpha}} \Big\} \|\varphi - \psi\| = r\|\varphi - \psi\|.
\end{aligned} \tag{3.7}$$

By the assumption (ii), it follows that S is a contractive mapping in D^*. So there exists a unique almost periodic solution $x(t)$ of (2.1). □

Proof of Assertion 2. Let $y(t)$ be an arbitrary solution of (2.1). It follows that

$$\begin{aligned}
y(t) - x(t) =& R(t,t_0)\big(y(t_0) - x(t_0)\big) + \int_{t_0}^{t} R(t,s)\big[F(s,y(s)) - F(s,x(s))\big]\, \mathrm{d}s \\
&+ \sum_{t_0<\tau_k<t} R(t,\tau_k)\big(I_k(y(\tau_k) - I_k(x(\tau_k))\big).
\end{aligned}$$

Taking the norm, we get

$$\begin{aligned}
&\big|y(t) - x(t)\big| \le Ke^{-\alpha(t-t_0)}\big|y(t_0) - x(t_0)\big| + \max_i \Big(\int_{t_0}^{t} Ke^{-\alpha(t-s)} L_1 \\
&\times \sum_{j=1}^{n} \overline{\alpha}_{ij} \big|y_i(s) - x_i(s)\big|\, \mathrm{d}s\Big) + \sum_{t_0<\tau_k<t} Ke^{-\alpha(t-\tau_k)} L_2 \big|y(\tau_k) - x(\tau_k)\big|.
\end{aligned}$$

Setting $u(t) = \big|y(t) - x(t)\big|e^{\alpha t}$ and applying Gronwall–Bellman's inequality we have

$$\big|y(t) - x(t)\big| \le K\big|y(t_0) - x(t_0)\big|(1 + KL_2)^{i(t_0,t)} \\ \times \exp\big(-\alpha + KL_1 \max_i \sum_{j=1}^{n} \overline{\alpha}_{ij}\big)(t - t_0).$$

Thus, proof of Theorem 1 is finished. □

Example 1. Let us consider the following model of impulsive integro–differential neural networks

$$\begin{cases} \dfrac{dx_i(t)}{dt} = -x_i(t) + \displaystyle\int_0^t e^{-3(t-s)} x_i(s)\,\mathrm{d}s + \sum_{j=1}^{n} \alpha_{ij} f_j(x_j(t)) + \gamma_i(t), \;\; t \ne \tau_k, \\ \Delta x(t) = p_k x(t) + \gamma_k, \; t = \tau_k, \; i = 1, 2, \dots, n, \; k \in \mathbb{Z}, \end{cases} \tag{3.8}$$

where $t \in \mathbb{R}, \; \alpha_{ij} \in \mathbb{R}, \; f_j(t), \; \gamma_i(t), \in C(\mathbb{R}, \mathbb{R}), \{\tau_k\} \in \mathcal{B}$ and

$$p_k = (p_{1k}, p_{2k}, \dots, p_{nk}), \; -1 < p_{ik} < 0, \; \gamma_k \in \mathbb{R}^n, \; i, j = 1, 2, \dots, n, \;\; k \in \mathbb{Z}.$$

It is easy to check that if the sequence $\{p_k\}$ is almost periodic then for the following integro-differential system

$$\begin{cases} \dfrac{dx_i(t)}{dt} = -x_i(t) + \displaystyle\int_0^t e^{-3(t-s)} x_i(s)\,\mathrm{d}s, \; t \ne \tau_k, \;\; i = 1, 2, \dots, n, \\ \Delta x(t) = p_k x(t), \;\; t = \tau_k, \; k \in \mathbb{Z}, \end{cases}$$

for the functions $R(t, 0) = R(t)$, $R(t) = (R_1(t), R_2(t), \dots, R_n(t))$, where

$$R_i(t) = \frac{1}{2\sqrt{2}}\Big[\prod_{l=1}^{k}(1 + p_{il})\Big(\sqrt{2} + 1 + \frac{\sqrt{2} - 1}{e^{2\sqrt{2}\tau_l}}\Big)e^{-(2-\sqrt{2})\tau_l}\Big]e^{-(2-\sqrt{2})t}, \; i = 1, 2, \dots, n$$

the inequality (3.1) holds. Let conditions H9–H10 hold. Then

$$K^* L_1 (2 - \sqrt{2})^{-1} \max_i \sum_{j=1}^{n} \alpha_{ij} < 1 \;\text{ and }\;\; 2 - \sqrt{2} - K^* L_1 \max_i \sum_{j=1}^{n} \alpha_{ij} > 0,$$

where

$$K^* = \frac{1}{2\sqrt{2}}\Big[\prod_{l=1}^{k}(1 + p_{il})\Big(\sqrt{2} + 1 + \frac{\sqrt{2} - 1}{e^{2\sqrt{2}\tau_l}}\Big)e^{-(2-\sqrt{2})\tau_l}\Big].$$

Thus all assumptions of Theorem 1 are satisfied. Therefore, there exists a unique almost periodic solution for equation (3.8) which is exponentially stable.

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