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Picard-Reproducing Kernel Hilbert Space Method for Solving Generalized Singular Nonlinear Lane-Emden Type Equations

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Abstract. An iterative method is discussed with respect to its effectiveness and capability of solving singular nonlinear Lane-Emden type equations using reproducing kernel Hilbert space method combined with the Picard iteration. Some new error estimates for application of the method are established. We prove the convergence of the combined method. The numerical examples demonstrates a good agreement between numerical results and analytical predictions.

Keywords: reproducing kernel Hilbert space, Picard iteration, error estimate, convergence, singular nonlinear Lane-Emden type equations.

AMS Subject Classification: 65L05; 46E22; 34B16.

1 Introduction

Generalized Lane-Emden equations arise in the modeling of several phenomena in physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere, theory of thermionic currents, pattern formation, population evolution [2, 3, 6, 16, 17] and have attracted much attention in recent years. Lane-Emden equation is a singular nonlinear differential equation which describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. This equation is one of the basic equations in the theory of stellar structure and has been the focus of many studies. In recent years, the approximate solutions to the Lane-Emden equation were given by homotopy perturbation method [15, 23], the Legendre wavelets [24], perturbation method [10], the Adomian decomposition method [21], the Bessel collocation method [25], the Pade series method [19], the rational Legendre pseudospectral method [14], the Taylor series method [11], the nonperturbative approximate method [18], and the Hermite functions collocation method [13]. The numerical solving of the Lane-Emden problem, is challenging because of the nonlinearity and singular behavior at the origin. Most of the methods, which used to solve nonlinear differential equations, transform the equation into a system of nonlinear equations. It is cumbersome to solve these systems, or the solution may be unreliable. In order to overcome the nonlinearity of problem we use the Picard iteration. The convergence and an error estimate for implementation of Picard iteration are established. After linearization the reproducing kernel Hilbert space method generates a rapid convergent series solution with easily computable components. We also obtained the truncation error estimate of the series solution. In fact this work presents reproducing kernel Hilbert space method combined with the Picard iteration method for solving singular nonlinear Lane-Emden type equations and the effectiveness and performance of the method is studied. Picard-Reproducing kernel Hilbert space method, combines advantages of these two methods and therefore can be used to solve efficiently singular nonlinear Lane-Emden type equations. We prove the convergence of the combined method. The kernel based methods for approximating solutions of differential equations, are a recent and fast growing research area that spans many different fields in applied mathematics, science and engineering. The reproducing kernels have successfully been applied to several nonlinear problems such as, nonlinear system of boundary value problems, singular nonlinear initial and boundary value problems, singular nonlinear two-point periodic boundary value problem, nonlinear systems of partial differential equations and multiple solutions of nonlinear boundary value problems [1, 7, 8, 9, 12, 20, 22]. In this article, we discuss the numerical method for the generalized Lane-Emden equation

$$\begin{cases} y'' + \frac{\alpha}{x} y' + f(x,y) = g(x), \quad 0 < x \le 1,\ \alpha \ge 0, \\ y(0) = A,\ y'(0) = 0, \end{cases} \tag{1.1}$$

where A is a constant and f is real valued continuous function and $g \in C[0,1]$. Several numerical examples are given to show applicability and accuracy of the proposed numerical method.

2 Picard iteration

Definition 1. *For $u \in L^\infty[0,1]$ let*

$$(\mathfrak{D}u)(x) := \int_0^x t^{-\alpha} \int_0^t s^\alpha u(s) ds dt, \quad \forall x > 0.$$

Applying the operator $\mathfrak{D}$ to the first equation (1.1), we have

$$y(x) = A + \mathfrak{D}(g(x) - f(x, y(x))) =: (\mathcal{T}y)(x), \tag{2.1}$$

where $A = y(0)$.

Proposition 1. *Let $g \in L^\infty[0,1]$ and $f(x,y(x)) \in L^\infty(\mathbb{D})$, where $\mathbb{D} = [0,1] \times [0, y_{\max}]$ and $y_{\max} = \max_{0\le x\le 1} |y(x)|$. Every solution y of the fixed point problem (2.1) has the following properties*

1. $\lim_{x\downarrow 0} y(x) = A.$

2. $y \in C^1[0,1], \quad y'(x) = -\int_0^x \left(\frac{s}{x}\right)^\alpha \big(f(s,y(s)) - g(s)\big)ds, \quad \forall x > 0.$

Especially $\lim_{x\downarrow 0} y'(x) = 0$ *and if* $f(x,y(x)) - g(x) > 0$ *a.e. for* $x \in [0,1]$ *then* $y'(x) < 0$ *and* $0 \le y(x) \le A$ *for all* $x \in [0,1]$.

Proof. 1. Let $\delta > 0, c_0 = \sup_{x\in[0,\delta]} |f(x,y(x)) - g(x)|$. By assumption $0 \le c_0 < \infty$. Then for all $0 < x \le \delta$

$$|y(x)| = |A + \mathfrak{D}(g(x) - f(x,y(x)))| = |A + \int_0^x t^{-\alpha} \int_0^t s^\alpha \big(g(s) - f(s,y(s)\big) ds dt|$$
$$\le A + \int_0^x t^{-\alpha} \int_0^t s^\alpha |g(s) - f(s,y(s)| ds dt \le A + c_0 \int_0^x t^{-\alpha} \int_0^t s^\alpha ds dt = A + \frac{c_0 x^2}{2(\alpha+1)}.$$

Thus $\lim_{x\downarrow 0} y(x) = A$.

2. For all $x > 0$, let $z = \frac{s}{x}$ then

$$y'(x) = -\int_0^x (\frac{s}{x})^\alpha (f(s,y(s)) - g(s)) ds = -x \int_0^1 z^\alpha (f(zx, y(zx)) - g(zx)) dz.$$

Suppose δ and c_0 are similar to part 1. Then for all $0 < x \le \delta$,

$$|y'(x)| \le c_0 x \int_0^1 z^\alpha dz = c_0 \frac{x}{\alpha+1},$$

thus $\lim_{x\downarrow 0} y'(x) = 0$. □

Theorem 1. *Suppose that $f(x,y)$ satisfies a Lipschitz condition with respect to its second argument*

$$|f(x,u) - f(x,v)| \le k|u - v|, \quad k < \infty,$$

for all $x \in [0,1]$ and $u, v \in L^\infty[0,1]$. Then the fixed point problem (2.1) has the unique fixed point $y^* = \mathcal{T} y^*$. Moreover, the iteration method $y_{m+1} = \mathcal{T} y_m$ converges to y^* and

$$\|y^* - y_m\|_\infty \le \frac{\cosh(\sqrt{k}) L_0}{2(\alpha+1)} \times \frac{k^m}{(2m)!},$$

where $L_0 = \max_{0\le x\le 1} |g(x) - f(x,A)|$.

Proof. From (2.1) for any $x \in (0,1]$ we have

$$|\mathcal{T}u - \mathcal{T}v| = |\mathfrak{D}(f(x,u)) - \mathfrak{D}(f(x,v))|$$
$$\le \int_0^x t^{-\alpha} \int_0^t s^\alpha |f(s,u) - f(s,v)| ds dt \le \frac{kx^2}{2(\alpha+1)} \|u - v\|_\infty. \quad (2.2)$$

In the next step of proof, we need to establish the inequality

$$|\mathcal{T}^n u - \mathcal{T}^n v| \leq \frac{k^n x^{2n}}{(2n)!}\|u - v\|_\infty. \tag{2.3}$$

First note that $(\mathfrak{D}x^j) = \frac{x^{j+2}}{(j+2)(\alpha+j+1)}$. By induction:

1. $n = 1$: From (2.2) the inequality (2.3) is true.

2. Step $n \to n+1$: Assume that (2.3) is valid for an n, then from Proposition 1 and (2.2) we have

$$|\mathcal{T}^{n+1}u - \mathcal{T}^{n+1}v| \leq |\mathcal{T}(\mathcal{T}^n u) - \mathcal{T}(\mathcal{T}^n v)| = |\mathfrak{D}(f(x, \mathcal{T}^n u)) - \mathfrak{D}(f(x, \mathcal{T}^n v))|$$

$$= |\mathfrak{D}(f(x, \mathcal{T}^n u) - f(x, \mathcal{T}^n v))| \leq k\mathfrak{D}|(\mathcal{T}^n u) - (\mathcal{T}^n v)| \leq k\mathfrak{D}(\frac{k^n x^{2n}}{(2n)!}\|u - v\|_\infty)$$

$$= \frac{k^{n+1}x^{2n+2}}{(2n+2)(\alpha+2n+1)(2n)!}\|u - v\|_\infty \leq \frac{k^{n+1}x^{2(n+1)}}{(2(n+1))!}\|u - v\|_\infty,$$

i.e. inequality (2.3) is valid for $n+1$. Since k is a constant and $0 < x \leq 1$, from inequality (2.3) we deduce that there exist $m \in \mathbb{N}$ such that $\mathcal{T}^m$ is a contraction on $L^\infty[0,1]$. Consequently, the Banach fixed point theorem implies that operator $\mathcal{T}^m$ has a unique fixed point $y^* \in L^\infty[0,1]$ and $\lim_{n\to\infty} \mathcal{T}^{mn}y_0 = y^*$ for any $y_0 \in L^\infty[0,1]$. Let y^* be the unique fixed point of $\mathcal{T}^m$, then

$$\mathcal{T}^m y^* = y^* \Rightarrow \mathcal{T}^m(\mathcal{T}y^*) = \mathcal{T}^{m+1}y^* = \mathcal{T}(\mathcal{T}^m y^*) = \mathcal{T}y^*,$$

then by uniqueness of the fixed point of $\mathcal{T}^m$ we deduce that $\mathcal{T}y^* = y^*$. So y^* is also the fixed point of $\mathcal{T}$. Let $\bar{y}$ be another fixed point of $\mathcal{T}$, then

$$\mathcal{T}^m\bar{y} = \mathcal{T}^{m-1}(\mathcal{T}\bar{y}) = \mathcal{T}^{m-1}\bar{y} = ... = \mathcal{T}\bar{y} = \bar{y},$$

then uniqueness of the fixed point of $\mathcal{T}^m$ implies uniqueness of fixed point of $\mathcal{T}$. Moreover for a fixed $y_0 \in L^\infty[0,1]$ let $y_k = \mathcal{T}^k y_0, (k = 0, 1, .., m-1)$, we see that $\lim_{n'\to\infty} \mathcal{T}^{mn'+k}y_0 = y^*, (k = 0, 1, .., m-1)$ and since for any $n \in \mathbb{N}$ there exist a unique $n' \in \mathbb{N}$ and $k = 0, 1, .., m-1$ such that $n = mn' + k$ then we have $\lim_{n\to\infty} \mathcal{T}^n y_0 = y^*$. In addition,

$$|\mathcal{T}^{m+n}y_0 - \mathcal{T}^m y_0| \leq \Sigma_{i=0}^{n-1}|\mathcal{T}^{m+i+1}y_0 - \mathcal{T}^{m+i}y_0|$$
$$\leq \Sigma_{i=0}^{n-1}\frac{k^{m+i}x^{2(m+i)}}{(2(m+i))!}\|y_1 - y_0\|_\infty \leq (\Sigma_{i=0}^{\infty}\frac{k^i}{(2i)!})\frac{k^m}{(2m)!}\|y_1 - y_0\|_\infty.$$

Let $y_0(x) = A$ then we have

$$|y_1(x) - y_0(x)| = |\mathcal{T}y_0 - y_0| = |\mathfrak{D}(g(x) - f(x, A))| \leq \frac{L_0}{2(\alpha+1)}.$$

Since $\lim_{n\to\infty} \mathcal{T}^{m+n}y_0 = y^*$ and $\Sigma_{i=0}^{\infty}\frac{k^i}{(2i)!} = \cosh(\sqrt{k})$ the proof of theorem is complete. □

Each iteration of $y_{m+1} = \mathcal{T}y_m$ gives us the solution of a linear problem

$$\begin{cases} y''_{m+1} + \frac{\alpha}{x}y'_{m+1} + f(x, y_m) = g(x), \quad 0 < x \leq 1,\ \alpha \geq 0, \\ y_{m+1}(0) = A, y'_{m+1}(0) = 0. \end{cases} \tag{2.4}$$

3 Solution procedure and error estimate

Instead of nonlinear problem (1.1) we applied the reproducing kernel Hilbert space method to linear one (2.4) iteratively. Put $Ly \equiv y'' + \frac{\alpha}{x}y'$, after homogenization(such a homogenization can be found in [4]), the problem (2.4) can be convert into the following form

$$\begin{cases} Lu_{m+1} = F(x, u_m), & 0 < x \leq 1,\ \alpha \geq 0, \\ u_{m+1}(0) = 0,\ u'_{m+1}(0) = 0, \end{cases} \tag{3.1}$$

where $F(x, y) = g(x) - f(x, y + A)$ and $y_{m+1} = u_{m+1} + A$. In order to solve problem (3.1), reproducing kernel space $W_2^3[0,1]$ is defined in the following, for more details and proofs we refer to [4].

Definition 2. *The inner product space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x)|$ u, u', u'' are absolutely continuous real valued functions, u, u', u'' ,$u^{(3)} \in L^2[0,1]$, $u(0) = 0$, $u'(0) = 0\}$. The inner product in $W_2^3[0,1]$ is given by*

$$(u(.), v(.))_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(3)}(x)v^{(3)}(x)dx,$$

and the norm $\|u\|_{W_2^3}$ is denoted by $\|u\|_{W_2^3} = \sqrt{(u, u)_{W_2^3}}$,where $u, v \in W_2^3[0,1]$.

In [4], the authors proved that $W_2^3[0,1]$ is a reproducing kernel space. The reproducing kernel $R_x(.) \in W_2^3[0,1]$ can be denoted by

$$R_x(t) = \begin{cases} \frac{x^2t^2}{4} + \frac{x^2t^3}{12} - \frac{xt^4}{24} + \frac{t^5}{120}, & t \leq x, \\ \frac{x^5}{120} - \frac{x^4t}{24} + \frac{(3x^2+x^3)t^2}{12}, & t > x. \end{cases}$$

For the method of obtaining reproducing kernel $R_x(t)$, refer to [4, 7, 8, 9]. For any fixed $x_i \in [0,1]$, Let $\psi_i(x) = L_tR_x(t)|_{t=x_i}$. The subscript t by the operator L indicates that the operator L applies to the function of t.

Theorem 2. *Let $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$.*

Proof. Clearly $\psi_i(x) \in W_2^3[0,1]$. For each fixed function $u \in W_2^3[0,1]$, let we have $(u(.), \psi_i(.))_{W_2^3} = 0, (i = 1, 2, ...)$, which means that

$$\begin{aligned}(u(x), \psi_i(x))_{W_2^3} &= (u(x), L_tR_x(t)|_{t=x_i})_{W_2^3} \\ &= L_t(u(x), R_x(t))_{W_2^3}|_{t=x_i} = (Lu)(x_i) = 0.\end{aligned}$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$, from the existence of L^{-1} in $W_2^3[0,1]$. □

The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad \beta_{ii} > 0,\ \ i = 1, 2, \ldots.$$

Theorem 3. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$ *and the solution of (3.1) is unique, then the solution of (3.1) satisfies the form*

$$u_{m+1}(x,s) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u_m(x_k))\overline{\psi}_i(x).$$

Proof. u_{m+1} in (3.1) can be expanded to Fourier series in terms of orthogonal basis $\overline{\psi}_i \in W_2^3[0,1]$. Note that $(f(x),\varphi_i(x)) = f(x_i)$ for each $f \in W_2^1[0,1]$.

$$\begin{aligned}
u_{m+1}(x) &= \sum_{i=1}^{\infty}(u_{m+1}(x),\overline{\psi}_i(x))\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(u_{m+1}(x),L^*\varphi_k(x))\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(Lu_{m+1}(x),\varphi_k(x))\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(F(x,u_m),\varphi_k(x))\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u_m(x_k))\overline{\psi}_i(x).
\end{aligned}$$

□

Now, the approximate solution $y_{m+1,N}$ can be obtained by taking N terms in the series representation of $y_{m+1} = u_{m+1} + A$ and

$$y_{m+1,N}(x) = A + u_{m+1,N}(x) = A + \sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}F(x_k,u_m(x_k))\overline{\psi}_i(x).$$

The L^{∞}-estimate of the truncation errors is stated as follows.

Theorem 4. *For any* $u \in W_2^3[0,1]$ *we have*

$$\max_{0\le x\le 1}|u_N(x) - u(x)| \le \mathcal{C}_N\,\|u\|, \tag{3.2}$$

and the constant $\mathcal{C}_N \to 0$ *as* $N \to \infty$.

Proof. Let $R_{N,x}(t) = \sum_{i=1}^{N}\overline{\psi}_i(x)\overline{\psi}_i(t)$ then from the orthogonality of functions $\{\overline{\psi}_i\}_{i=1,\dots,N}$ it is easy to see that $u_N(x) = (u(.),R_{N,x}(.))$. By the Cauchy-Schwartz inequality and for any $x \in [0,1]$,

$$\begin{aligned}
|u_N(x) - u(x)| = |&\sum_{i=N+1}^{\infty}(u,\overline{\psi}_i)\overline{\psi}_i(x)|\\
&= |(u(y),\sum_{i=N+1}^{\infty}\overline{\psi}_i(y)\overline{\psi}_i(x))| \le \|u\|\|\sum_{i=N+1}^{\infty}\overline{\psi}_i(y)\overline{\psi}_i(x)\|\\
&= \|u\|\,|\sum_{i=N+1}^{\infty}\overline{\psi}_i^2(x)| = \|u\|\,|R_x(x) - R_{N,x}(x)|.
\end{aligned}$$

Let $\mathcal{C}_N(x) = |R_x(x) - R_{N,x}(x)|$ and $\mathcal{C}_N = \max_{0\le x\le 1}\mathcal{C}_N(x)$, then for any $x \in [0,1]$ we have

$$\begin{aligned}\mathcal{C}_{N+1}(x) = |R_x(x) - R_{N+1,x}(x)| &= \| \sum_{i=N+2}^{\infty} \overline{\psi}_i(y)\overline{\psi}_i(x)\| \\ &= | \sum_{i=N+2}^{\infty} \overline{\psi}_i^2(x)| \le | \sum_{i=N+1}^{\infty} \overline{\psi}_i^2(x)| = \mathcal{C}_N(x),\end{aligned}$$

then we have

$$\mathcal{C}_{N+1} = \max_{0\le x\le 1} \mathcal{C}_{N+1}(x) \le \max_{0\le x\le 1} \mathcal{C}_N(x) = \mathcal{C}_N.$$

So $\{\mathcal{C}_N\}_{N=1}^{\infty}$ is a decreasing positive sequence in $\mathbb{R}$ and obviously converges to zero. □

The equation (3.2) describes the worst-case error behavior of the truncation, and the error is given as a percentage of $u \in W_2^3[0,1]$, which is the only unknown quantity in the error bound. Notice that value of the constant $\mathcal{C}_N$ is calculable.

Theorem 5. *Suppose the conditions of Theorem 1 hold. If y is the true solution of (1.1), then we have*

$$\max_{0\le x\le 1} |y - y_{N,m}| \le \frac{\cosh(\sqrt{k})L_0}{2(\alpha+1)} \times \frac{k^m}{(2m)!} + k_0 C_N, \tag{3.3}$$

where $k_0 = \max_{0\le x\le 1} \sqrt{R_x(x)}$ and $C_N \to 0$ as $N \to 0$.

Proof. Let $u_m = \sum_{i=1}^{\infty} c_i\overline{\psi}_i(x)$, then we have

$$\|y_m - y_{N,m}\| = \|u_m - u_{N,m}\| = \Big(\sum_{i=N+1}^{\infty} c_i^2\Big)^{1/2} = C_N.$$

It is easy to see that $C_{N+1} \le C_N$ i.e C_N is monotonically decreasing with the increasing of N and $C_N \to 0$ as $N \to 0$. By the Cauchy-Schwartz inequality and for any $x \in [0,1]$,

$$\begin{aligned}|y_m(x) - y_{N,m}(x)| = |u_m(x) - u_{N,m}(x)| &= (R_x(.), u_m(.) - u_{N,m}(.)) \\ &\le \|R_x(.)\|\|u_m(.) - u_{N,m}\| = \sqrt{R_x(x)}C_N.\end{aligned}$$

From the Theorem 1 we have

$$\begin{aligned}|y(x) - y_{N,m}(x)| &\le |y(x) - y_m(x)| + |y_m(x) - y_{N,m}(x)| \\ &\le \frac{\cosh(\sqrt{k})L_0}{2(\alpha+1)} \times \frac{k^m}{(2m)!} + k_0 C_N.\end{aligned}$$

□

The error estimate (3.3) contains two parts, the first part is related to iterative scheme (2.4) and vanish as $m \to \infty$ and the second part is truncation error and vanishes as $N \to \infty$.

4 Application of the Method

In this section by considering some numerical examples, we show the capability and versatility of our new method. In order to show the accuracy of approximate solutions in the absence of exact solutions, we shall consider the integral of the squared residual error over the domain,

$$E = \int_0^1 (Res(x))^2 dx,$$

where $Res(x)$ denotes the residual error at the point $x \in [0, 1]$.

Example 1. For $f(x, y) = y^n$, $g(x) = 0$ and $\alpha = 2$ the problem (1.1) is the standard Lane-Emden equation.

$$\begin{cases} y'' + \frac{2}{x}y' + y^n = 0, & 0 < x \leq 1, \\ y(0) = 1, \quad y'(0) = 0. \end{cases}$$

Exact solutions exist only for $n = 0, 1$ and 5, that are given respectively by

$$y(x) = 1 - \frac{x^2}{6},\ y(x) = \frac{\sin(x)}{x}, \quad y(x) = \left(1 + \frac{x^2}{3}\right)^{-1/2}.$$

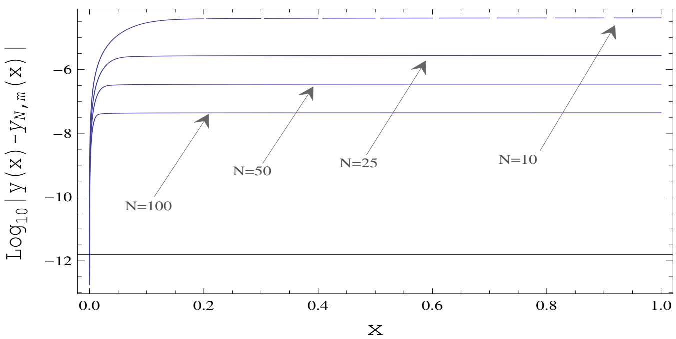

Figure 1. Absolute errors (in logarithmic scale) of approximations, for Example 1 with $n = 0$, m=10 iterations and $N = 10, 25, 50, 100$.

Figures 1, 2 and 3 illustrate the absolute errors (in logarithmic scale) of approximations for $n = 0, 1, 5$ with $m = 10$ iterations and $N = 10, 25, 50, 100$. The integrals of the squared residual errors are reported in Table 1 for $n = 0, 1, 2, 3, 4, 5$ with $m = 10$ iterations and various N.

Example 2. Consider the following Lane-Emden equation

$$\begin{cases} y'' + \frac{1}{x}y' + e^{h(y)} = 0, & 0 < x \leq 1, \\ y(0) = 1, \quad y'(0) = 0. \end{cases}$$

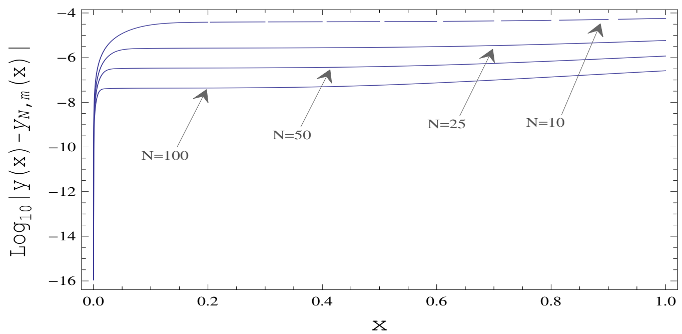

Figure 2. Absolute errors (in logarithmic scale) of approximations, for Example 1 with $n = 1$, m=10 iterations and $N = 10, 25, 50, 100$.

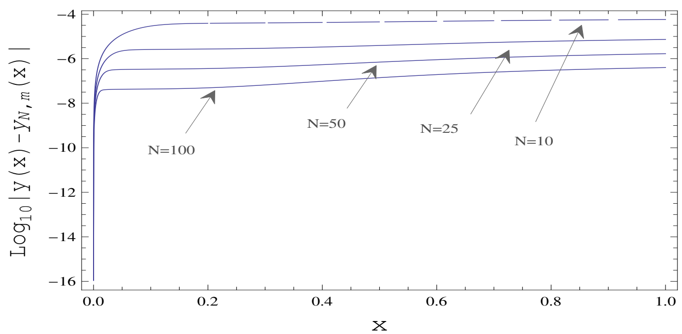

Figure 3. Absolute errors (in logarithmic scale) of approximations, for Example 1 with $n = 5$, m=10 iterations and $N = 10, 25, 50, 100$.

Table 1. Integrals of the squared residual errors for Example 1 with m=10 iterations and various $N = 10, 25$ and 50.

n	N=10	N=25	N=50
0	4.02746×10^{-5}	1.93114×10^{-6}	1.17886×10^{-7}
1	4.02755×10^{-5}	1.93182×10^{-6}	1.17944×10^{-7}
2	4.03258×10^{-5}	1.93249×10^{-6}	1.17991×10^{-7}
3	4.04028×10^{-5}	1.93266×10^{-6}	1.17998×10^{-7}
4	4.0521×10^{-5}	1.93272×10^{-6}	1.17989×10^{-7}
5	4.06908×10^{-5}	1.933×10^{-6}	1.17983×10^{-7}

Figure 4 is the graph of $\text{Log}_{10}|\mathcal{C}_N(x)| = \text{Log}_{10}|R_x(x) - R_{N,x}(x)|$ for Example 2 with $N = 10, 25, 50, 100$. The Integrals of the squared residual errors are reported in Table 2 for various $h(y)$. Figure 5, shows several of the approximate

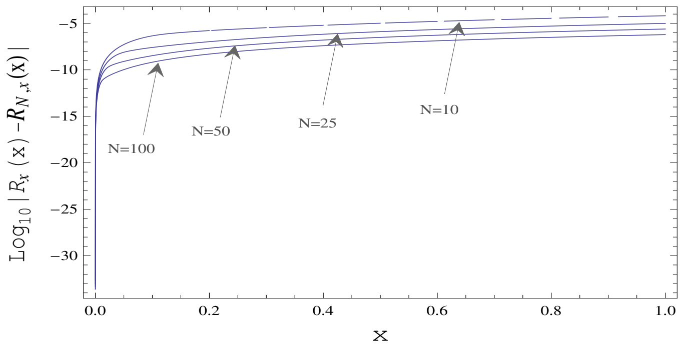

Figure 4. Graph of $\text{Log}_{10}|R_x(x) - R_{N,x}(x)|$ for Example 2 with $N = 10, 25, 50, 100$.

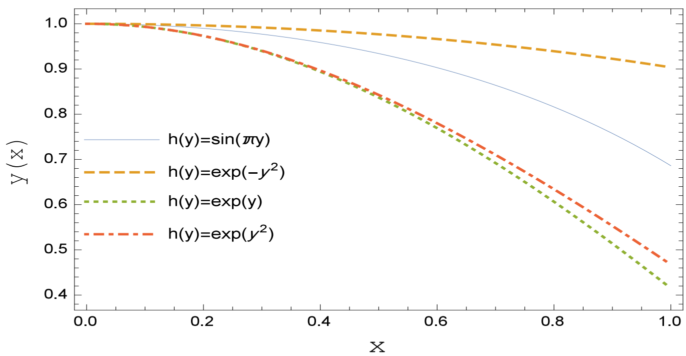

Figure 5. Graph of approximate solutions of the Example 2 for various $h(y)$ with m=10 iterations and $N = 50$.

Table 2. Integrals of the squared residual errors for Example 2 with m=10 iterations, $N = 10, 25, 50$ and various nonlinearity .

$h(y)$	N=10	N=25	N=50
y	5.20584×10^{-4}	2.51154×10^{-5}	1.63545×10^{-6}
$-y^2$	9.13069×10^{-6}	4.56534×10^{-7}	2.98878×10^{-8}
y^2	5.6771×10^{-4}	2.55454×10^{-5}	1.64482×10^{-6}
$sin(\pi y)$	7.74872×10^{-5}	3.60194×10^{-6}	2.33742×10^{-7}
$cos(\pi y)$	9.18103×10^{-6}	4.56752×10^{-7}	2.98794×10^{-8}
$cos^2(\pi y)$	4.6141×10^{-4}	2.52981×10^{-5}	1.68488×10^{-6}

solutions on the interval [0,1].

Example 3. Consider the following Lane-Emden equation

$$\begin{cases} y'' + \frac{1}{x}y' + xy = x^5 - x^4 + 44x^2 - 30x, & x \geq 0, \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$$

The exact solution is given by $y(x) = x^4 - x^3$.

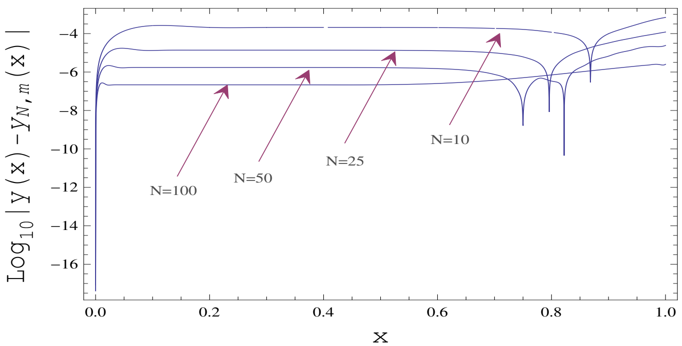

Figure 6. Absolute errors (in logarithmic scale) of approximations, for Example 3 with m=10 iterations and $N = 10, 25, 50, 100$.

Figure 6 shows the absolute errors (in logarithmic scale) of approximations with $m = 10$ iterations and $N = 10, 25, 50, 100$.

Example 4. The isothermal gas sphere equation is modeled by Lane-Emden equation of the second kind [5],

$$\begin{cases} y'' + \frac{2}{x}y' + e^y = 0, & 0 < x \leq 1, \\ y(0) = 0, \quad y'(0) = 0, \end{cases}$$

Table 3. Integrals of the squared residual errors for Lane-Emden equation of the second kind with m=10 iterations and various values for N.

N=10	N=25	N=50
4.0277×10^{-5}	1.93147×10^{-6}	1.1792×10^{-7}

The integrals of the squared residual errors for Lane-Emden equation of the second kind are reported in Table 3 with $m = 10$ iterations and various values of N. Figure 7 demonstrates approximate solution of the Lane-Emden equation of the second kind with m=10 iterations and $N = 50$.

The results presented in this section show that as the number of terms N increases the error in the approximations continues to decrease and this confirms our prediction for convergence and error estimates in the previous section.

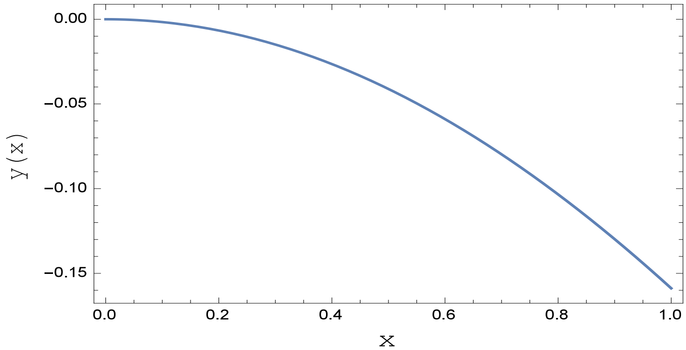

Figure 7. Graph of approximate solution of the Example 4 with m=10 iterations and $N = 50$.

5 Conclusions

In this paper, the reproducing kernel Hilbert space method combined with the Picard's iteration, is employed successfully to obtain the analytical approximate solutions for generalized nonlinear singular Lane Emden type equations. The implementation of the proposed iterative method is simple and the convergence of this method is proved. The Picard iteration is used to deal with the nonlinearity and it's error estimation is established. After linearization, the reproducing kernel Hilbert space method generates a rapid convergent series solution with easily computable components. We also obtained the truncation error estimate of the series solution. To demonstrate the computation efficiency, mentioned method is implemented for several examples and the results show the validity, accuracy and applicability of the method.

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