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Gradient Based Iterative Algorithm to Solve General Coupled Discrete-Time Periodic Matrix Equations over Generalized Reflexive Matrices

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Abstract. The discrete-time periodic matrix equations are encountered in periodic state feedback problems and model reduction of periodic descriptor systems. The aim of this paper is to compute the generalized reflexive solutions of the general coupled discrete-time periodic matrix equations. We introduce a gradient-based iterative (GI) algorithm for finding the generalized reflexive solutions of the general coupled discrete-time periodic matrix equations. It is shown that the introduced GI algorithm always converges to the generalized reflexive solutions for any initial generalized reflexive matrices. Finally, two numerical examples are investigated to confirm the efficiency of GI algorithm.

Keywords: discrete-time periodic matrix equation, iterative algorithm, gradient based iterative algorithm, generalized reflexive solution .

AMS Subject Classification: 15A24; 39B42; 65F10; 65F30.

1 Introduction

Let us begin with some notations and definitions. The symbols A^T, $\mathrm{tr}(A)$ and $\|A\|$ will stand for the transpose, the trace and the Frobenius norm of a matrix $A \in \mathbb{R}^{m\times n}$, respectively. For a matrix $A \in \mathbb{R}^{m\times n}$, the so-called stretching function $\mathrm{vec}(A)$ is defined by $\mathrm{vec}(A) = (a_1^T,\ a_2^T, \ldots,\ a_n^T)^T$, where a_k is the k-th column of A. The notation $A \otimes B$ represents the Kronecker product of matrices A and B. A matrix $P \in \mathbb{R}^{n\times n}$ is called a generalized reflection matrix if $P = P^T$ and $P^2 = I$. Throughout, we always suppose that $P, Q \in \mathbb{R}^{n\times n}$ are given generalized reflection matrices. If $A = PAQ$ then $A \in \mathbb{R}^{n\times n}$ is called a generalized reflexive matrix with respect to (P, Q) [5]. The symbol $\mathbb{R}_r^{n\times n}(P, Q)$ denotes the set of $n \times n$ generalized reflexive matrices with respect to (P, Q). Obviously every matrix $A \in \mathbb{R}^{n\times n}$ is also a generalized reflexive matrix with respect to (I, I). In [5], three important applications of

the generalized reflexive matrices were proposed.
The linear systems and linear matrix equations have several applications in several problems of applied mathematics and engineering [6, 26, 29, 32, 33]. For example, the stability of discrete-time linear periodic system

$$x(k+1) = A_k x(k) + B_k u(k), \quad \forall k \in \mathbb{Z}$$

is closely related with the following discrete-time periodic Lyapunov matrix equations [3, 35]

$$A_k P_k A_k^T - P_{k+1} = -B_k B_k^T, \quad \forall k \in \mathbb{Z} \tag{1.1}$$

and

$$A_k^T P_{k+1} A_k - P_k = -Q_k, \quad \forall k \in \mathbb{Z}. \tag{1.2}$$

In the model reduction and stability analysis of the linear periodic time-varying descriptor systems

$$E_i x_{i+1} = A_i x_i + B_i u_i, \qquad y_i = C_i x_i, \quad \forall i \in \mathbb{Z}$$

we need to solve the following generalized projected periodic discrete-time algebraic Lyapunov matrix equations [2, 6, 27]

$$\begin{cases} A_i X_i A_i^T - E_i X_{i+1} E_i^T = Q_l(i) B_i B_i^T Q_l(i)^T, \\ X_i = Q_r(i) G_i Q_r(i)^T. \end{cases} \tag{1.3}$$

The applications of linear matrix equations have motivated both mathematicians and engineers to construct methods catering to solve linear matrix equations [1, 4, 6, 7, 8, 9, 19, 23, 25]. Based on Smith iterations [24], iterative methods were developed for periodic standard Lyapunov matrix equations and projected generalized Lyapunov matrix equations [27, 28]. Kressner introduced new variants of the squared Smith iteration and Krylov subspace based methods for the approximate solution of discrete-time periodic Lyapunov matrix equations [20]. In [17], Granat et al. presented novel recursive blocked algorithms for solving various periodic triangular matrix equations. In this paper, we propose a GI algorithm to find the generalized reflexive solutions of the general coupled discrete-time periodic matrix equations

$$\begin{cases} A_{1,i} X_i B_{1,i} + C_{1,i} X_{i+1} D_{1,i} = E_{1,i}, \\ A_{2,i} X_i B_{2,i} + C_{2,i} X_{i+1} D_{2,i} = E_{2,i}, \end{cases} \tag{1.4}$$

for $i = 1, 2, ...$, where the coefficient matrices $A_{1,i}, C_{1,i} \in \mathbb{R}^{p_1 \times n}$, $A_{2,i}, C_{2,i} \in \mathbb{R}^{p_2 \times n}$, $B_{1,i}, D_{1,i} \in \mathbb{R}^{n \times q_1}$, $B_{2,i}, D_{2,i} \in \mathbb{R}^{n \times q_2}$, $E_{1,i} \in \mathbb{R}^{p_1 \times q_1}$, $E_{2,i} \in \mathbb{R}^{p_2 \times q_2}$ and the generalized reflexive solutions $X_i \in \mathbb{R}_r^{n \times n}(P, Q)$ are periodic with period θ, i.e., $A_{1,i+\theta} = A_{1,i}$, $A_{2,i+\theta} = A_{2,i}$, $C_{1,i+\theta} = C_{1,i}$, $C_{2,i+\theta} = C_{2,i}$, $D_{1,i+\theta} = D_{1,i}$, $D_{2,i+\theta} = D_{2,i}$, $E_{1,i+\theta} = E_{1,i}$, $E_{2,i+\theta} = E_{2,i}$ and $X_{i+\theta} = X_i$. It is worth mentioning that the generalized reflexive solutions of the general coupled discrete-time periodic matrix equations (1.4) have not been dealt with yet. Meanwhile the general coupled discrete-time periodic matrix equations (1.4) contain various linear discrete-time periodic matrix equations as special cases such as (1.1),

(1.2) and (1.3).
The remaining parts of this paper are organized as follows. In Section 2, first a GI algorithm is proposed for solving (1.4) over the generalized reflexive matrices. Then by analysis of convergence we prove that the proposed algorithm consistently converges to the generalized reflexive solutions for any initial generalized reflexive matrices. Theoretical results are verified on the relevant numerical examples in Section 3. Section 4 ends this paper with a brief conclusion.

2 Main results

In this section, first we obtain the conditions for solvability of (1.4) over the generalized reflexive matrices. Then a GI algorithm and its convergence analysis are given.
It is easily shown that the general coupled discrete-time periodic matrix equations (1.4) over the generalized reflexive matrixes are equivalent to the following general coupled matrix equations

$$\begin{cases} \mathcal{A}_1\mathcal{X}\mathcal{B}_1 + \mathcal{C}_1\mathcal{X}\mathcal{D}_1 = \mathcal{E}_1, \\ \mathcal{A}_2\mathcal{X}\mathcal{B}_2 + \mathcal{C}_2\mathcal{X}\mathcal{D}_2 = \mathcal{E}_2, \\ \mathcal{A}_1\mathcal{P}\mathcal{X}\mathcal{Q}\mathcal{B}_1 + \mathcal{C}_1\mathcal{P}\mathcal{X}\mathcal{Q}\mathcal{D}_1 = \mathcal{E}_1, \\ \mathcal{A}_2\mathcal{P}\mathcal{X}\mathcal{Q}\mathcal{B}_2 + \mathcal{C}_2\mathcal{P}\mathcal{X}\mathcal{Q}\mathcal{D}_2 = \mathcal{E}_2, \end{cases} \tag{2.1}$$

where

$$\mathcal{A}_j = \begin{pmatrix} 0 & \cdots & 0 & A_{j,1} \\ A_{j,2} & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{j,\theta} & 0 \end{pmatrix}, \quad \mathcal{B}_j = \begin{pmatrix} 0 & B_{j,2} & & 0 \\ \vdots & & \ddots & \\ 0 & & & B_{j,\theta} \\ B_{j,1} & 0 & \cdots & 0 \end{pmatrix},$$

$$\mathcal{C}_j = \operatorname{diag}\left(C_{j,1}, C_{j,2}, ..., C_{j,\theta}\right), \mathcal{D}_j = \operatorname{diag}\left(D_{j,1}, D_{j,2}, ..., D_{j,\theta}\right),$$

$$\mathcal{E}_j = \operatorname{diag}\left(E_{j,1}, E_{j,2}, ..., E_{j,\theta}\right), \quad \mathcal{X} = \operatorname{diag}\left(X_2, X_3, ..., X_\theta, X_1\right),$$

$$\mathcal{P} = \operatorname{diag}\left(P, P, ..., P\right), \quad \mathcal{Q} = \operatorname{diag}\left(Q, Q, ..., Q\right),$$

for $j = 1, 2$. By using Kronecker product and vectorization operator, the general coupled matrix equations (2.1) can be transformed into the linear system $Ax = b$ with the following parameters:

$$A = \begin{pmatrix} \mathcal{B}_1^T \otimes \mathcal{A}_1 + \mathcal{D}_1^T \otimes \mathcal{C}_1 \\ \mathcal{B}_2^T \otimes \mathcal{A}_2 + \mathcal{D}_2^T \otimes \mathcal{C}_2 \\ \mathcal{B}_1^T\mathcal{Q} \otimes \mathcal{A}_1\mathcal{P} + \mathcal{D}_1^T\mathcal{Q} \otimes \mathcal{C}_1\mathcal{P} \\ \mathcal{B}_2^T\mathcal{Q} \otimes \mathcal{A}_2\mathcal{P} + \mathcal{D}_2^T\mathcal{Q} \otimes \mathcal{C}_2\mathcal{P} \end{pmatrix}, \quad x = \operatorname{vec}(\mathcal{X}), \quad b = \begin{pmatrix} \operatorname{vec}(\mathcal{E}_1) \\ \operatorname{vec}(\mathcal{E}_2) \\ \operatorname{vec}(\mathcal{E}_1) \\ \operatorname{vec}(\mathcal{E}_2) \end{pmatrix}. \tag{2.2}$$

By applying (2.2), we can present the following lemma.

Lemma 1. *The general coupled discrete-time periodic matrix equations (1.4) have a unique generalized reflexive solution group* $(X_1, X_2, ..., X_\theta)$ *if and only* *if* rank$((A,b))$ = rank(A) *and* A *has a full column rank; in this case, the homogenous general coupled discrete-time periodic matrix equations*

$$\left\{\begin{array}{l} A_{1,i}X_iB_{1,i}+C_{1,i}X_{i+1}D_{1,i}=0, \\ A_{2,i}X_iB_{2,i}+C_{2,i}X_{i+1}D_{2,i}=0, \end{array}\right. \qquad i=1,2,...$$

have a unique generalized reflexive solution group $(X_1,X_2,...,X_\theta)=0$.

Obviously the size of the coefficient matrices of the general coupled matrix equations (2.1) and the linear system (2.2) is large. When the size of coefficient matrices is large, the iterative methods such as [10, 18, 22] will consume more computer time and memory space. Also in this case, the obtained solutions are not accurate enough. To overcome the complications, we directly extend the GI algorithm to solve (1.4) over the generalized reflexive matrices. One of the famous method for solving the linear system $Ax=b$ is the GI algorithm [12, 13, 14] as follows:

$$x^{(k+1)}=x^{(k)}+\delta A^T(b-Ax^{(k)}), \qquad 0<\delta<\frac{2}{\|A\|^2}. \tag{2.3}$$

In recent years the GI algorithms have gained much attention for solving linear matrix equations [14, 15, 16]. In [11, 12, 13], Ding and Chen proposed the GI algorithms for solving matrix equations. Zhou et al. constructed a GI algorithm to approximate the solutions to the coupled linear matrix equations [34]. By defining a relaxation parameter, Niu et al. proposed a relaxed GI algorithm for solving Sylvester matrix equations [21]. Different from the GI algorithm presented in [11] and the relaxed GI algorithm given in [21], Wang et al. introduced a modified GI algorithm for solving Sylvester matrix equations [30]. In [31], Wang and Liao obtained the optimal convergence factor of the GI algorithm for linear matrix equations. Based on the (2.1), (2.2) and (2.3), we present the following GI algorithm for solving (1.4) over the generalized reflexive matrices:

Algorithm 1. (**GI algorithm to solve (1.4) over the generalized reflexive matrices**)

Step 1 *Choose the initial generalized reflexive matrices* $X_i(1)\in\mathbb{R}_r^{n\times n}(P,Q)$ *for* $i=1,2,...,\theta$ *and a parameter* $\delta>0$*;*

Step 2 *Set* $X_{\theta+1}(1)=X_1(1)$*,* $X_0(1)=X_\theta(1)$*,* $C_{j,0}=C_{j,\theta}$ *and* $D_{j,0}=D_{j,\theta}$ *for* $j=1,2$*;*

Step 3 *Compute*

$$R_{j,i}(1)=E_{j,i}-A_{j,i}X_i(1)B_{j,i}-C_{j,i}X_{i+1}(1)D_{j,i}, \qquad i=1,2,...,\theta, \qquad j=1,2,$$

and set $R_{j,0}(1)=R_{j,\theta}(1)$ *for* $j=1,2$*;*

Step 4 *For* $k=1,2,...,$ *compute*

$$X_i(k+1)=X_i(k)+\frac{\delta}{2}\Big[A_{1,i}^TR_{1,i}(k)B_{1,i}^T+A_{2,i}^TR_{2,i}(k)B_{2,i}^T+C_{1,i-1}^TR_{1,i-1}(k)D_{1,i-1}^T$$

$$+C_{2,i-1}^T R_{2,i-1}(k) D_{2,i-1}^T + PA_{1,i}^T R_{1,i}(k) B_{1,i}^T Q + PA_{2,i}^T R_{2,i}(k) B_{2,i}^T Q$$
$$+PC_{1,i-1}^T R_{1,i-1}(k) D_{1,i-1}^T Q + PC_{2,i-1}^T R_{2,i-1}(k) D_{2,i-1}^T Q\Big], \quad i = 1, 2, ..., \theta,$$
$$X_{\theta+1}(k+1) = X_1(k+1), \qquad X_0(k+1) = X_\theta(k+1),$$
$$R_{j,i}(k+1)=E_{j,i} - A_{j,i}X_i(k+1)B_{j,i} - C_{j,i}X_{i+1}(k+1)D_{j,i}, \; i=1,2,\ldots,\theta, \;\; j=1,2,$$
$$R_{j,0}(k+1) = R_{j,\theta}(k+1), \qquad j = 1, 2.$$

Stopping criterion. To check convergence, we use the stopping criterion

$$\sqrt{\sum_{i=1}^{\theta} \left(\|R_{1,i}(k)\|^2 + \|R_{2,i}(k)\|^2 \right)} \leq \texttt{tol},$$

where `tol` is a chosen fixed threshold.

Remark 1. From the above algorithm, we can easily see that $X_i(k) \in \mathbb{R}_r^{n\times n}(P, Q)$ for $i = 1, 2, \ldots, \theta$.

In the following theorem, we proceed to prove the convergence of Algorithm 1 to the generalized reflexive solutions of (1.4).

Theorem 1. *Suppose that the general coupled discrete-time periodic matrix equations (1.4) have a unique generalized reflexive solution group* $(X_1^*, X_2^*, \ldots, X_\theta^*)$. *If the parameter* δ *satisfies the inequality*

$$0 < \delta < \frac{2}{\sum_{i=1}^{\theta} \left(\left\| A_{1,i}B_{1,i} \right\|^2 + \left\| C_{1,i}D_{1,i} \right\|^2 + \left\| A_{2,i}B_{2,i} \right\|^2 + \left\| C_{2,i}D_{2,i} \right\|^2 \right)}, \quad (2.4)$$

then for any initial generalized reflexive matrix group $(X_1(1), X_2(1), ..., X_\theta(1))$, *the iterative solution group* $(X_1(k), X_2(k), ..., X_\theta(k))$ *generated by Algorithm 1 converges to the generalized reflexive group* $(X_1^*, X_2^*, ..., X_\theta^*)$, *that is*

$$\lim_{k\to\infty} X_i(k) = X_i^*, \qquad \text{for} \quad i = 1, 2, ..., \theta.$$

Proof. To prove this theorem, first we define the error matrices in the k-th iteration of Algorithm 1 as

$$\widetilde{X}_i(k) = X_i(k) - X_i^*, \qquad \text{for} \quad i = 1, 2, ..., \theta.$$

By using the error matrices, we can obtain the residual matrices in the k-th iteration as the following form

$$R_{j,i}(k) = -A_{j,i}\widetilde{X}_i(k)B_{j,i} - C_{j,i}\widetilde{X}_{i+1}(k)D_{j,i}, \quad \text{for} \;\; i = 1, 2, ..., \theta, \;\; j = 1, 2.$$

This implies that

$$\widetilde{X}_i(k+1) = \widetilde{X}_i(k) - \frac{\delta}{2}\Big[A_{1,i}^T\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i} + C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)B_{1,i}^T$$

$$+A_{2,i}^T\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)B_{2,i}^T$$
$$+C_{1,i-1}^T\Big(A_{1,i-1}\widetilde{X}_{i-1}(k)B_{1,i-1}+C_{1,i-1}\widetilde{X}_i(k)D_{1,i-1}\Big)D_{1,i-1}^T$$
$$+C_{2,i-1}^T\Big(A_{2,i-1}\widetilde{X}_{i-1}(k)B_{2,i-1}+C_{2,i-1}\widetilde{X}_i(k)D_{2,i-1}\Big)D_{2,i-1}^T$$
$$+PA_{1,i}^T\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)B_{1,i}^TQ$$
$$+PA_{2,i}^T\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)B_{2,i}^TQ$$
$$+PC_{1,i-1}^T\Big(A_{1,i-1}\widetilde{X}_{i-1}(k)B_{1,i-1}+C_{1,i-1}\widetilde{X}_i(k)D_{1,i-1}\Big)D_{1,i-1}^TQ$$
$$+PC_{2,i-1}^T\Big(A_{2,i-1}\widetilde{X}_{i-1}(k)B_{2,i-1}+C_{2,i-1}\widetilde{X}_i(k)D_{2,i-1}\Big)D_{2,i-1}^TQ\Big]. \qquad (2.5)$$

For $i=1,2,...,\theta$, by applying (2.5) we can obtain

$$\|\widetilde{X}_i(k+1)\|^2=\mathrm{tr}\Big(\widetilde{X}_i(k+1)^T\widetilde{X}_i(k+1)\Big)$$
$$=\|\widetilde{X}_i(k)\|^2-\delta\mathrm{tr}\Big(\widetilde{X}_i(k)^TA_{1,i}^T\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)B_{1,i}^T$$
$$+\widetilde{X}_i(k)^TA_{2,i}^T\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)B_{2,i}^T$$
$$+\widetilde{X}_i(k)^TC_{1,i-1}^T\Big(A_{1,i-1}\widetilde{X}_{i-1}(k)B_{1,i-1}+C_{1,i-1}\widetilde{X}_i(k)D_{1,i-1}\Big)D_{1,i-1}^T$$
$$+\widetilde{X}_i(k)^TC_{2,i-1}^T\Big(A_{2,i-1}\widetilde{X}_{i-1}(k)B_{2,i-1}+C_{2,i-1}\widetilde{X}_i(k)D_{2,i-1}\Big)D_{2,i-1}^T$$
$$+\widetilde{X}_i(k)^TPA_{1,i}^T\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)B_{1,i}^TQ$$
$$+\widetilde{X}_i(k)^TPA_{2,i}^T\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)B_{2,i}^TQ$$
$$+\widetilde{X}_i(k)^TPC_{1,i-1}^T\Big(A_{1,i-1}\widetilde{X}_{i-1}(k)B_{1,i-1}+C_{1,i-1}\widetilde{X}_i(k)D_{1,i-1}\Big)D_{1,i-1}^TQ$$
$$+\widetilde{X}_i(k)^TPC_{2,i-1}^T\Big(A_{2,i-1}\widetilde{X}_{i-1}(k)B_{2,i-1}+C_{2,i-1}\widetilde{X}_i(k)D_{2,i-1}\Big)D_{2,i-1}^TQ\Big)$$
$$+\frac{\delta^2}{4}\Big\|A_{1,i}^T\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)B_{1,i}^T$$
$$+A_{2,i}^T\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)B_{2,i}^T$$
$$+C_{1,i-1}^T\Big(A_{1,i-1}\widetilde{X}_{i-1}(k)B_{1,i-1}+C_{1,i-1}\widetilde{X}_i(k)D_{1,i-1}\Big)D_{1,i-1}^T$$
$$+C_{2,i-1}^T\Big(A_{2,i-1}\widetilde{X}_{i-1}(k)B_{2,i-1}+C_{2,i-1}\widetilde{X}_i(k)D_{2,i-1}\Big)D_{2,i-1}^T$$
$$+PA_{1,i}^T\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)B_{1,i}^TQ$$
$$+PA_{2,i}^T\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)B_{2,i}^TQ$$

$$
\begin{aligned}
&+PC_{1,i-1}^T\Big(A_{1,i-1}\widetilde{X}_{i-1}(k)B_{1,i-1}+C_{1,i-1}\widetilde{X}_i(k)D_{1,i-1}\Big)D_{1,i-1}^TQ\\
&+PC_{2,i-1}^T\Big(A_{2,i-1}\widetilde{X}_{i-1}(k)B_{2,i-1}+C_{2,i-1}\widetilde{X}_i(k)D_{2,i-1}\Big)D_{2,i-1}^TQ\Big\|^2\\
&=\|\widetilde{X}_i(k)\|^2-2\delta\mathrm{tr}\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)^T\\
&+A_{2,i}\widetilde{X}_i(k)B_{2,i}\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)^T\\
&+C_{1,i-1}\widetilde{X}_i(k)D_{1,i-1}\Big(A_{1,i-1}\widetilde{X}_{i-1}(k)B_{1,i-1}+C_{1,i-1}\widetilde{X}_i(k)D_{1,i-1}\Big)^T\\
&+C_{2,i-1}\widetilde{X}_i(k)D_{2,i-1}\Big(A_{2,i-1}\widetilde{X}_{i-1}(k)B_{2,i-1}+C_{2,i-1}\widetilde{X}_i(k)D_{2,i-1}\Big)^T\Big)\\
&+\delta^2\Big\|A_{1,i}^T\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)B_{1,i}^T\\
&+C_{1,i-1}^T\Big(A_{1,i-1}\widetilde{X}_{i-1}(k)B_{1,i-1}+C_{1,i-1}\widetilde{X}_i(k)D_{1,i-1}\Big)D_{1,i-1}^T\\
&+A_{2,i}^T\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)B_{2,i}^T\\
&+C_{2,i-1}^T\Big(A_{2,i-1}\widetilde{X}_{i-1}(k)B_{2,i-1}+C_{2,i-1}\widetilde{X}_i(k)D_{2,i-1}\Big)D_{2,i-1}^T\Big\|^2\\
&=\|\widetilde{X}_i(k)\|^2-2\delta\mathrm{tr}\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)^T\\
&+A_{2,i}\widetilde{X}_i(k)B_{2,i}\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)^T\\
&+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)^T\\
&+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)^T\Big)\\
&+\delta^2\Big\|A_{1,i}^T\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)B_{1,i}^T\\
&+C_{1,i}^T\Big(A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big)D_{1,i}^T\\
&+A_{2,i}^T\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)B_{2,i}^T\\
&+C_{2,i}^T\Big(A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big)D_{2,i}^T\Big\|^2\\
&\le\|\widetilde{X}_i(k)\|^2-2\delta\Big(\Big\|A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big\|^2\\
&+\Big\|A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big\|^2\Big)\\
&+\delta^2\Big(\Big\|A_{1,i}B_{1,i}\Big\|^2+\Big\|C_{1,i}D_{1,i}\Big\|^2+\Big\|A_{2,i}B_{2,i}\Big\|^2+\Big\|C_{2,i}D_{2,i}\Big\|^2\Big)\\
&\times\Big(\Big\|A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big\|^2+\Big\|A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big\|^2\Big).
\end{aligned}
$$

By defining the nonnegative definite function $Z(k)$ as follows

$$Z(k)=\sum_{i=1}^{\theta}\|\widetilde{X}_i(k)\|^2$$

we have

$$\begin{aligned}
Z(k+1)&=\sum_{i=1}^{\theta}\|\widetilde{X}_i(k+1)\|^2\\
&\leq\sum_{i=1}^{\theta}\|\widetilde{X}_i(k)\|^2-2\delta\sum_{i=1}^{\theta}\Big(\Big\|A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big\|^2\\
&\quad+\Big\|A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big\|^2\Big)\\
&\quad+\delta^2\sum_{i=1}^{\theta}\Big(\Big\|A_{1,i}B_{1,i}\Big\|^2+\Big\|C_{1,i}D_{1,i}\Big\|^2+\Big\|A_{2,i}B_{2,i}\Big\|^2+\Big\|C_{2,i}D_{2,i}\Big\|^2\Big)\\
&\quad\times\sum_{i=1}^{\theta}\Big(\Big\|A_{1,i}\widetilde{X}_i(k)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(k)D_{1,i}\Big\|^2\\
&\quad+\Big\|A_{2,i}\widetilde{X}_i(k)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(k)D_{2,i}\Big\|^2\Big)\\
&\leq Z(0)-\delta\Big[2-\delta\sum_{i=1}^{\theta}\Big(\Big\|A_{1,i}B_{1,i}\Big\|^2+\Big\|C_{1,i}D_{1,i}\Big\|^2+\Big\|A_{2,i}B_{2,i}\Big\|^2+\Big\|C_{2,i}D_{2,i}\Big\|^2\Big)\Big]\\
&\quad\times\sum_{r=1}^{k}\sum_{i=1}^{\theta}\Big(\Big\|A_{1,i}\widetilde{X}_i(r)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(r)D_{1,i}\Big\|^2\\
&\quad+\Big\|A_{2,i}\widetilde{X}_i(r)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(r)D_{2,i}\Big\|^2\Big).
\end{aligned}$$

If the convergence factor δ is chosen to satisfy in (2.4) then we can conclude that

$$\begin{aligned}
&\sum_{r=1}^{\infty}\sum_{i=1}^{\theta}\Big(\Big\|A_{1,i}\widetilde{X}_i(r)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(r)D_{1,i}\Big\|^2\\
&+\Big\|A_{2,i}\widetilde{X}_i(r)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(r)D_{2,i}\Big\|^2\Big)<\infty.
\end{aligned}$$

It follows from the necessary condition of the above series convergence that

$$\begin{aligned}
&\lim_{r\to\infty}\sum_{i=1}^{\theta}\Big(\Big\|A_{1,i}\widetilde{X}_i(r)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(r)D_{1,i}\Big\|^2\\
&+\Big\|A_{2,i}\widetilde{X}_i(r)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(r)D_{2,i}\Big\|^2\Big)=0.
\end{aligned}$$

Hence we deduce that

$$\lim_{r\to\infty}\left(A_{1,i}\widetilde{X}_i(r)B_{1,i}+C_{1,i}\widetilde{X}_{i+1}(r)D_{1,i}\right)=0, \qquad \text{for} \ \ i=1,2,...,\theta$$

and

$$\lim_{r\to\infty}\left(A_{2,i}\widetilde{X}_i(r)B_{2,i}+C_{2,i}\widetilde{X}_{i+1}(r)D_{2,i}\right)=0, \qquad \text{for} \quad i=1,2,...,\theta.$$

Now according to Lemma 1, it can be obtained that

$$\lim_{r\to\infty}\widetilde{X}_i(r)=0, \qquad \text{for} \ \ i=1,2,...,\theta.$$

This finishes the proof of Theorem 1. □

3 Numerical examples

In this section, two numerical examples are proposed for the validation of the proposed method. We performed our computations using Matlab software on a Pentium IV.

Example 1. We consider the discrete-time periodic matrix equations

$$A_iX_i+X_{i+1}B_i=C_i, \quad i=1,2,3$$

over the generalized reflexive matrices $X_1, X_2, X_3 \in \mathbb{R}_r^{5\times5}(P,Q)$ where

$$A_1=\begin{pmatrix} 2.6756 & 0.3840 & 0.6085 & 0.0576 & 0.0841\\ 0 & 2.4508 & 0.0158 & 0.3676 & 0.4544\\ 0 & 0 & 2.2324 & 0.6315 & 0.4418\\ 0 & 0 & 0 & 2.0784 & 0.3533\\ 0 & 0 & 0 & 0 & 2.9943 \end{pmatrix},$$

$$A_2=\begin{pmatrix} -3.2475 & 0.5915 & 0 & 0 & 0\\ 0.3400 & -3.7362 & 0.2644 & 0 & 0\\ 0.3142 & 0.0381 & -3.2519 & 0.6649 & 0\\ 0.3651 & 0.4586 & 0.8729 & -2.7797 & 0.8903\\ 0.3932 & 0.8699 & 0.2379 & 0.0099 & -2.9985 \end{pmatrix},$$

$$A_3=\begin{pmatrix} -7.4617 & 0.9200 & 0.1939 & 0.5488 & 0.6273\\ 0.0099 & -6.6666 & 0.9048 & 0.9316 & 0.6991\\ 0.4199 & 0.3678 & -7.2374 & 0.3352 & 0.3972\\ 0.75370.6208 & 0.6318 & -6.4845 & 0.4136 & \\ 0.7939 & 0.7313 & 0.2344 & 0.3919 & -6.4036 \end{pmatrix},$$

$$B_1=\begin{pmatrix} 9.1529 & 0.7621 & 0.6154 & 0.4057 & 0.0579\\ 0.2311 & 9.2033 & 0.7919 & 0.9355 & 0.3529\\ 0.6068 & 0.0185 & 9.4470 & 0.9169 & 0.8132\\ 0.4860 & 0.8214 & 0.7382 & 8.7898 & 0.0099\\ 0.8913 & 0.4447 & 0.1763 & 0.8936 & 8.3285 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 8.8962 & 0.6979 & 0 & 0 & 0 \\ 0.6822 & 9.3352 & 0.8998 & 0 & 0 \\ 0.3028 & 0.8600 & 9.0740 & 0.2897 & 0 \\ 0.5417 & 0.8537 & 0.6449 & 8.5403 & 0.5681 \\ 0.1509 & 0.5936 & 0.8180 & 0.5341 & 9.3587 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 8.5536 & 0.2259 & 0 & 0 & 0 \\ 0.4235 & 8.2233 & 0.3798 & 0 & 0 \\ 0.5155 & 0.7604 & 8.0513 & 0.0592 & 0 \\ 0.3340 & 0.5298 & 0.6808 & 8.2317 & 0.0150 \\ 0.4329 & 0.6405 & 0.4611 & 0.0503 & 8.3431 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 4.7707 & 11.0062 & 2.2473 & 14.7423 & 37.2686 \\ 40.4523 & 3.4924 & 15.8879 & 3.2050 & 1.6293 \\ 3.1070 & 19.8112 & 3.2951 & 23.4979 & 16.1410 \\ 10.5080 & 1.1985 & 7.9500 & 1.4258 & 0.9173 \\ 3.0847 & 28.6604 & 3.8242 & 26.0354 & 17.0594 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 5.4803 & 17.7438 & 9.5312 & 17.6358 & 59.4288 \\ 54.2694 & 8.1642 & 18.1172 & 1.5983 & 1.7068 \\ 5.7146 & 30.8417 & 8.6997 & 30.0182 & 28.3407 \\ 17.0218 & 6.4641 & 11.3355 & 4.4598 & 4.2447 \\ 9.4985 & 44.6355 & 10.9847 & 33.1244 & 30.2029 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 9.2292 & -2.9557 & 4.4280 & -7.0361 & -37.8309 \\ -32.3081 & 7.3022 & -10.3963 & 6.7477 & 5.2847 \\ 4.4957 & -13.3424 & 2.7645 & -18.3820 & -12.0716 \\ -3.6924 & 5.6008 & -4.7002 & 5.8530 & 9.0744 \\ 7.5784 & -18.1886 & 3.9682 & -16.2704 & -7.4623 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By applying Algorithm 1 with the initial generalized reflexive matrices $X_1(1) = X_2(1) = X_3(1) = 0$ and several values of parameter δ, we obtain results presented in Figure 1 where

$$r(k) = \log_{10}\left(\sqrt{\sum_{i=1}^{3} \|C_i - A_i X_i(k) - X_{i+1}(k) B_i\|^2}\right).$$

After 125 iterations, we obtain the generalized reflexive solutions of the discrete time periodic matrix equations as follows:

$$X_1^* = \begin{pmatrix} 0 & 0.3757 & 0 & 0.4919 & 1.8847 \\ 1.9034 & 0 & 0.6286 & 0 & 0 \\ 0 & 0.8185 & 0 & 1.0121 & 0.8036 \\ 0.4947 & 0 & 0.3500 & 0 & 0 \\ 0 & 1.2219 & 0 & 1.0828 & 0.8231 \end{pmatrix} \in \mathbb{R}_r^{5\times 5}(P,Q),$$

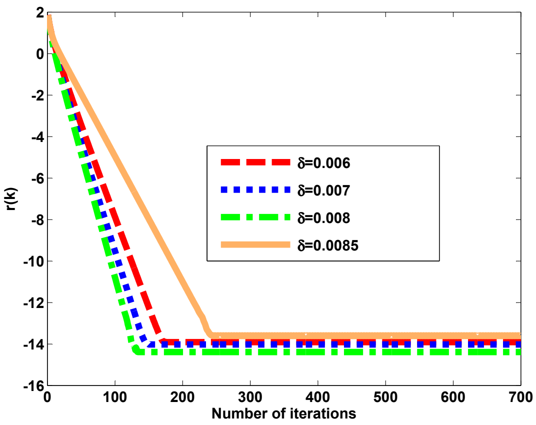

Figure 1. The residuals for Example 1

$$X_2^* = \begin{pmatrix} 0 & 0.7514 & 0 & 0.9838 & 3.7693 \\ 3.8068 & 0 & 1.2571 & 0 & 0 \\ 0 & 1.6371 & 0 & 2.0242 & 1.6072 \\ 0.9893 & 0 & 0.7001 & 0 & 0 \\ 0 & 2.4438 & 0 & 2.1657 & 1.6463 \end{pmatrix} \in \mathbb{R}_r^{5\times 5}(P,Q),$$

$$X_3^* = \begin{pmatrix} 0 & 1.5029 & 0 & 1.9677 & 7.5386 \\ 7.6135 & 0 & 2.5142 & 0 & 0 \\ 0 & 3.2742 & 0 & 4.0484 & 3.2144 \\ 1.9786 & 0 & 1.4002 & 0 & 0 \\ 0 & 4.8875 & 0 & 4.3314 & 3.2925 \end{pmatrix} \in \mathbb{R}_r^{5\times 5}(P,Q).$$

The results show that Algorithm 1 can quickly obtain the solutions of the discrete-time periodic matrix equations.

Example 2. We study the coupled discrete-time periodic matrix equations

$$\begin{cases} X_i + A_i X_{i+1} B_i = C_i, \\ D_i X_i E_i + X_{i+1} = F_i, \end{cases} \qquad i = 1, 2, 3$$

with the following parameters

$$A_1 = \begin{pmatrix} 8.6756 & 0.3840 & 0.6085 & 0.0576 & 0.0841 \\ 0 & 8.4508 & 0.0158 & 0.3676 & 0.4544 \\ 0 & 0 & 8.2324 & 0.6315 & 0.4418 \\ 0 & 0 & 0 & 8.0784 & 0.3533 \\ 0 & 0 & 0 & 0 & 8.9943 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 7.1271 & 0.5915 & 0 & 0 & 0 \\ 0.3400 & 6.9757 & 0.2644 & 0 & 0 \\ 0.3142 & 0.0381 & 6.5725 & 0.6649 & 0 \\ 0.3651 & 0.4586 & 0.8729 & 7.5205 & 0.8903 \\ 0.3932 & 0.8699 & 0.2379 & 0.0099 & 7.4683 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -5.4617 & 0.9200 & 0.1939 & 0.5488 & 0.6273 \\ 0.0099 & -4.6666 & 0.9048 & 0.9316 & 0.6991 \\ 0.4199 & 0.3678 & -5.2374 & 0.3352 & 0.3972 \\ 0.7537 & 0.6208 & 0.6318 & -4.4845 & 0.4136 \\ 0.7939 & 0.7313 & 0.2344 & 0.3919 & -4.4036 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 3.2526 & -0.7621 & -0.6154 & -0.4057 & -0.0579 \\ -0.2311 & 4.2903 & -0.7919 & -0.9355 & -0.3529 \\ -0.6068 & -0.0185 & 3.6033 & -0.9169 & -0.8132 \\ -0.4860 & -0.8214 & -0.7382 & 3.9692 & -0.0099 \\ -0.8913 & -0.4447 & -0.1763 & -0.8936 & 4.0508 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 9.7027 & 0.6979 & 0.4966 & 0.6602 & 0.7271 \\ 0 & 9.9568 & 0.8998 & 0.3420 & 0.3093 \\ 0 & 0 & 9.2523 & 0.2897 & 0.8385 \\ 0 & 0 & 0 & 9.1991 & 0.5681 \\ 0 & 0 & 0 & 0 & 9.9883 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 8.5536 & 0.2259 & 0 & 0 & 0 \\ 0.4235 & 8.2233 & 0.3798 & 0 & 0 \\ 0.5155 & 0.7604 & 8.0513 & 0.0592 & 0 \\ 0.3340 & 0.5298 & 0.6808 & 8.2317 & 0.0150 \\ 0.4329 & 0.6405 & 0.4611 & 0.0503 & 8.3431 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 3.9317 & -0.5711 & -0.4319 & -0.9159 & -0.7327 \\ -0.5485 & 3.7974 & -0.6343 & -0.6020 & -0.4222 \\ -0.2618 & -0.9623 & 3.9515 & -0.2536 & -0.9614 \\ -0.5973 & -0.7505 & -0.0839 & 3.7528 & -0.0721 \\ -0.0493 & -0.7400 & -0.9455 & -0.5134 & 3.9206 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 3.6206 & 0.1614 & 0.8121 & 0.3756 & 0.9566 \\ 0 & 3.4906 & 0.6101 & 0.1662 & 0.1472 \\ 0 & 0 & 3.6084 & 0.8332 & 0.8699 \\ 0 & 0 & 0 & 3.4648 & 0.7694 \\ 0 & 0 & 0 & 0 & 3.4116 \end{pmatrix},$$

$$D_3 = \begin{pmatrix} -7.2833 & 0.1122 & 0 & 0 & 0 \\ 0.3941 & -6.8592 & 0.2816 & 0 & 0 \\ 0.5030 & 0.4668 & -7.3344 & 0.9028 & 0 \\ 0.7220 & 0.0147 & 0.7085 & -7.4818 & 0.5208 \\ 0.3062 & 0.6641 & 0.7839 & 0.8045 & -6.6376 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 10.1171 & -0.9327 & -0.2093 & -0.3193 & -0.1998 \\ 0 & 10.9492 & -0.4551 & -0.3749 & -0.0495 \\ 0 & 0 & 10.9331 & -0.8678 & -0.5667 \\ 0 & 0 & 0 & 10.2905 & -0.1219 \\ 0 & 0 & 0 & 0 & 10.0954 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} -3.0278 & -0.8194 & 0 & 0 & 0 \\ -0.2882 & -3.6955 & -0.7536 & 0 & 0 \\ -0.8167 & -0.5602 & -4.3628 & -0.1834 & 0 \\ -0.9855 & -0.2440 & -0.2141 & -3.9130 & -0.6773 \\ -0.0174 & -0.8220 & -0.6021 & -0.1703 & -4.8245 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} -10.8912 & 0 & 0 & 0 & 0 \\ 0 & -10.5019 & 0 & 0 & 0 \\ 0 & 0 & -10.1112 & 0 & 0 \\ 0 & 0 & 0 & -10.5195 & 0 \\ 0 & 0 & 0 & 0 & -10.5216 \end{pmatrix},$$

$$C_1 = 10^3 \begin{pmatrix} -0.2241 & -0.2708 & 0.2276 & 1.0767 & 0.0841 \\ 0.9796 & 0.0200 & -0.2286 & -0.1756 & -0.0375 \\ -0.2072 & -0.0106 & 0.8505 & -0.0594 & 0.1004 \\ 0.3911 & 0.8389 & -0.2531 & -0.2562 & -0.0840 \\ 0.4749 & 0.4448 & -0.1961 & -0.1827 & -0.0544 \end{pmatrix},$$

$$C_2 = 10^3 \begin{pmatrix} 0.3894 & 0.0856 & 1.7398 & 4.7341 & 1.1657 \\ 4.6264 & 1.0147 & 0.5306 & 0.6037 & 0.4854 \\ 0.2283 & 0.3661 & 3.9528 & 1.3614 & 1.7330 \\ 2.9085 & 4.4464 & 1.1209 & 0.7294 & 0.6241 \\ 3.0508 & 2.4571 & 0.5878 & 0.5811 & 0.4184 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 168.0550 & 37.2777 & -298.8015 & -717.0216 & -94.5945 \\ -482.8643 & 58.0143 & 121.2561 & 33.3031 & 35.8865 \\ 52.3395 & -13.3380 & -592.5257 & -106.8510 & -179.0708 \\ -158.3376 & -374.7244 & 111.1639 & 131.8542 & 40.3635 \\ -152.6549 & -162.7658 & 72.9395 & 123.7053 & 25.5157 \end{pmatrix},$$

$$F_1 = \begin{pmatrix} -233.8164 & -182.3232 & 229.7091 & 727.3289 & 67.2928 \\ 601.4525 & -65.0355 & -158.1209 & -132.0087 & -45.6560 \\ -267.8252 & -112.8876 & 699.5537 & 90.0978 & 161.7304 \\ 179.6600 & 522.0632 & -81.1046 & -129.9805 & -19.9160 \\ 185.4702 & 213.6419 & -177.5114 & -48.7498 & -41.4132 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} -282.2199 & -206.4360 & -351.0661 & -472.2401 & -202.9035 \\ -333.6208 & -195.0378 & -105.5166 & -24.2335 & -31.2039 \\ -216.4398 & -247.9265 & -482.4659 & -126.7748 & -165.6702 \\ -202.4713 & -378.4970 & -76.5490 & 0 & 0 \\ -155.3920 & -199.6648 & -36.9034 & 0 & 0 \end{pmatrix},$$

$$F_3 = 10^3 \begin{pmatrix} -0.0829 & -0.0115 & 1.9069 & 5.4131 & 0.7457 \\ 5.0859 & 0.7071 & -0.2809 & -0.3410 & -0.0962 \\ -0.6546 & -0.5434 & 4.5224 & 0.9083 & 1.4102 \\ 2.3706 & 3.9606 & -0.6364 & -0.6582 & -0.2144 \\ 1.6942 & 1.4985 & -0.5756 & -0.3633 & -0.1869 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We apply Algorithm 1 with the initial matrices $X_1(1) = X_2(1) = X_3(1) = 0$ and several values of parameter δ to solve the coupled discrete-time periodic matrix equations. Figure 2 shows the performance of Algorithm 1 with the residuals

$$r(k) = \log_{10}\left(\sqrt{\sum_{i=1}^{3}[\|C_i - X_i(k) - A_i X_{i+1}(k) B_i\|^2 + \|F_i - D_i X_i(k) E_i - X_{i+1}(k)\|^2]}\right).$$

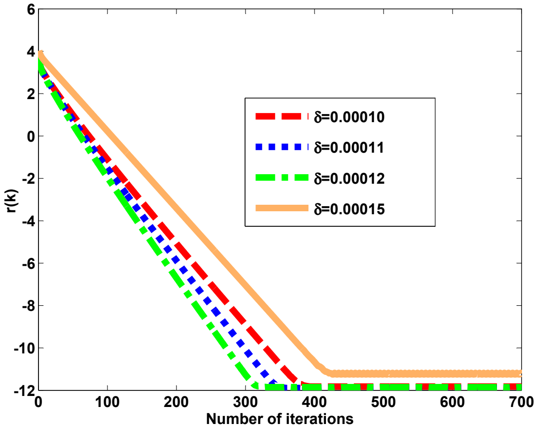

Figure 2. The residuals for Example 2

After 315 iterations, Algorithm 1 can compute the generalized reflexive solutions of the coupled discrete-time periodic matrix equations by the following:

$$X_1^* = \begin{pmatrix} 0 & 0 & 6.4515 & 17.6055 & 2.4249 \\ 16.9630 & 2.4456 & 0 & 0 & 0 \\ 0 & 0 & 15.6354 & 4.1373 & 4.7196 \\ 7.8725 & 13.0642 & 0 & 0 & 0 \\ 8.4808 & 7.1767 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}_r^{5\times 5}(P, Q),$$

$$X_2^* = \begin{pmatrix} 0 & 0 & 12.9031 & 35.2109 & 4.8498 \\ 33.9259 & 4.8911 & 0 & 0 & 0 \\ 0 & 0 & 31.2709 & 8.2746 & 9.4392 \\ 15.7451 & 26.1284 & 0 & 0 & 0 \\ 16.9616 & 14.3533 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}_r^{5\times 5}(P, Q),$$

$$X_3^* = \begin{pmatrix} 0 & 0 & 25.8061 & 70.4219 & 9.6997 \\ 67.8519 & 9.7822 & 0 & 0 & 0 \\ 0 & 0 & 62.5417 & 16.5491 & 18.8784 \\ 31.4902 & 52.2568 & 0 & 0 & 0 \\ 33.9232 & 28.7066 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}_r^{5\times 5}(P, Q).$$

From Figure 2, we can see that Algorithm 1 is effective to solve discrete-time periodic matrix equations.

4 Conclusions

In this paper, the generalized reflexive solutions of general coupled discrete-time periodic matrix equations (1.4) were studied. We proposed a gradient based iterative method to solve (1.4) over the generalized reflexive matrices. It was proven that the iterative solution converges to the generalized reflexive solutions for any initial generalized reflexive matrices. The numerical examples demonstrated the potential of this method in solving (1.4) over the generalized reflexive matrices.

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