View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by CERN Document Server

Further issues in fundamental interactions

J.-M. Gérard
Centre for Particle Physics and Phenomenology (CP3),
Université catholique de Louvain, Belgium

Abstract
Driven by the mass problem, we raise some issues of the fundamental interactions in terms of non-trivial commutation relations implemented within toy theories.

Introduction

The four known basic forces of nature turn out to proceed from a universal gauge principle. In particular, Einstein's general theory of relativity can be considered as the first Yang–Mills theory. Indeed photons do not carry an electric charge but gravitons appear to gravitate the way gluons glue in quantum chromodynamics. The confinement and spontaneous symmetry breaking mechanisms put forward to prevent long-range nuclear forces nowadays form the cornerstone of the Standard Model for particle physics. Such subtle issues to get round gauge invariance are highly suspected to be responsible for an explicit violation of the invariance under time-reversal in both strong and weak interactions.

The following three lectures are built upon the problem of mass.

- In the first lecture, a geometrical interpretation of non-Abelian gauge invariance is outlined from the striking fact that somebody in a free-falling lift would experience no apparent weight. Our main goal here is to display, through the concepts of mass and energy, how universal the basic forces may be. For that purpose, we mostly rely on a *scalar* theory for gravity which allows us to avoid tedious tensor calculus.
- In the second lecture, we make use of an effective theory for strong interactions to explain the origin of nucleon masses. We limit ourselves to the case of *two light flavours* and emphasize that the observed proton–neutron mass splitting might imply a large electric dipole moment for the neutron.
- In the last lecture, inspired by the chiral symmetry breaking at work in the theory for strong interactions, we consider an effective theory for electroweak interactions to explain the origin of boson and fermion masses. We illustrate how Yukawa interactions allow in principle a matter–antimatter asymmetry, *regardless of the flavour mixing pattern.*

1 Gauge invariance and $[D_\mu\, , D_\nu] \neq 0$

1.1 Weight of compact bodies

The beauty of modern physics lies in the fact that it allows us not only to relate seemingly different phenomena, such as the fall of a ripe apple and the motion of the full moon, or electricity and magnetism, but also to unify apparently independent everyday concepts such as rest and uniform motion, space and time, or gravitation and acceleration. In this way, we now have at our disposal a well-defined theoretical frame to explain why we do not feel the gravitational field of the Sun, but also to formulate rather precise questions about the origin of our weight, at least within the precision of our usual bathroom scales.

Our weight is obviously contingent upon the gravitational force exerted by the Earth:

$$\vec{W} = m_{\text{gr}}\, \vec{g} \qquad (g = \frac{GM_\oplus}{R_\oplus^2}) \tag{1.1}$$

and its precise value depends on our location (altitude but also latitude). As opposed to weight, mass appears to be an intrinsic property of matter which relates its manifest response (acceleration) to an abstract cause (force) in classical mechanics:

$$\vec{F} = m_{\text{in}}\,\vec{a}. \tag{1.2}$$

Stevin's dropping of lead spheres from the top of the Delft churchtower, Galileo's observations of wooden balls rolling down sloping planes, and Newton's experiments with pendulums made of various materials already indicated that all bodies tend to fall with the same acceleration at the surface of the Earth, no matter what their constitution may be, i.e.,

$$m_{\text{gr}} = m_{\text{in}}, \tag{1.3}$$

with an accuracy of about 10^{-3}. More accurate torsion balance experiments initiated by the Hungarian Baron Roland von Eötvös around 1890 nicely confirmed such a correlation between gravity and inertia at the level of 10^{-9}. Nowadays, this equality between gravitational and inertial masses is firmly established at the level of 10^{-12}. The so-called 'weak' equivalence principle rests upon Eq. (1.3).

From the striking universality of free fall (see the apple and the Moon falling towards the Earth) Einstein inferred, as far back as 1907, that his law which links mass to rest energy, i.e.,

$$m = \frac{E_0}{c^2}, \tag{1.4}$$

"holds not only for inertial but also for gravitational mass" [1]. In other words, *energy has weight*. So, electromagnetic binding energies do contribute equally to the inertial and gravitational mass such that all atoms (H, H*, $\overline{\text{H}}$, etc.) fall with the same acceleration. In particular, matter and antimatter fall the same way since both represent positive energies. The amazing accuracy of modern experiments extends this 'Einstein' equivalence principle to strong and weak nuclear binding energies since atoms are made of protons, neutrons, and electrons. But what about gravitational bound states?

For an homogeneous and spherical distribution of matter, the gravitational binding energy

$$\Omega \equiv -\frac{1}{2}\sum_{i\neq j} G\frac{m_i\, m_j}{r_{ij}} \tag{1.5}$$

is simply given by

$$\Omega = -\frac{3}{5}\frac{GM^2}{R}. \tag{1.6}$$

From the useful relation between the Newton constant G, the light velocity c, and the solar mass $M_\odot \approx 2\times 10^{30}$ kg,

$$\frac{2GM_\odot}{c^2} \approx 3\text{ km}, \tag{1.7}$$

which warns you that the Sun confined inside a (Schwarzschild) radius of 3 km would simply be a black hole, we get a ratio of the internal gravitational binding energy to the total mass energy scaling like

$$s \equiv |\frac{\Omega}{Mc^2}| \approx \frac{3}{10}(\frac{3\text{ km}}{R})(\frac{M}{2\times 10^{30}\text{ kg}}). \tag{1.8}$$

For a typical ball (say, R = 10 cm, M = 2 kg) we obtain in this manner a 'sensitivity' (or compactness factor) of the order of 10^{-26}. Consequently, present Eötvös-like laboratory experiments are totally unable to tell us whether the gravitational binding energy contributes equally to the inertial and to the gravitational mass. Let us therefore define the mass ratio for gravitational bound states as follows:

$$\frac{m_{\text{gr}}}{m_{\text{in}}} \equiv 1 + \eta\frac{\Omega}{Mc^2}, \tag{1.9}$$

with η a dimensionless parameter measuring any departure from universality for compact bodies in free fall. For an homogeneous Earth ($R \approx$ 6400 km, $M \approx 6 \times 10^{24}$ kg) and Moon ($R \approx$ 1700 km, $M \approx 7 \times 10^{22}$ kg), the compactness factors are roughly 4×10^{-10} and 2×10^{-11}, respectively. The observational fact that the Moon's orbit around the Earth does not appear to be continuously polarized towards the Sun [2] guarantees that they both fall towards the Sun at equal rates with an accuracy of about 2×10^{-13}. From the relation

$$|\frac{a_{\oplus} - a_{☾}}{g}| = \eta | \frac{\Omega_{\oplus}}{M_{\oplus} c^2} - \frac{\Omega_{☾}}{M_{☾} c^2}|, \tag{1.10}$$

we infer that their gravitational binding energy contributes equally to the inertial and to the gravitational mass with an accuracy of about 5×10^{-4}. A more careful analysis, taking into account the inhomogeneous distribution of matter in the Earth and Moon, gives the range [3]

$$|\eta^{\text{exp}}| = (4.0 \pm 4.3) \times 10^{-4}. \tag{1.11}$$

Let us raise this empirical fact at the level of a 'strong' equivalence principle (SEP) which simply states that the free fall of a compact body is also independent of its gravitational binding energy, i.e.,

$$\eta_{\text{SEP}} \equiv 0. \tag{1.12}$$

The SEP can be considered as a physical principle which limits the choice of our theory for gravitation among all the possible metric theories one can construct.

1.2 Mass versus energy in gravitational interactions

The relativistic (Lorentz-invariant) action for a free elementary particle reads

$$S^{\text{free}} = \int \{-m_{\text{in}}\, c^2\} d\tau = \int \{-m_{\text{in}} c^2 + m_{\text{in}} \frac{v^2}{2} + \mathcal{O}(\frac{1}{c^2})\} dt. \tag{1.13}$$

If this massive particle carries an electric charge q and freely propagates in an electromagnetic vector field $A^\mu(t, \vec{x})$, then the action becomes

$$S^{\text{e.m.}} = \int \{-m_{\text{in}} c^2 - \frac{q}{c} \frac{dx_\mu}{d\tau} A^\mu\} d\tau. \tag{1.14}$$

By straight analogy with the Coulomb potential A^0 in the static limit ($dx_0 = c\, dt, d\vec{x} = 0$), the trajectory of an elementary particle propagating in a scalar gravitational field $V(t, \vec{x})$ might simply be defined by

$$S^{\text{gr.}} = \int \{-m_{\text{in}}\, c^2 - m_{\text{gr}} V\} d\tau. \tag{1.15}$$

If the weak equivalence principle (1.3) applies, this action can equivalently be written as

$$S^{\text{gr.}} = \int -m_{\text{in}}\, c\, ds, \tag{1.16}$$

ds being the invariant distance (or arclength) given by

$$ds^2 = (1 + \frac{V}{c^2})^2 \eta_{\mu\nu} dx^\mu dx^\nu. \tag{1.17}$$

Proposed by the Finnish physicist G. Nordström in 1913, i.e., two years before the birth of general relativity [4], this background-dependent scalar theory is thus characterized by a specific, conformally flat, space–time defined by

$$g_{\mu\nu} = (1 + \frac{V}{c^2})^2 \eta_{\mu\nu}. \tag{1.18}$$

In other words, the physical metric $g_{\mu\nu}(t, \vec{x})$ has only one degree of freedom, a scalar graviton field, the rest being fixed a priori by the flat Minkowski metric $\eta_{\mu\nu}$ which acts here as an absolute background in a way consistent with the Einstein equivalence principle. As a direct consequence, any massless particle plunged in this scalar gravitational field keeps on propagating along the light-cone

$$ds^2 \propto \eta_{\mu\nu} dx^\mu dx^\nu = 0. \tag{1.19}$$

In particular, the massless scalar graviton itself does not feel gravity and the strong equivalence principle (1.12) obviously holds true since the gravitational binding energy does not interfere in the free fall of a body.

Nordström's theory with its prior space–time geometry [5] was the first, mathematically consistent, theory resolving the clash between Newton's instantaneous gravity and Einstein's special relativity. However, this theory was in fact definitively falsified no more than six years after its elaboration. Following Nordström, the massless photon does not gravitate either and there is thus no possible light-bending at the limb of the Sun, in 'flat' contradiction with the direct observations [6] made by Dyson and Eddington during a total solar eclipse in 1919. Yet, since it embodies the strong equivalence principle, we shall rely on this rather simple toy theory in which only *mass* can feel the gravitational degree of freedom. For a more realistic theory where gravity couples to all kinds of *energy* in a way also compatible with the SEP, one should introduce a formalism which is free of any prior space–time geometry, i.e., background-independent.

Inspired by Nordström's theory where the equivalence principle has simply been geometrized, let us assume the gravitational interactions of matter (and light) to be characterized by the universal coupling to a metric field. For a free massive particle, this simply amounts to substituting $g_{\mu\nu}(q)$ for the Minkowski metric $\eta_{\mu\nu}$ in Eq. (1.13):

$$c^2 d\tau^2 \to ds^2 = g_{\mu\nu}(q) dq^\mu dq^\nu. \tag{1.20}$$

The relativistic principle of 'maximal ageing', originally set forth for twins, extends to curved space–time if a local inertial frame can be defined on every segment of the free body world line. In this case, the variational principle

$$\delta \int ds = 0 \tag{1.21}$$

implies that the track $q^\mu(\lambda)$ of a free particle plunged in a given gravitational field is always the shortest path (or geodesic) of the curved space–time, regardless of its (inertial) mass. Setting $d\lambda = ds$ on the unvaried path *after* all partial derivatives have been evaluated in the generalized Euler–Lagrange equations of motion,

$$\{\frac{d}{d\lambda}\frac{\partial}{\partial q'^\rho} - \frac{\partial}{\partial q^\rho}\} \{g_{\mu\nu}(q) q'^\mu q'^\nu\}^{\frac{1}{2}} = 0, \tag{1.22}$$

one easily obtains

$$\frac{d^2 q^\sigma}{ds^2} + \Gamma^\sigma_{\ \mu\nu} \frac{dq^\mu}{ds}\frac{dq^\nu}{ds} = 0, \tag{1.23}$$

where

$$\Gamma^\sigma_{\ \mu\nu} \equiv \frac{1}{2} g^{\sigma\rho}(\partial_\nu g_{\mu\rho} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}) \tag{1.24}$$

are the Christoffel symbols, also known as the components of the (affine) connection.

For illustration, let us consider the stationary, inhomogeneous gravitational field

$$V(r) = -\frac{GM}{r} \tag{1.25}$$

induced by the Sun on the Earth which is 150 million kilometres away. It is enough that the mixed space–time components of the Γ connection obey the approximate relation

$$\Gamma^i_{00} = \frac{1}{c^2}\delta^{ik}\partial_k V + \mathcal{O}(\frac{1}{c^4}) \tag{1.26}$$

in the weak field approximation

$$|\frac{V}{c^2}| \approx \frac{1.5 \text{ km}}{150 \times 10^6 \text{ km}} = 10^{-8} \ll 1 \tag{1.27}$$

to recover the Newtonian equation of motion

$$\frac{d^2\vec{q}}{dt^2} + \vec{\nabla} V \approx \vec{0}. \tag{1.28}$$

As a consequence, the weak equivalence principle is automatically implemented through the *kinematics* of test particles (space–time tells small mass how to move), without any reference to the specific *dynamics* of gravity (large mass tells space–time how to curve). In the particular case of the Nordström scalar theory, one has indeed the exact relation

$$\Gamma^i{}_{00} = \delta^{ik}\ \partial_k \ln(1 + \frac{V}{c^2}). \tag{1.29}$$

So, what then privileges Einstein's non-linear field equations which are supposed to determine the geometry around the Sun as well as the dynamics of the whole Universe? Here, we would like to emphasize that the free fall for compact bodies (i.e., bodies containing non-negligible gravitational binding energy, in contrast to test bodies) may give us a clue.

"If a person falls freely he will not feel his own weight" [7]. From this early "happiest thought", Einstein inferred that all physical laws of special relativity (electromagnetism included) should remain valid in a sufficiently small free falling laboratory to eventually establish his quite successful general theory of relativity, more than eight years later. The geodesic equations of motion we have derived in Eq. (1.23) nicely illustrate this remarkable property. Indeed they can be interpreted as a generalized Newtonian first law of classical mechanics in the presence of gravitational forces:

$$Dp^\sigma \equiv (\partial_\nu p^\sigma + \Gamma^\sigma{}_{\mu\nu} p^\mu) dq^\nu = 0 \tag{1.30}$$

with $p^\sigma \equiv m dq^\sigma/d\tau$, the relativistic 4-momentum of a test particle. In an inertial (free falling) frame, the Christoffel symbols $\Gamma^\sigma{}_{\mu\nu}$, which are *not* the components of a general coordinate tensor, identically vanish and the reduced equations of motion

$$(\frac{d^2x^\sigma}{d\tau^2})\,|_{\Gamma\to 0} = 0 \tag{1.31}$$

remain covariant with respect to (linear) Lorentz transformations, in full agreement with Einstein's equivalence principle. Similarly, in the limit of non-relativistic velocities the proper-time interval $d\tau$ reduces to the coordinate-time interval dt and the resulting equations of motion for a free particle

$$(\frac{d^2x^i}{dt^2})\,|_{\frac{v}{c}\to 0} = 0 \tag{1.32}$$

are only covariant with respect to Galileo transformations.

Now, on the basis of Eq. (1.30), we assume that the gravitational field interacts with matter *and* radiation through the general covariance which simply turns the ordinary derivative ∂_ν acting on any vector into the covariant derivative D_ν defined by

$$(D_\nu)^\sigma{}_\mu \equiv \partial_\nu \delta^\sigma{}_\mu + \Gamma^\sigma{}_{\mu\nu}. \tag{1.33}$$

In general, covariant derivatives do not commute in a curved space–time and we have

$$[D_\mu, D_\nu]^\sigma_{\ \lambda} \equiv -R^\sigma_{\ \lambda\mu\nu} \tag{1.34}$$

where

$$R^\sigma_{\ \lambda\mu\nu} \equiv \partial_\nu \Gamma^\sigma_{\ \lambda\mu} - \partial_\mu \Gamma^\sigma_{\ \lambda\nu} + \Gamma^\rho_{\ \lambda\mu}\Gamma^\sigma_{\ \rho\nu} - \Gamma^\rho_{\ \lambda\nu}\Gamma^\sigma_{\ \rho\mu} \tag{1.35}$$

is the Riemann tensor. In the weak field approximation $|\frac{V}{c^2}| \ll 1$, the following space–time components of this curvature tensor

$$R^i_{\ 00j} = -\frac{1}{c^2}\delta^{ik}\partial_k\partial_j V(r) + \mathcal{O}(\frac{1}{c^4}) \tag{1.36}$$

encode the first non-trivial gravitational effects of the Sun (and of the Moon) that one 'feels' on Earth, i.e., the tides:

$$V_{\text{tide}}(\vec{x}) \equiv \frac{1}{2}\sum_{k,j} x^k x^j \partial_k\partial_j V(\vec{0}). \tag{1.37}$$

But what does determine the full Riemann tensor in general:

$$R^\sigma_{\ \lambda\mu\nu} \neq 0? \tag{1.38}$$

Within a metric theory one can raise (lower) the space–time indices of any tensor. In particular, the anti-symmetry property of the Riemann tensor under a $\mu \leftrightarrow \nu$ interchange implies

$$D_\nu D_\mu R^{\sigma\ \mu\nu}_{\ \lambda} = 0! \tag{1.39}$$

These tensorial identities are most easily derived by working in a local inertial frame (i.e., $\Gamma \to 0$), as allowed by the Einstein equivalence principle. It seems therefore quite interesting to focus our attention on the *first* covariant derivatives of the Riemann tensor.

In a conformally flat space–time, the metric $g_{\mu\nu} = A^2(V)\eta_{\mu\nu}$ only depends on a scalar gravitational field $V(t, \vec{x})$ and one easily derives the relation

$$D_\mu R^{\sigma\ \mu\nu}_{\ \lambda} = \frac{1}{6}[\eta^{\sigma\nu}\eta_{\lambda\rho} - \delta^\nu_{\ \lambda}\delta^\sigma_{\ \rho}]\partial^\rho R \tag{1.40}$$

with

$$R \equiv g^{\lambda\mu}R^\nu_{\ \lambda\mu\nu} = -6A^{-3}\,\Box_\eta\, A \tag{1.41}$$

the curvature scalar. In our toy theory, i.e., the Nordström scalar theory based on Eq. (1.18), $A(V) = 1 + \frac{V}{c^2}$ and massless gravitons freely propagate in a Minkowski fixed background. Consequently, $R = 0$ and the Riemann tensor has to fulfil the non-trivial constraints

$$D_\mu R^{\sigma\ \mu\nu}_{\ \lambda} = 0 \tag{1.42}$$

in the vacuum. Contrary to Eq. (1.39), such non-linear constraints do *not* result from the Einstein equivalence principle. We may thus conjecture that they are necessary to guarantee the strong version of the equivalence principle in any metric theory for gravitation [8].

It turns out that Einstein's theory of gravity also complies with the tensorial constraints (1.42) in empty space. This property, due to the purely geometrical Bianchi identities, is quite remarkable since the gravitational fields of general relativity are known to interact with themselves, even when propagating in the vacuum. But in the presence of matter, what is then

$$D_\mu R^{\sigma\ \mu\nu}_{\ \lambda} \equiv j^{\sigma\ \nu}_{\ \lambda} \tag{1.43}$$

geometrically? Well, astrophysics tells us that the Universe might be dominated by some dark matter at galactic distance scales and by some dark energy at cosmological distance scales. But these interpretations rely on the validity of general relativity at all scales, while direct evidence for such exotic substances is still missing. Consequently, alternative identifications of the $j^{\sigma}{}_{\lambda}{}^{\nu}$ tensor are still allowed nowadays.

In the Nordström scalar theory, we note from Eq. (1.40) that the conformally flat space–time background implies a genuine (mass) conservation law

$$\partial_\nu j^{\sigma}{}_{\lambda}{}^{\nu} = \frac{1}{6}[\eta^{\sigma\nu}\eta_{\lambda\rho} - \delta^{\nu}{}_{\lambda}\delta^{\sigma}{}_{\rho}]\partial_\nu\partial^\rho R = 0 \tag{1.44}$$

in a way analogous to the theory for electromagnetism. Indeed, the anti-symmetry property of the field strength in the inhomogeneous Maxwell equations

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{1.45}$$

automatically implies the (charge) conservation law

$$\partial_\nu j^\nu = 0, \tag{1.46}$$

whatever the nature of the source at work may be (a Dirac electron, a Klein–Gordon charged pion, etc.). However, a covariant conservation law like

$$D_\nu j^{\sigma}{}_{\lambda}{}^{\nu} = 0 \tag{1.47}$$

does not imply, in general, an exact differential conservation law [9]. This is known to apply also for any non-Abelian gauge theory to which we now turn.

In 1954, Yang and Mills examined what would happen if the isospin symmetry introduced to explain similarities of protons and neutrons were a local, i.e., space–time dependent, symmetry. For that purpose, they explored the possibility that the relative orientation of isospin at two distinct points of space–time has no physical meaning, (once of course electromagnetism is neglected). The local Lorentz frames of general relativity (labelled by Greek space–time indices) are thus simply replaced by local SU(2) frames (labelled by Latin internal indices) and a connection is needed to compare nucleons located at distinct points of space–time. In particular, the covariant derivative acting on any spinor Ψ^b is introduced via the minimal substitution

$$(D_\nu)^a_b \equiv \partial_\nu \delta^a_b - \; ig\, A^a_{b\nu}, \tag{1.48}$$

with g the relevant coupling constant. To display the geometric nature of non-Abelian gauge interactions, let us rescale the Yang–Mills Hermitian matrix A_ν as follows:

$$g\, A \to A. \tag{1.49}$$

So, the components $\Gamma^{\sigma}_{\mu\nu}$ of the connection are replaced by the massless gauge fields $A^{a}{}_{b\nu}$ and the Riemann–Christoffel curvature tensor $R^{\sigma}{}_{\lambda\mu\nu}$ by the non-Abelian field strength $F^a_{b\mu\nu}$ such that:

$$[D_\mu \, , \, D_\nu]^a_b \equiv -i\, F^a_{b\mu\nu} \tag{1.50}$$

with

$$F^a_{b\mu\nu} \equiv \partial_\mu A^{a}{}_{b\nu} - \partial_\nu A^{a}{}_{b\mu} + iA^{c}{}_{b\mu}A^{a}{}_{c\nu} - iA^{c}{}_{b\nu}A^{a}{}_{c\mu} \tag{1.51}$$

and a, b, c, isospin (or colour) indices. This parallel drawn between the space–time curvature in Eq. (1.35) and the non-Abelian field strength in Eq. (1.51) is quite striking. Note here that the appearance of a factor i in the substitution

$$\Gamma \to -iA \tag{1.52}$$

stems from the hermiticity of the $i\hbar\partial_\mu$ operator in quantum field theory. The identities

$$D_\nu\, D_\mu\, F^{a\;\mu\nu}_{b} = 0 \tag{1.53}$$

suggest that the universality of free fall (i.e., the strong equivalence principle) is on an equal footing with the universality of coupling (i.e., the gauge principle). To pursue such a parallel between gravitation and gauge interactions, we introduce external current densities

$$D_\mu \, F^{a\ \mu\nu}_{\ b} = j^{a\ \nu}_{\ b}. \tag{1.54}$$

But again, what is $D_\mu \, F^{a\ \mu\nu}_{\ b}$ geometrically [10]? Well, here high-energy particle physics convincingly tells us that the gluons couple to (spin-$\frac{1}{2}$) matter fields, i.e, the coloured quarks. If we define the current as the first variation of the quantum chromodynamics (QCD) action with respect to the gauge fields, we obtain

$$j^{a\ \nu}_{\ b} = \bar{q}_b \gamma^\nu q^a. \tag{1.55}$$

It is then a direct consequence of the Dirac equation and its conjugate that this current indeed satisfies a covariant conservation law given by

$$D_\nu j^{a\ \nu}_{\ b} = \partial_\nu j^{a\ \nu}_{\ b} - iA^a_{\ c\mu} \ j^{c\ \mu}_{\ b} + iA^c_{\ b\mu} \ j^{a\ \mu}_{\ c} = 0. \tag{1.56}$$

Yet, the current is not conserved in the ordinary sense because gauge fields carry the colours with which they interact.

To summarize, the concepts of mass and energy in gravity provide us with a deep connection between general coordinate transformations and gauge transformations, and in particular between general relativity and non-Abelian gauge theories. Einstein gravitational fields carry energy and thus gravitate the way Yang–Mills gauge fields carry colours and thus self-interact. This has to be contrasted with the Nordström massless graviton which couples only to mass and the Maxwell neutral photon which couples only to electric charge.

1.3 Mass versus energy in electromagnetic, weak, and strong interactions

Today, Einstein's famous question

Does the inertia of a body depend upon its energy content?

applies to all forms of binding energy Ω that contribute to the inertial mass M of bound states:

$$M = \sum_i m_i + \frac{\Omega}{c^2}. \tag{1.57}$$

For compact spherical bodies of radius R, we already know from Eq. (1.8) that the gravitational contribution to the binding energy per unit mass scales like $\frac{M}{R}$:

$$\begin{aligned}
s_{\text{grav}} \equiv \left|\frac{\Omega_{\text{grav}}}{Mc^2}\right| \ &\approx 10^{-26} \quad \text{(sphere.................} M = 2 \times 10^0 \text{ kg.......} R = 10 \text{ cm)} \\
&\approx 10^{-10} \quad \text{(Earth....................} M = 6 \times 10^{24} \text{ kg......} R = 6400 \text{ km)} \\
&\approx 10^{-6} \quad \text{(Sun......................} M = 2 \times 10^{30} \text{ kg......} R = 700\,000 \text{ km)} \\
&\approx 10^{-3} \quad \text{(white dwarf..........} M = 2 \times 10^{30} \text{ kg......} R = 1000 \text{ km)} \\
&\approx 10^{-1} \quad \text{(neutron star...........} M = 2 \times 10^{30} \text{ kg......} R = 10 \text{ km).}
\end{aligned} \tag{1.58}$$

The ultimate stage of a heavy star, a stellar black hole, may thus be regarded as the extreme case where the binding energy is of the same order as the rest mass energy. For a dense stellar object with $R \approx \frac{2GM}{c^2}$, one indeed guesses $s_{\text{grav}} \approx 0.3$ from Eq. (1.6).

In the case of microscopic black holes, quantum arguments plead in favour of a mass directly proportional to the Planck scale,

$$M_{\text{bh}} \propto (\frac{\hbar c}{G})^{\frac{1}{2}} \approx 10^{19} \text{ GeV}, \tag{1.59}$$

excluding thus any production at the LHC if space–time is only 4-dimensional. For a binding energy proportional to the Newton constant G, Eq. (1.57) allows us to re-express the sensitivity as

$$s_{\text{grav}} = -\frac{G}{M}\frac{\partial M}{\partial G}. \tag{1.60}$$

Consequently, from Eq. (1.59) one derives now a firm upper bound for the ratio of internal gravitational binding energy to the total mass energy:

$$s_{\text{grav}} \leq \frac{1}{2}, \tag{1.61}$$

in full agreement with the field equations around a black hole for a tensor-scalar theory of gravity [11]. What about the other fundamental interactions?

At the molecular level, mass defects in chemical reactions are known to be quite negligible since Lavoisier (1789):

$$2\text{H}_2 + \text{O}_2 \rightarrow 2\text{H}_2\text{O} + Q \quad \text{with } \frac{Q}{Mc^2} \approx 10^{-13}. \tag{1.62}$$

At the (sub) atomic level, such is not the case anymore. Electromagnetic, nuclear, and strong interactions lead, respectively, to

$$\begin{aligned}
&- \quad \left|\frac{\Omega_{\text{em}}}{Mc^2}\right| \approx 10^{-8} \quad (m_{\text{hydrogen}} \approx m_{\text{proton}} + m_{\text{electron}} - 13.6\frac{\text{eV}}{c^2})\\
&- \quad \left|\frac{\Omega_{\text{nucl}}}{Mc^2}\right| \approx 10^{-3} \quad (m_{\text{deuterium}} \approx m_{\text{proton}} + m_{\text{neutron}} - 2.2\frac{\text{MeV}}{c^2})\\
&- \quad \frac{E_{\text{strong}}}{Mc^2} \approx 1 \qquad (m_{\text{proton}} \approx m_{\text{neutron}} = +\,940\frac{\text{MeV}}{c^2}).
\end{aligned} \tag{1.63}$$

As a result, the origin of the bulk of our mass and, consequently, of our weight is the kinetic energy of the massless gluons and nearly massless quarks confined in the nucleons. In technical words, our bathroom scales simply react to the fact that the QCD vacuum behaves like a paramagnetic medium [12]. The anti-screening effect of virtual gluons at 10^{-18} m is also a superb answer of the strong interactions to Einstein's question about the inertia of a body: gravitational self-interactions cannot saturate the mass of black holes the way strong interactions do for the mass of nucleons.

2 Confinement and $[q_i, p_j] \neq 0$

2.1 Nucleon mass

In Lecture 1, we have seen that the strong interactions are based on a gauge-invariant theory with Dirac particles (quarks) acting as colour sources:

$$L_{\text{fundamental}}(\text{gluons; quarks}) = \bar{q}\,(i\gamma_\mu D^\mu - m)\,q. \tag{2.1}$$

In the limit of two massless (up and down) quark flavours, chirality is conserved:

$$\begin{aligned}
q_L &= \tfrac{1}{2}(1-\gamma_5)q \quad , \quad \gamma_5 q_L = -q_L\\
q_R &= \tfrac{1}{2}(1+\gamma_5)q \quad , \quad \gamma_5 q_R = +q_R
\end{aligned} \tag{2.2}$$

and the corresponding Lagrangian

$$L_{\text{fundamental}}(\text{gluons; quarks} = u, d) = \bar{q}_L\, i\gamma_\mu D^\mu\, q_L + \bar{q}_R\, i\gamma_\mu D^\mu\, q_R \tag{2.3}$$

is invariant under a global U(2)$_L\times$ U(2)$_R$ symmetry:

$$\begin{aligned}
q_L \rightarrow g_L\, q_L \quad &\sim \quad (2_L, 1_R)\\
q_R \rightarrow g_R\, q_R \quad &\sim \quad (1_L, 2_R).
\end{aligned} \tag{2.4}$$

In Nature, one doublet of nucleon states ($J^P = \frac{1}{2}^+$) turns out to be massive (1 GeV) while one triplet of light pions ($J^P = 0^-$) is observed around 100 MeV. In other words, the chiral symmetry appears to be spontaneously broken down to an (approximate) SU(2) isospin symmetry through the confinement mechanism. In order to label the vacuum states, we introduce an effective (colour singlet) two-by-two complex matrix χ which, by construction, transforms according to

$$\chi \to g_L \, \chi \, g_R^\dagger \sim (2_L, 2^*{}_R) \tag{2.5}$$

with respect to the underlying chiral symmetry group. We may of course simply consider a bilinear in the up and down quark fields,

$$\chi^b{}_a \div \bar{q}_a (1 - \gamma_5) q^b, \tag{2.6}$$

though what really matters here are the chiral transformation properties. The complex field χ can always be expressed as a linear combination of two independent Hermitian matrix fields σ and π:

$$\chi \equiv \frac{(\sigma + i\pi)}{\sqrt{2}} \qquad (\sigma = \sigma^\alpha \tau_\alpha \quad , \quad \pi = \pi^\alpha \tau_\alpha) \tag{2.7}$$

with τ_0 the two-by-two unity matrix and $\tau_{1,2,3}$ the standard Pauli spin matrices:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.8}$$

The effective Lagrangian for this field reads in general

$$L_{\text{effective}}(\chi) = \frac{1}{2} \, \mathrm{Tr} \, (\partial_\mu \chi \partial^\mu \chi^\dagger) - V[\mathrm{Tr} \, (\chi \chi^\dagger)^n] \tag{2.9}$$

and the chiral invariant potential V should provide χ with a non-zero real vacuum expectation value (v.e.v.) proportional to the unity matrix in order to preserve the isospin SU(2) subgroup characterized by $g_L = g_R$ vectorial transformations.

For illustration, we may consider a minimal linear sigma model

$$L_{\text{linear}}(\chi) = \frac{1}{2} \, \mathrm{Tr} \, (\partial_\mu \chi \partial^\mu \chi^\dagger) - \frac{\lambda}{4} \, \mathrm{Tr} \, (\chi \chi^\dagger - \frac{f^2}{2})^2 \quad , \lambda > 0 \tag{2.10}$$

where

$$\begin{aligned} &< 0|\sigma|0 > = f \mathbb{1} \\ &< 0|\pi|0 > = 0. \end{aligned} \tag{2.11}$$

A suitable redefinition of the σ field,

$$\sigma \to \sigma - < 0|\sigma|0 >, \tag{2.12}$$

leads then to the following physical mass spectrum

$$\begin{aligned} m_{\sigma^\alpha} &= \sqrt{\lambda} f \\ m_{\pi^\alpha} &= 0. \end{aligned} \tag{2.13}$$

If, in addition, we assume that the nucleon doublet

$$N = \begin{pmatrix} p \\ \\ n \end{pmatrix} \tag{2.14}$$

transforms as

$$\begin{aligned} N_L \to g_L N_L &\quad \sim \quad (2_L, 1_R) \\ N_R \to g_R N_R &\quad \sim \quad (1_L, 2_R) \end{aligned} \tag{2.15}$$

under the chiral symmetry group, we may also consider

$$\begin{aligned} L_{\text{linear}}(N) &= \overline{N}_L\, i\gamma_\mu \partial^\mu\, N_L + \overline{N}_R\, i\gamma_\mu \partial^\mu N_R - g_{\pi\text{NN}}(\overline{N}_L \chi N_R + \text{h.c.}) \\ &= \overline{N}\, i\gamma_\mu \partial^\mu N - \frac{g_{\pi\text{NN}}}{\sqrt{2}}(\overline{N}\sigma N + i\overline{N}\gamma_5 \pi N) \end{aligned} \tag{2.16}$$

with $\frac{g_{\pi\text{NN}}}{\sqrt{2}} \approx 13.5$, the measured pseudoscalar coupling. The σ and π are then identified as scalar (0^+) and pseudoscalar (0^-) fields, respectively, while the nucleon mass is driven by the v.e.v. of the σ field given in Eq. (2.11) to fulfil the relation

$$M_N = g_{\pi\text{NN}} \frac{f}{\sqrt{2}}. \tag{2.17}$$

The rather simple linear sigma model defined by (2.10) and (2.16) seems to correctly implement the chiral symmetry breaking since it produces a (semi) realistic mass spectrum for the pseudoscalar triplet π, the nucleon doublet N, and the scalar triplet a_0:

$$140\ \text{MeV} = m_\pi \ll M_N \approx m_{a_0} = 980\ \text{MeV}. \tag{2.18}$$

However, at the experimental level, the full scalar multiplet around the nucleon mass scale is not settled yet. Moreover, at the theoretical level, chiral transformations of baryons are ambiguous. This latter fact becomes particularly obvious in the generalized case of three massless quark flavours (u, d, s). The Gell-Mann baryon octet $(J^P = \frac{1}{2}^+)$

$$B = \begin{pmatrix} \Sigma^0 + \frac{\Lambda}{\sqrt{3}} & \sqrt{2}\Sigma^+ & \sqrt{2}p \\ \sqrt{2}\Sigma^- & -\Sigma^0 + \frac{\Lambda}{\sqrt{3}} & \sqrt{2}n \\ \sqrt{2}\Xi^- & \sqrt{2}\Xi^0 & \frac{-2\Lambda}{\sqrt{3}} \end{pmatrix} \tag{2.19}$$

may indeed transform either as

$$\begin{array}{lcl} B_L \to g_L B_L g_L^\dagger & \sim & (8_L, 1_R) \\ B_R \to g_R B_R g_R^\dagger & \sim & (1_L, 8_R) \end{array} \tag{2.20}$$

or as

$$\begin{array}{lcl} B_L \to g_L B_L g_R^\dagger & \sim & (3_L, 3^*_R) \\ B_R \to g_R B_R g_L^\dagger & \sim & (3^*_L, 3_R) \end{array} \tag{2.21}$$

under SU(3)$_L\times$ SU(3)$_R$ since only the transformation properties of the baryon under the vectorial subgroup SU(n_F) (isospin symmetry, eightfold way, etc.) really matter [13]. So, let us turn to a non-linear effective theory to get rid of the elusive scalars and couple baryons to pseudoscalars in a unique way. For that purpose, we make use of the polar theorem which tells us that any arbitrary matrix $(\xi^\dagger\chi)$ can be written as the product of a Hermitian matrix (σ) and a unitary matrix (ξ):

$$\chi \equiv \xi(\pi)\frac{\sigma}{\sqrt{2}}\xi(\pi). \tag{2.22}$$

The main advantage of this new parametrization is that now the most general potential only depends on the scalar fields:

$$V[\text{Tr}\,(\chi\chi^\dagger)^n] = V(\sigma). \tag{2.23}$$

The chiral transformations (2.5) of χ require

$$\xi \to g_L \xi h^\dagger = h\, \xi g_R^\dagger \tag{2.24}$$

such that these scalars transform linearly with respect to h:

$$\sigma \to h(x)\sigma\, h(x)^{\dagger}. \tag{2.25}$$

The vectorial transformations h are not broken by the v.e.v. of the σ field given in Eq. (2.11). Moreover, being non-linear functions of g_L, g_R *and* $\pi(x)$, they depend in general on the space–time coordinates. Yet, for $g_L = g_R$, we have $h = g_L = g_R$ and we recover the successful SU(2)$_I\times$ U(1)$_B$ global symmetry. It is thus quite natural to extend these linear, though local, transformations to all the other hadron isospin multiplets to describe their interactions with the light pseudoscalar one. In particular, we shall impose this local *hidden symmetry* on the nucleons:

$$N \to h(x)\, N. \tag{2.26}$$

From the following transformation laws

$$\begin{array}{l}(\xi^{\dagger}\partial_{\mu}\xi) \to h(\xi^{\dagger}\partial_{\mu}\xi)h^{\dagger} + h(\partial_{\mu})h^{\dagger} \\ (\xi\partial_{\mu}\xi^{\dagger}) \to h(\xi\partial_{\mu}\xi^{\dagger})h^{\dagger} + h(\partial_{\mu})h^{\dagger},\end{array} \tag{2.27}$$

one can indeed easily build a gauge-invariant effective Lagrangian for the nucleon–pion interactions. At the leading order in the derivative couplings, it reads

$$L_{\text{non-linear}}(\overline{N},\pi) = \overline{N}(i\,\gamma^{\mu}D_{\mu} - M_N)N + g_A\,\overline{N}\,\gamma^{\mu}\gamma_5\,A_{\mu}\,N \tag{2.28}$$

with

$$D_{\mu}\,N \equiv [\partial_{\mu} + \frac{1}{2}(\xi^{\dagger}\partial_{\mu}\xi + \xi\partial_{\mu}\xi^{\dagger})]N \to h(x)D_{\mu}\,N \tag{2.29}$$

the effective covariant derivative acting on the nucleon doublet and

$$A_{\mu} \equiv \frac{i}{2}(\xi^{\dagger}\partial_{\mu}\xi - \xi\partial_{\mu}\xi^{\dagger}) \to hA_{\mu}h^{\dagger} \tag{2.30}$$

an effective field coupled to the axial-vector nucleon current.

If the elusive scalar degrees of freedom are frozen at their v.e.v., they simply decouple and we are left with an effective theory for the light pseudoscalar fields alone:

$$L_{\text{non-linear}}(\pi) = -f^2\,\text{Tr}\,(A_{\mu}A^{\mu}) = \frac{f^2}{4}\,\text{Tr}\,(\partial_{\mu}U\,\partial^{\mu}U^{\dagger}) \tag{2.31}$$

with

$$U \equiv \xi^2 \to g_L\,U\,g_R^{\dagger}. \tag{2.32}$$

This minimal effective Lagrangian contains in fact all the necessary features of the spontaneous chiral symmetry breaking pattern

$$\text{U}(2)_L \times \text{U}(2)_R \to \text{SU}(2)_{\text{isospin}} \times \text{U}(1)_{\text{baryon}}. \tag{2.33}$$

Indeed, if we expand the U field as follows:

$$U(\pi) = 1\!\!1 + i(\frac{\pi}{f}) - \frac{1}{2}(\frac{\pi}{f})^2 - i\,a(\frac{\pi}{f})^3 + (a - \frac{1}{8})(\frac{\pi}{f})^4 + \mathcal{O}(\pi^5), \tag{2.34}$$

– the vacuum expectation value of U is invariant under the unbroken vectorial subgroup U(2)$_{L\,+\,R}$ defined by $g_L = g_R$:

$$g_L < 0|U|0 > g_R^{\dagger} = < 0|U|0 >; \tag{2.35}$$

– the excited fields out of the vacuum are the four pseudoscalar Goldstone bosons associated with the four broken generators:

$$\pi = \begin{pmatrix} \pi_3 + \eta_0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi_3 + \eta_0 \end{pmatrix}. \tag{2.36}$$

The unitarity condition

$$UU^\dagger = \mathbb{1} \tag{2.37}$$

is fulfilled for any real value of a. It is quite convenient (and also standard) to fix the value of this free parameter to

$$a = \frac{1}{6} \tag{2.38}$$

with

$$U(\pi) = \exp(\frac{i\pi}{f}). \tag{2.39}$$

But you may as well choose for example the value

$$a = \frac{1}{4} \tag{2.40}$$

with

$$U(\pi) = \frac{(1 + \frac{i\pi}{2f})}{(1 - \frac{i\pi}{2f})} \tag{2.41}$$

since chiral invariance ensures that any physical quantity is independent of a.

Expanding A_μ defined in Eq. (2.30) to first order in π, we note that the derivative nucleon–pion interaction is related to the standard pseudoscalar one through the Dirac equation of motion and implies the Goldberger–Treiman relation [14]

$$\frac{g_{\pi \mathrm{NN}}}{\sqrt{2}} = g_A \frac{M_N}{f}. \tag{2.42}$$

From $g_A \approx 1.27$, the axial-vector coupling measured in parity-violating $(n \to pe\bar{\nu})$ β decays, and $M_N \approx 940$ MeV, the average mass of the nucleons, one can already infer that $f \approx 90$ MeV for the v.e.v. of the σ field defined in Eq. (2.11). However, a more precise estimate of the remaining free parameter f is directly obtained from weak interactions. Indeed, gauging SU(2)$_L$ requires, as usual, the introduction of a covariant derivative. At the fundamental level, it amounts to the minimal substitution

$$D_\mu \to D_\mu - iW^L{}_\mu \tag{2.43}$$

for the left-handed component of the quark fields in Eq. (2.1), such that

$$L_{\text{fund.}}(q) \ni \bar{q}_L^a \gamma^\mu W^L{}_\mu{}^{ab} q_L^b \equiv J^\mu{}_L(q) W^L{}_\mu \tag{2.44}$$

with

$$(J^\mu{}_L)^{ba}(q) = \bar{q}_L^a \gamma^\mu q_L^b. \tag{2.45}$$

At the effective level, we have to consider the minimal substitution

$$\partial_\mu U \to D_\mu U = \partial_\mu U - iW^L{}_\mu U \tag{2.46}$$

in Eq. (2.31) since

$$U \to g_L(x) U \tag{2.47}$$

under SU(2)$_L$ gauge transformations. The interaction terms are given by

$$L_{\text{eff.}}(\pi) \ni -i\frac{f^2}{4} \,\mathrm{Tr}\,(W^L{}_\mu\, U \partial^\mu U^\dagger - \partial^\mu U U^\dagger W^L{}_\mu) \equiv J^\mu_L(\pi) W^L_\mu \tag{2.48}$$

with

$$(J^\mu{}_L)^{ba}(\pi) = i\frac{f^2}{2}(\partial^\mu U U^\dagger)^{ba} \ni -\frac{f}{2}\partial^\mu \pi^{ba} \tag{2.49}$$

the left-handed hadronic current. Consequently, we obtain the vacuum-to-pion hadronic matrix element

$$< 0|(J^\mu{}_L)^{ud}|\pi^+ > = -i\frac{f}{\sqrt{2}}p^\mu \tag{2.50}$$

with

$$f = f_\pi \approx 93\,\text{MeV} \tag{2.51}$$

extracted from the measured $\pi^+ \to e^+\nu_e$ decay amplitude.

So, now we dispose of a rather elegant and very efficient frame to incorporate all the well-known results originally obtained from standard current algebra techniques, and in particular the theorems on electromagnetic quantum corrections derived in the 1960s. Electromagnetism is indeed introduced through the minimal substitution

$$\partial_\mu U \to D_\mu U = \partial_\mu U - iV_\mu[Q, U], \qquad Q = e\ \text{diag}\ (+\frac{2}{3}, -\frac{1}{3}) \tag{2.52}$$

since

$$U \to g(x)\ U g(x)^\dagger \tag{2.53}$$

under vectorial $\text{U}(1)_{QED}$ gauge transformations. In the Landau gauge for the photon propagator, the only relevant one-loop diagram is a quadratically divergent tadpole produced by the contact term

$$L(U) \ni \frac{e^2 f^2}{2}\ \text{Tr}\ (QUQU^\dagger)V_\mu\ V^\mu, \tag{2.54}$$

with $U(\pi)$ defined in Eq. (2.34). Consequently,

- expanding the U field at $\mathcal{O}(\pi^2)$, we obtain the combination $\text{Tr}(QQ\pi^2 - Q\pi Q\pi)$ which implies a mass correction for the charged pion only:

$$m^2_{\pi^+} - m^2_{\pi^0} = \frac{3\alpha}{4\pi}\Lambda^2 \ \ ; \tag{2.55}$$

- expanding the U field at $\mathcal{O}(\pi^4)$, with $a = 0$ to get rid of the cubic term, we obtain the combination $\text{Tr}(QQ\pi^4 - Q\pi^2 Q\pi^2)$ which does not allow an iso-singlet to decay into three pions, i.e.,

$$A^{\text{e.m.}}(\eta_0 \to \pi^+\pi^0\pi^-) = 0. \tag{2.56}$$

The knowledge of the underlying QCD theory helped us in understanding these two puzzling results. On the one hand, the quadratic dependence on the ultraviolet momentum cut-off Λ in the $\pi^+ - \pi^0$ mass difference is tamed in a natural way by one-loop diagrams involving the vector and axial vector resonances at work around 0.8GeV. (Interestingly, the composite structure of the pion softens its electromagnetic self-energy the way a composite structure for the Higgs scalar would naturally protect its mass in an effective theory of electroweak interactions.) On the other hand, the observed isospin-violating $\eta^{(\prime)} \to \pi\pi\pi$ decays are induced by the up–down quark mass difference to which we now turn.

2.2 Nucleon mass splitting

At the fundamental level, isospin violation beyond electromagnetism arises from the mass term

$$\Delta L_m(u, d) = -\bar{q}_{La}\ m^a_{\ b}\ q^b_R + \text{h.c.} \tag{2.57}$$

If the two-by-two quark mass matrix m is first treated as a spurion field, it has to transform under the chiral $\mathrm{U}(2)_L \times \mathrm{U}(2)_R$ group according to the rule

$$m(x) \rightarrow g_L\, m(x) g_R^\dagger. \tag{2.58}$$

At the effective level, the leading mass correction for the nucleons arises from the chiral invariant

$$\Delta L_m(N) = -\frac{b}{2}\overline{N}(\xi^\dagger m \xi^\dagger + \xi m^\dagger \xi)N \ni -\frac{b}{2}\overline{N}(m + m^\dagger)N. \tag{2.59}$$

Once the quark mass matrix is frozen to its real eigenvalues

$$m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \tag{2.60}$$

a neutron–proton splitting then takes place with

$$m_n - m_p = b(m_d - m_u) \approx 1.3 \text{ MeV} \tag{2.61}$$

if electromagnetic self-interaction corrections (in principle favourable to the proton) are neglected. Correspondingly, the leading mass correction for the (pseudo-) Goldstone bosons arises from

$$\Delta L_m(\pi) = \frac{f^2 r}{4} \text{ Tr } (mU^\dagger + Um^\dagger) \ni -\frac{r}{4} \text{ Tr } (m\pi^2). \tag{2.62}$$

For the charged pions, we obtain

$$m_{\pi^\pm}^2 = \frac{r}{2}(m_u + m_d) \approx 140 \text{ MeV}. \tag{2.63}$$

From the trace of the neutral pseudoscalars squared mass matrix

$$m_{\text{neutral}}^2 = \frac{r}{2}\begin{pmatrix} m_u + m_d & m_u - m_d \\ \\ m_u - m_d & m_u + m_d \end{pmatrix}, \tag{2.64}$$

we also obtain a quadratic mass relation

$$m_{\pi^0}^2 + m_{\eta'}^2 = 2m_{\pi^\pm}^2 \tag{2.65}$$

in clear contradiction with the observed mass spectrum

$$\begin{aligned} m_{\pi^0} &= 135 \text{ MeV} \\ m_{\eta'} &= 958 \text{ MeV}. \end{aligned} \tag{2.66}$$

The fact that the η' mass is close to the nucleon (and scalar) mass scale given in Eq. (2.18) strongly suggests the way to solve this problem [15]: assume the symmetry breaking pattern to be

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B \rightarrow \mathrm{SU}(2)_I \times \mathrm{U}(1)_B \tag{2.67}$$

instead of (2.33), such that only three Goldstone bosons are produced and not four. To implement the explicit breaking of the axial U(1), we thus add an $\mathcal{O}(1 \text{ GeV})$ mass term for the iso-singlet η_0:

$$\Delta L_{\mathrm{U}(1)} = -\frac{1}{2} m_0^2\, \eta_0{}^2. \tag{2.68}$$

The squared mass matrix then becomes

$$m_{\text{neutral}}^2 = \begin{pmatrix} m_{\pi^\pm}^2 & \Delta \\ \\ \Delta & m_0^2 + m_{\pi^\pm}^2 \end{pmatrix} ; |\Delta| \equiv \frac{r}{2}|m_u - m_d| \ll m_0^2 \tag{2.69}$$

and the resulting quadratic mass relations

$$
\begin{aligned}
m^2_{\pi^0} &\approx m^2_{\pi^\pm} - \frac{\Delta^2}{m_0^2} \\
m^2_{\eta'} &\approx m_0^2 + m^2_{\pi^\pm} + \frac{\Delta^2}{m_0^2}
\end{aligned} \tag{2.70}
$$

are in agreement with the electromagnetic self-interaction correction given in Eq. (2.55). But we still have to check that the modification (2.68) of the effective theory for strong interactions is compatible with what we know from the underlying QCD dynamics. Let us for that purpose consider the conservation law of the iso-singlet current

$$J^\mu_5 \equiv \bar{u}\gamma^\mu\gamma_5 u + \bar{d}\gamma^\mu\gamma_5 d \tag{2.71}$$

associated with the axial U(1) symmetry. At the effective level, the right-handed current J^μ_R is directly obtained from the simple parity transformation

$$U(\pi) \xrightarrow{P} U^\dagger(\pi) = U(-\pi) \tag{2.72}$$

applied on the left-handed hadronic current J^μ_L already derived in Eq. (2.49), such that

$$J^\mu_5 \equiv \text{ Tr } (J^\mu_R - J^\mu_L) = i\frac{f^2}{2} \text{ Tr } (\partial^\mu U^\dagger U - \partial^\mu U U^\dagger). \tag{2.73}$$

From the identity

$$\text{Tr } (\partial_\mu U U^+) = \frac{2i}{f}\partial_\mu \eta_0, \tag{2.74}$$

we infer that the iso-singlet current is *not* conserved in the massless limit $m_u = m_d = 0$:

$$\partial_\mu J^\mu_5 = 2f \,\Box\, \eta_0 = -2f m_0^2 \eta_0. \tag{2.75}$$

At the fundamental level, the same violation of a classical conservation law is induced by quantum effects and the so-called axial U(1) anomaly is precisely given by

$$\partial_\mu J^\mu_5 = n_F \frac{\alpha_s}{4\pi} G^{\alpha\beta} \widetilde{G}_{\alpha\beta} \tag{2.76}$$

with

$$\widetilde{G}_{\alpha\beta} \equiv \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} G^{\gamma\delta} \tag{2.77}$$

the dual of the gluon field strength. But this is not the end of the story since, as we shall see, the Standard Model of electroweak interactions provides us in principle with a *complex* quark mass matrix

$$m \neq m^\dagger \tag{2.78}$$

via the Higgs mechanism. Up to now, specific chiral $g^0_{L,R}$ unitary transformations had been implicitly used to write this mass matrix as a diagonal *and* real one. But the axial U(1) anomaly implies that one phase cannot be rotated away and that we end up at best with

$$g^0_L m g^0_R{}^\dagger = \exp(\frac{i\theta_M}{4}) \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \exp(\frac{i\theta_M}{4}). \tag{2.79}$$

The presence of a physical phase is in principle the signal for a violation under time reversal. The corresponding T operator is indeed anti-unitary, as is most easily seen from its effect on the Heisenberg commutator

$$[q_i, p_j] = i\hbar\ \delta_{ij} \xrightarrow{T} -i\hbar\, \delta_{ij} = [q_i, -p_j]. \tag{2.80}$$

Note that this microscopic irreversibility has to be distinguished from macroscopic ones which originate in quite peculiar boundary conditions: a Bunsen burner for heat propagation or the Lemaître Big Bang for an expanding Universe.

A simple way to convince ourselves that the strong axial anomaly indeed implies T violation is through the field redefinition

$$U \rightarrow \exp(\frac{i\theta_M}{2})U \tag{2.81}$$

or, equivalently,

$$\eta_0 \rightarrow \eta_0 + \frac{f}{2}\theta_M \tag{2.82}$$

with

$$\theta_M \equiv \arg\det m. \tag{2.83}$$

This field redefinition renders m totally real in Eq. (2.62) but modifies of course the anomalous part (2.68) of the effective Lagrangian,

$$\Delta L_{\mathrm{U(1)}} \rightarrow \Delta L_{\mathrm{U(1)}} - \frac{f}{2} m_0^2 \theta_M \eta_0, \tag{2.84}$$

in such a way that the η_0 pseudoscalar field now gets a non-zero v.e.v.:

$$< 0|\eta_0(0^{-+})|0 > \approx -[\frac{m_0^2}{m_0^2 + m_\pi^2}]\frac{f\theta_M}{2}. \tag{2.85}$$

Accordingly, both T *and* P violations occur in strong interactions once $m_0^2 \neq 0$. The identification of the axial anomaly, expressed at the effective level in Eq. (2.75) and at the fundamental one in Eq. (2.76), together with the shift in Eq. (2.84) requires a corresponding modification of the QCD action itself:

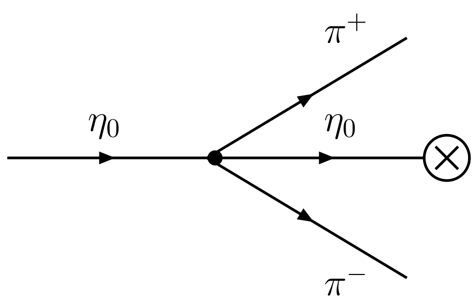
$$L_{\mathrm{QCD}} \rightarrow L_{\mathrm{QCD}} + \frac{\alpha_s}{8\pi}\theta_M G^{\alpha\beta}\widetilde{G}_{\alpha\beta}. \tag{2.86}$$

So, this new pseudoscalar term implies physical effects despite the fact that $G\widetilde{G}$ can be written as a total derivative. Let us illustrate this rather surprising result with a first example.

If we choose $a = \frac{1}{6}$ in Eq. (2.34), we may ignore the kinetic term in Eq. (2.31) and focus on the mass term in Eq. (2.62), with

$$\Delta L_m(\pi) \ni \frac{m_\pi^2}{2f^2}\, \eta_0 \pi^+ \pi^- \eta_0 \tag{2.87}$$

in the isospin limit, to get a non-zero T-violating $\eta' \rightarrow \pi^+\pi^-$ decay amplitude. Indeed, let one of the two η_0's propagate and then annihilate into the vacuum via the linear term in η_0 introduced in Eq. (2.84):



This non-local 'tadpole' contribution amounts to substituting *directly* $< 0|\eta_0|0 >$ for one η_0 in Eq. (2.87) and we obtain in that manner the local amplitude

$$|A(\eta' \rightarrow \pi^+\pi^-)| = \frac{m_\pi^2}{2f}[\frac{m_0^2}{m_0^2 + m_\pi^2}]\theta_M. \tag{2.88}$$

The corresponding two-body decay width reads

$$\Gamma(\eta' \to \pi^+\pi^-) \equiv \frac{1}{16\pi m_{\eta'}}|A(\eta' \to \pi^+\pi^-)|^2[1 - 4\frac{m_\pi^2}{m_{\eta'}^2}]^{\frac{1}{2}} \approx 0.2 \times \theta_M^2 \text{ MeV}. \tag{2.89}$$

Taking into account the measured η' total width, we obtain

$$\text{Br } (\eta' \to \pi^+\pi^-) \approx \theta_M^2 \tag{2.90}$$

such that the present experimental limit on this branching ratio

$$\text{Br } (\eta' \to \pi^+\pi^-) < 2 \times 10^{-2} \tag{2.91}$$

provides a rather weak bound

$$\theta_M \lesssim 10^{-1}. \tag{2.92}$$

Note, however, that the sizeable η_0 component in η(548) extracted from the non-linear effective theory with three light quark flavours (u, d, s) [16],

$$\begin{aligned} \eta &= \eta_8 \cos\phi - \eta_0 \sin\phi \\ \eta' &= \eta_8 \sin\phi + \eta_0 \cos\phi \end{aligned} \qquad (\phi \approx -22^\circ), \tag{2.93}$$

allows us to get a stronger bound, namely

$$\theta_M < 3 \times 10^{-4}, \tag{2.94}$$

from the new experimental limit

$$\text{Br } (\eta \to \pi^+\pi^-) < 1.3 \times 10^{-5}. \tag{2.95}$$

Significant improvements on these tree-level bounds are not foreseen since branching ratios are *quadratic* in the theta angle. So, let us turn to a second application with the ($p\pi^-$ loop-induced) neutron electric dipole moment *linear* in θ_M.

2.3 Nucleon electric dipole moment

Working in the isospin limit $m_u = m_d \equiv m_q$, the T-conserving effective interaction

$$\Delta L_m(N) \ni \frac{b\, m_q}{f^2}\overline{N}\pi N\eta_0, \tag{2.96}$$

derived this time from Eq. (2.59), leads to a *scalar* (i.e., T-violating) coupling if, again, η_0 is replaced by its v.e.v. given in Eq. (2.85). Consequently, the full pion–nucleon interaction is now defined by

$$L_{\pi NN} = -\frac{1}{\sqrt{2}}\overline{N}(g_{\pi NN}\, i\, \gamma_5 + g^\theta_{\ \pi NN})\pi N \tag{2.97}$$

with

$$g_{\pi NN} \approx \sqrt{2}\frac{M_N}{f} \tag{2.98}$$

and

$$g^\theta_{\ \pi NN} \approx \frac{m_q}{\Delta m_q}\frac{\Delta M_N}{\sqrt{2}f}[\frac{m_0^2}{m_0^2 + m_\pi^2}]\theta_M, \tag{2.99}$$

the T-conserving and T-violating effective couplings, respectively. For particles with spin $\vec{s}$ moving in an electromagnetic field $(\vec{E}, \vec{B})$, the classical dipole interactions are described by

$$H = -(d\vec{E} + \mu\vec{B}) \cdot \vec{s}. \tag{2.100}$$

At this level the spin can be viewed as an intrinsic angular momentum such that its transformation laws under T and P are the same as for the magnetic field, but opposite to the ones for the electric field. Accordingly, only a magnetic moment is allowed in any T-invariant theory. The Dirac relativistic equation alone tells us that the electron should have a magnetic moment given by

$$\mu_e = -\frac{e\hbar}{2m_e c}. \tag{2.101}$$

Were the proton and neutron elementary particles, the Dirac theory would then also predict

$$\mu_p^{(D)} = +\frac{e\hbar}{2m_p c}\ ,\ \mu_n^{(D)} = 0. \tag{2.102}$$

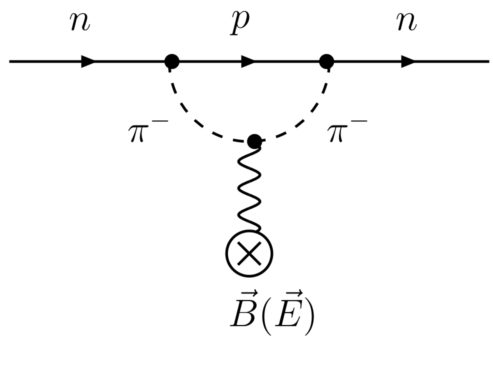
In fact, measurements yield anomalous magnetic moments:

$$\mu_p \approx +2.79\mu_N\ ,\ \mu_n \approx -1.91\mu_N \tag{2.103}$$

with

$$\mu_N \equiv \frac{e\hbar}{2M_N c} \approx 10^{-14}\text{e.cm.} \tag{2.104}$$

the nuclear magneton (remember, $\hbar c \approx 197$ MeV Fermi). These large departures from the predicted Dirac values are consequences of the fractional charge of the quarks confined in the nucleons. At the effective level, the magnetic dipole moment of the neutron can be associated with its charged pion cloud. In this heuristic picture, the neutron electric dipole moment obtained by substituting $g^{\theta}_{\pi NN}$ for the left or right $g_{\pi NN}$ vertex of the associated diagram



is thus expected to scale like

$$d_n \approx 2[\frac{g^{\theta}_{\pi NN}}{g_{\pi NN}}]\mu_n. \tag{2.105}$$

Neglecting again electromagnetic contributions to the proton–neutron mass difference, we obtain then

$$d_n \approx [\frac{m_n - m_p}{m_n + m_p}] \cdot [\frac{m_d + m_u}{m_d - m_u}]\theta_M\ \mu_n. \tag{2.106}$$

Compared with the present experimental limit,

$$|d_n| < 2.9 \times 10^{-26}\text{e.cm.}, \tag{2.107}$$

the approximate expression (2.106) confirms the quite impressive bound first derived in Ref. [17]:

$$\theta_M < 10^{-9}. \tag{2.108}$$

Such a strong constraint has been challenging theoreticians for decades. The fine-tuning we face here for the time-reversal violation in a quantum theory of strong interactions (QCD) is rather similar to the fine-tuning for the vacuum energy density in a relativistic theory of gravitational interactions (GR):

$$\theta \equiv \arg\det m + \theta_{\text{QCD}} \approx 0 \Leftrightarrow (2 \times 10^{-3}\ \text{eV})^4 \approx \frac{c^2}{8\pi G}\Lambda_{GR} + \frac{f^2 m_\pi^2}{2} \equiv \rho_{\text{vac.}}. \tag{2.109}$$

Two *ad hoc* parameters, θ_{QCD} and Λ_{GR}, are indeed introduced by hand to reconcile our theoretical prejudices with observations. Possible issues for the strong θ-puzzle are in fact inspired by attempts to solve the cosmological Λ-problem. Let us briefly consider two of them.

Firstly, by analogy with quintessence models which promote the cosmological constant Λ at the level of a field, one may transform the θ-parameter into a new dynamical variable

$$\theta(x) = \frac{2a_0(x)}{F}. \tag{2.110}$$

All the pseudoscalar fields then have a zero v.e.v. since

$$< 0|\eta_0|0 >=< 0|a_0|0 >= 0 \tag{2.111}$$

corresponds to the minimum of the new effective theory

$$L(\eta_0, a_0) = \frac{1}{2}\partial_\mu\eta_0\partial^\mu\eta_0 + \frac{1}{2}\partial_\mu a_0\partial^\mu a_0 - \frac{1}{2}\frac{m_0^2}{F^2}[F\eta_0 + fa_0]^2 \tag{2.112}$$

obtained after a field redefinition analogous to Eq. (2.81), i.e.,

$$U \to \exp(\frac{ia_0}{F})U. \tag{2.113}$$

As a consequence, the T and P discrete transformations are conserved in strong interactions but the spectrum of light pseudoscalars is modified. In the limit of massless quarks, the heavy iso-singlet pseudoscalar present in Eq. (2.112) is indeed given by

$$\eta' = \frac{[F\eta_0 + fa_0]}{(F^2 + f^2)^{\frac{1}{2}}} \tag{2.114}$$

while a new Goldstone boson, the axion, appears as the orthogonal combination:

$$a = \frac{[-f\eta_0 + Fa_0]}{(F^2 + f^2)^{\frac{1}{2}}}. \tag{2.115}$$

The axion can be treated as the light brother of η'. Yet, despite an efficient $\Delta I = \frac{1}{2}$ contribution through its η_0 component which implies

$$\text{Br}\,(K^+ \to \pi^+ a) \approx \frac{f^2}{F^2}\text{Br}\,(K^0 \to \pi^+\pi^-), \tag{2.116}$$

it has not been seen so far and the present bound is

$$\text{Br}\,(K^+ \to \pi^+ a) < 6 \times 10^{-11}. \tag{2.117}$$

This direct limit from particle physics already puts a rather severe constraint on the scale F, namely

$$F > 10^4 \text{ GeV}. \tag{2.118}$$

Consequently, the scale F associated with the spontaneous symmetry breaking at the origin of the axion cannot be identified with the Fermi scale and the original Peccei–Quinn scenario [18] is excluded by this simple exercise. Once the u and d quark masses are taken into account, the neutral squared mass matrix

$$m^2_{\text{neutral}} = \begin{pmatrix} m^2_{\pi^\pm} & 0 & 0 \\ 0 & m_0^2 + m^2_{\pi^\pm} & (\frac{f}{F})m_0^2 \\ 0 & (\frac{f}{F})m_0^2 & (\frac{f}{F})^2 m_0^2 \end{pmatrix} \tag{2.119}$$

implies that the axion is in fact a pseudo-Goldstone boson with a mass given by

$$m_a \approx \frac{f}{F} m_\pi < 1 \text{ keV}. \tag{2.120}$$

Being a light cousin of the neutral pion, it only decays into two photons (in a P-wave) and its lifetime scales like

$$\tau(a \to \gamma\gamma) \approx (\frac{m_\pi}{m_a})^3 (\frac{F}{f})^2 \tau(\pi \to \gamma\gamma) \approx (\frac{F}{f})^5 \times 10^{-16} s. \tag{2.121}$$

Axions couple to electromagnetic fields just as neutral pions do via the well-known Primakoff effect. So, if axions exist, they should be produced at the solar core and immediately leave the Sun without further scattering, carrying an energy of the order of T_{core} = 10^7 K ($\approx$ 1 keV). This has to be contrasted with photons which scatter for about 10^7 years before reaching the surface of the Sun with an energy of the order of T_{surface} = 6000 K ($\approx$ 1 eV). From the known energy loss of the Sun, one infers the bound [19]

$$F > 10^7 \text{ GeV}. \tag{2.122}$$

This indirect astrophysical limit pushes the allowed lifetime of the axion far beyond the age of the Universe, promoting in this way the elusive particle at the level of a candidate for dark matter in cosmology if $F < 10^{12}$ GeV.

Secondly, by analogy with supersymmetry which ensures a vanishing vacuum energy, one may also impose an extra chiral symmetry which allows us to rotate away the θ-parameter if

$$\det m = 0. \tag{2.123}$$

However, the possibility of having a massless quark is hardly consistent with the isospin violation extracted from the mass spectrum of the full 0^{-+} nonet, i.e.

$$\frac{m_u}{m_d} \approx \frac{1}{2} \neq 0, \tag{2.124}$$

and is even ruled out by large-N_c arguments [20].

As a matter of fact, we now have to address the question of the origin of the quark (and lepton) masses beyond Newton's classical definitions:

– *the measure of inertia* ($\frac{F}{a}$): a body tends indeed to resist any change in its existing state of rest or uniform motion, but the confinement of coloured particles tells us that quarks are never at rest and never free;

– *the amount of matter* (ρV): it is obvious that an elephant weighs much more than a mouse because it is made of many more atoms than a mouse, but elementary particles like the electron and the top quark which appear on an equal footing in quantum field theory obey the hierarchy

$$\frac{m_{\text{electron}}}{m_{\text{top}}} \approx \frac{m_{\text{mouse}}}{m_{\text{elephant}}}. \tag{2.125}$$

In lecture 1, we have seen that the bulk of our weight is due to the mass of the nucleon. Why then should one worry about the mass of the electron? Well, the electron is the substance from which the chemical elements are built (see Mendeleev's Table). Its mass determines the size of atoms through the Bohr radius ($\propto\ {m_e}^{-1}$) or, to be more precise, the quantized energy levels with

$$13.6\frac{\text{eV}}{c^2} = [\frac{\alpha^2}{2} + \mathcal{O}(\alpha^4)]m_e c^2 \tag{2.126}$$

for the hydrogen atom in Dirac's theory. So, no electron mass, no atoms; but no atoms, no chemistry. Similarly, no (up and down) quark mass, no stable proton; but no stable proton, no chemistry again!

3 Spontaneous symmetry breaking and $[M_u, M_d] \neq 0$

3.1 Boson masses and mixing

Another way to solve the axial U(1) problem without introducing T-violation in the gauge theory for strong interactions is to assume the chiral symmetry breaking pattern

$$\text{SU(2)}_L \times \text{SU(2)}_R \rightarrow \text{SU(2)}_V \tag{3.1}$$

instead of (2.67). If such was the case, only *three* pseudoscalar Goldstone bosons would be produced out of an order parameter made of four degrees of freedom:

$$\chi \equiv \frac{(\sigma + i\pi)}{\sqrt{2}} \qquad (\sigma = \sigma^0\tau_0\ \ ,\ \ \pi = \pi^a\tau_a)\ , \tag{3.2}$$

the missing η_0 would not trigger the axial U(1) problem and strong interactions would respect time-reversal symmetry. However, we know that a full decoupling of the pseudoscalar η_0 is not compatible with the sizeable $\eta - \eta'$ mixing given in Eq. (2.93).

It turns out that the restricted chiral symmetry breaking (3.1) is quite relevant for the gauge theory of electroweak interactions. Indeed, the local invariance under SU(2)$_L\times$ U(1)$_Y$ of the Standard Model has to be spontaneously broken into U(1)$_Q$ with Q, the conserved electric charge:

$$Q \equiv T_{3L} + \frac{Y}{2}. \tag{3.3}$$

So, a set of *three* (eaten up) Goldstone bosons ($\pi = \pi^a\tau_a$) is precisely what is needed to preserve one local U(1) unbroken and, therefore, to guarantee that the zero photon mass is not a mere accident [21].

Let us again make use of the polar theorem [see Eq. (2.22)] to write

$$\chi \equiv \xi(\pi)\frac{\sigma}{\sqrt{2}}\xi(\pi) = \frac{\sigma^0}{\sqrt{2}}U(\pi). \tag{3.4}$$

In the limit where the iso-singlet scalar field σ^0 is frozen at its v.e.v.,

$$< 0|\sigma^0|0 >= v = (\sqrt{2}G_{\text{Fermi}})^{-\frac{1}{2}} \approx 246 \text{ GeV}, \tag{3.5}$$

an iso-triplet of Goldstone fields $(\pi^\pm, \pi^3)$ is embodied in the unitary field

$$U = \exp(\frac{i\pi}{v}) \tag{3.6}$$

which globally transforms as

$$U \to g_L U g_R^\dagger \tag{3.7}$$

under SU(2)$_L\times$ SU(2)$_R$. The chiral invariant kinetic term

$$L_{\text{kinetic}}(\pi) = \frac{v^2}{4}\,\text{Tr}\,(\partial_\mu U \partial^\mu U^\dagger) \tag{3.8}$$

analogous to Eq. (2.31) contains all the information about the scalar sector of the Standard Model, except of course for the elusive Higgs particle $(h = \sigma^0 - v)$.

Gauging now the subgroup SU(2)$_L\times$ U(1)$_Y$ with the following normalizations

$$T_{3L} = \frac{\tau_3}{2} \tag{3.9}$$

$$\frac{Y}{2} = T_{3R} + \frac{B}{2} \tag{3.10}$$

requires, as we know, the introduction of covariant derivatives. The baryon number B, by definition, vanishes for scalar fields. From the chiral transformations of U in Eq. (3.7), we therefore write a covariant derivative similar to Eq. (2.46):

$$D_\mu U = \partial_\mu U - i\frac{g}{2}W^L_{\ \mu} U + i\frac{g'}{2} U W^R_{\ \mu} \tag{3.11}$$

with

$$W^L_{\ \mu} = \begin{pmatrix} W_\mu^{\ 3} & \sqrt{2}W_\mu^{\ +} \\ \sqrt{2}W_\mu^{\ -} & -W_\mu^{\ 3} \end{pmatrix} \text{ and } W^R_{\ \mu} = \begin{pmatrix} B_\mu & 0 \\ 0 & -B_\mu \end{pmatrix}. \tag{3.12}$$

Note that the absence of charged gauge bosons in the $W^R_{\ \mu}$ matrix implies a (maximal) parity-violation through an explicit breaking of the SU(2)$_R$ global symmetry. Expanding the Goldstone field U to zero order (or, equivalently, working in the unitary gauge $U = \mathbb{1}$) in

$$L_{\text{non-linear}}(\pi) = \frac{v^2}{4}\,\text{Tr}\,(D_\mu U D^\mu U^\dagger), \tag{3.13}$$

we directly read the mass spectrum for the gauge bosons from

$$\text{Tr}\begin{pmatrix} (gW_\mu^{\ 3} - g'B_\mu) & g\sqrt{2}W_\mu^{\ +} \\ g\sqrt{2}W_\mu^{\ -} & -(gW_\mu^{\ 3} - g'B_\mu) \end{pmatrix}^2 \frac{v^2}{16} \equiv \frac{1}{2}M_Z^2 Z_\mu Z^\mu + M_W^2 W_\mu^+ W^{-\mu}. \tag{3.14}$$

The massive neutral gauge boson is the linear combination of $W_\mu^{\ 3}$ and B_μ present in this trace:

$$Z_\mu = \frac{(gW_\mu^{\ 3} - g'B_\mu)}{(g^2 + g'^2)^{\frac{1}{2}}} \tag{3.15}$$

with

$$M_Z = (g^2 + g'^2)^{\frac{1}{2}}\frac{v}{2}. \tag{3.16}$$

The orthogonal combination, absent from this trace, is naturally identified as the massless photon:

$$V_\mu = \frac{(g'W_\mu^{\ 3} + gB_\mu)}{(g^2 + g'^2)^{\frac{1}{2}}}. \tag{3.17}$$

From the electromagnetic minimal substitution

$$\partial_\mu U \to D_\mu U = \partial_\mu U - iV_\mu[Q, U], \quad Q = e(\frac{\tau_3}{2} + \frac{\tau_0}{6}) \tag{3.18}$$

already introduced in Eq. (2.52), we infer that

$$e = \frac{gg'}{(g^2 + g'^2)^{\frac{1}{2}}}. \tag{3.19}$$

If we define the electric charge of the positron as

$$e \equiv g \sin\theta_W, \tag{3.20}$$

then

$$\frac{g'}{g} = \tan\theta_W. \tag{3.21}$$

Finally, the charged gauge bosons $W_\mu^{\pm}$ have a mass given by

$$M_W = \frac{gv}{2}(\approx \frac{v}{3}) \tag{3.22}$$

such that a relation between physical quantities, i.e.,

$$\frac{M_W^2}{M_Z^2} = [\frac{g^2}{g^2 + g'^2}] = \cos^2\theta_W \tag{3.23}$$

holds true, in remarkable agreement with current precision data.

In the limit $g' \to 0$, the massive weak gauge bosons $(W^{\pm}, W^3)$ form another iso-triplet with respect to the global SU(2)$_V$ subgroup. Consequently, this unbroken hidden symmetry 'protects' the tree-level W–Z mass relation (3.23) against large radiative corrections. These one-loop quantum corrections grow logarithmically with the mass of the Higgs (which acts here as an ultraviolet cut-off), not quadratically. The 'custodial' symmetry being a successful feature of the Standard Model, it provides a rather severe constraint on any possible extension of its scalar sector. For illustration, a Two-Higgs-Doublet Model (2HDM) characterized by a pair ($H^{\pm}$) of physical charged scalars should display some degeneracy in its scalar mass spectrum. Indeed, yet another iso-triplet can be formed with either a CP-odd neutral scalar (A^0), as is the case for the Minimal Supersymmetric Standard Model (MSSM) in a decoupling limit:

$$M_{H^\pm}^2 = M_{A^0}^2 + M_W^2 \to M_{A^0}^2, \tag{3.24}$$

or with a CP-even one (H^0):

$$M_{H^\pm}^2 = M_{H^0}^2, \tag{3.25}$$

as is the case if a twisted custodial symmetry is imposed [22]. Note that new interesting LHC phenomenology may take place within the second scenario since the absence of a ZZA^0 coupling allows us to consider A^0 to be as light as 50 GeV.

3.2 Fermion masses, mixings, and phase

In the Standard Model of electroweak interactions, fermion masses are generated through arbitrary Yukawa interactions. *If* the global SU(2)$_L\times$ SU(2)$_R$ symmetry of the scalar sector is extended to the fermions, we have

$$L_{\text{Yukawa}} = -Y_{ij}\overline{\Psi}{}^{0}{}_L^{\,i}\chi\Psi^{0}{}_R^{\,j} + \text{ h.c.} \qquad (i, j = 1, ..., N_g). \tag{3.26}$$

The Latin indices assigned here to the (left and right-handed) quark doublets

$$\Psi^0_{L,R} = \begin{pmatrix} u^0 \\ d^0 \end{pmatrix}_{L,R} \tag{3.27}$$

run in the (three-dimensional) generation space. Once the order parameter is frozen at its SU(2)$_V$-invariant v.e.v.,

$$< 0|\chi|0 >= \frac{1}{\sqrt{2}} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}, \tag{3.28}$$

we necessarily obtain equal mass matrices for the up and down quarks,

$$M_{\text{up}} = M_{\text{down}}, \tag{3.29}$$

as happens for the nucleons [see Eq. (2.17)]. But here only the heaviest quark appears to satisfy an approximate Goldberger–Treiman relation,

$$m_{\text{top}} \approx y_{tt} \frac{v}{\sqrt{2}}, \tag{3.30}$$

with $y_{tt} \approx 1$. So, today the question is no longer why is the t quark so heavy but why are the other quarks and leptons so light (see the elephant and the mouse)? In the Standard Model, different Yukawa couplings to the right-handed quark fields are introduced to break the SU(2)$_V$ custodial symmetry. In other words, the invariance under the global SU(2)$_R$ is explicitly broken, as was already the case when gauging the kinetic term for the Goldstone bosons via Eq. (3.11). The order parameter χ transforms as

$$\chi \to g_L \chi g_R^\dagger \tag{3.31}$$

under the global SU(2)$_L \times$ SU(2)$_R$. With the local SU(2)$_L$ acting on χ from the left, let us write it in a bi-doublet matrix form

$$\chi = (H| - i\tau_2 H^*) = \begin{pmatrix} \phi^0 & -(\phi^-)^* \\ \phi^- & (\phi^0)^* \end{pmatrix} \tag{3.32}$$

with the help of two complex fields

$$\phi^0 = \frac{1}{\sqrt{2}}(\sigma^0 + i\pi^3) \tag{3.33}$$

$$\phi^- = \frac{1}{\sqrt{2}}(i\pi^1 - \pi^2). \tag{3.34}$$

Indeed, if H transforms as a doublet under SU(2),

$$H \to \exp(i\varepsilon^a \tau_a) H, \tag{3.35}$$

so does $(-i\tau_2 H^*)$ since the Pauli matrices (2.8) satisfy the identities

$$(-i\tau_2)(-\tau_a{}^*)(-i\tau_2)^{-1} = \tau_a. \tag{3.36}$$

Both the first column (H) and the second column $(-i\tau_2 H^*)$ of χ transform as complex doublets under the local SU(2)$_L \times$ U(1)$_Y$, with hypercharge $Y = -1$ and $+1$, respectively. Accordingly, the most general gauge-invariant Yukawa interactions are given by

$$L_{\text{Yukawa}} = -Y^{ij}_{\text{up}} \, \overline{\Psi}^0_{Li} \, H u^0_{Rj} - Y^{ij}_{\text{down}} \, \overline{\Psi}^0_{Li} (-i\tau_2 H^*) d^0_{Rj} + \text{ h.c.} \tag{3.37}$$

In this way, the up and down quark mass matrices are unrelated and independent diagonalizations are required. Assuming (for a while) Hermitian mass matrices, we write

$$M_{\text{up}} = Y_{\text{up}} \frac{v}{\sqrt{2}} = V_u{}^\dagger D_u V_u \tag{3.38}$$

$$M_{\text{down}} = Y_{\text{down}} \frac{v}{\sqrt{2}} = {V_d}^\dagger D_d V_d \tag{3.39}$$

with

$$D_u = \text{ diag } (m_u, m_c, m_t) \tag{3.40}$$

$$D_d = \text{ diag } (m_d, m_s, m_b) \tag{3.41}$$

and

$$V_u {V_u}^\dagger = V_d {V_d}^\dagger = \mathbb{1}. \tag{3.42}$$

In the Standard Model of electroweak interactions, mixing angles arise from a misalignment between the gauge interaction basis $\{q^0\}$ and the mass matrix basis $\{q\}$ for the quark fields:

$$\{q^0\} = V^\dagger \{q\}. \tag{3.43}$$

The left-handed charged current already introduced in Eq. (2.45) is now defined by

$$\bar{u}^{0\,i}_L \gamma^\mu \delta_{ij} d^{0\,j}_L = \bar{u}^i_L \gamma^\mu (V_{\text{CKM}})_{ij} d^j_L \tag{3.44}$$

and displays indeed a non-trivial Cabibbo–Kobayashi–Maskawa (CKM) mixing matrix

$$V_{\text{CKM}} \equiv V_u {V_d}^\dagger \neq \mathbb{1} \tag{3.45}$$

whenever

$$[M_u, M_d] \equiv {V_u}^\dagger [D_{\text{up}}, V_{\text{CKM}} D_{\text{down}} V^\dagger_{\text{CKM}}] V_u \neq 0. \tag{3.46}$$

Note that the trace of any power of this commutator,

$$\text{Tr } [M_u, M_d]^n = \text{Tr } [D_{\text{up}}, V_{\text{CKM}} D_{\text{down}} V^\dagger_{\text{CKM}}]^n, \tag{3.47}$$

defines an invariant which only depends on *physical* quantities, namely the quark masses in $D_{u,d}$, the mixing angles and, possibly, phases in V_{CKM}.

In the Standard Model of electroweak interactions, phases indeed arise from the arbitrary, i.e., complex, Yukawa couplings [23]:

$$Y_{\text{up,down}} \neq Y^*_{\text{up,down}}. \tag{3.48}$$

Under the anti-unitary time-reversal operator [see Eq. (2.80)], each entry of the CKM mixing matrix is complex-conjugated

$$V_{\text{CKM}} \xrightarrow{T} V^*_{\text{CKM}} \tag{3.49}$$

such that the invariant traces (3.47) transform as

$$\text{Tr } [M_u, M_d]^n \xrightarrow{T} (-1)^n \text{Tr } [M_u, M_d]^n. \tag{3.50}$$

Consequently, we have a T violation in weak interactions once

$$\text{Tr } [M_u, M_d]^{2n+1} \neq 0 \;,\; n \geq 1. \tag{3.51}$$

A well-known theorem named in honour of A. Cayley and W. Hamilton asserts that any $N \times N$ matrix C is the solution of its associated characteristic polynomial:

$$p(\lambda) = \det(C - \lambda \mathbb{1}) \Rightarrow p(C) = 0. \tag{3.52}$$

Let us apply this theorem for the Hermitian matrix

$$C = i[M_u, M_d]. \tag{3.53}$$

– In the case of two generations ($N_g = 2$), it simply implies that

$$p(C) = (C - c_1)(C - c_2) = C^2 - (\mathrm{Tr}\, C)C + \det C\mathbb{1} = 0. \tag{3.54}$$

The matrix C being traceless, we have

$$C^2 = -\det C\mathbb{1}. \tag{3.55}$$

After n iterations, we obtain

$$\mathrm{Tr}\, [M_u, M_d]^{2n+1} = (\det C)^n\, \mathrm{Tr}\, [M_u, M_d] = 0 \tag{3.56}$$

and time-reversal is always valid.

– In the case of three generations ($N_g = 3$),

$$p(C) = (C-c_1)(C-c_2)(C-c_3) = C^3-(\mathrm{Tr}\, C)C^2+\frac{1}{2}[(\mathrm{Tr}\, C)^2-\mathrm{Tr}(C^2)]C-\det C\mathbb{1} = 0 \tag{3.57}$$

and we now have

$$C^3 - \frac{1}{2}[\mathrm{Tr}(C^2)]C - \det C\mathbb{1} = 0. \tag{3.58}$$

Taking the trace, we conclude that

$$\mathrm{Tr}\, [M_u, M_d]^3 = 3\det[M_u, M_d] \neq 0 \tag{3.59}$$

and time-reversal is in principle violated.

Let us first consider a toy theory to illustrate a possible connection between mass generation and T violation. For that purpose, we simplify the flavour mixing pattern by assuming purely democratic transitions between the generations. In the two-generation case, it amounts to rotating the d–s frame by a $\frac{\pi}{4}$ angle relative to the u–c one and the Cabibbo mixing matrix is thus real:

$$V_{\mathrm{CKM}} = (\sqrt{2})^{-1} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{3.60}$$

The generalization to the three-generation case is not obvious. Indeed, any d–s–b frame rotation relative to the u–c–t one violates democracy. We are therefore forced to work in a complex space with the introduction of a phase

$$\omega = \exp(\frac{2i\pi}{3}) \tag{3.61}$$

to guarantee full democracy in the moduli of the unitary CKM matrix [24]:

$$V_{\mathrm{CKM}} = (\sqrt{3})^{-1} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix}. \tag{3.62}$$

This geometrical approach nicely confirms the previous mathematical theorem and exhibits the sharp difference between two and three generations of quarks as far as T violation is concerned.

Inspired by a rather successful mass relation in the charged lepton sector [25],

$$(m_e + m_\mu + m_\tau) = \frac{2}{3}\{\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\}^2, \tag{3.63}$$

let us implement the CKM mixing matrix (3.62) with a down-quark (Hermitian) mass matrix

$$M_{\mathrm{down}} = \begin{pmatrix} a & b & b^* \\ b^* & a & b \\ b & b^* & a \end{pmatrix} \tag{3.64}$$

invariant under cyclic permutation (d_3 discrete group) in the basis where the up-quark mass matrix is diagonal:

$$M_{\text{up}} = \text{diag }(m_u, m_c, m_t). \tag{3.65}$$

Here, the misalignment given in Eq. (3.43) is due to the impossibility of simultaneous diagonalization since the commutator of the two mass matrices reads

$$[M_u, M_d] = \begin{pmatrix} 0 & b(m_c - m_u) & b^*(m_t - m_u) \\ -b^*(m_c - m_u) & 0 & b(m_t - m_c) \\ -b(m_t - m_u) & -b^*(m_t - m_c) & 0 \end{pmatrix} \neq 0. \tag{3.66}$$

Moreover, a T violation occurs in this simple ansatz with democratic mixings since

$$\det[M_u, M_d] = (m_t - m_c)(m_t - m_u)(m_c - m_u)(b^{*3} - b^3) \neq 0. \tag{3.67}$$

The eigenvalues of the down mass matrix (3.64) are extracted from the relation

$$M_{\text{down}} V_{\text{CKM}} = V_{\text{CKM}} D_{\text{down}} \tag{3.68}$$

and are given by

$$\begin{aligned} m_d &= a + b\,\omega + b^*\omega^2 \\ m_s &= a + b + b^* \\ m_b &= a + b\,\omega^2 + b^*\omega. \end{aligned} \tag{3.69}$$

Consequently,

$$(b^{*3} - b^3) = \frac{2i}{9}(m_b - m_s)(m_b - m_d)(m_s - m_d)\Im(\omega^2) \tag{3.70}$$

and the determinant of the $[M_u, M_d]$ commutator depends only on *physical* quantities (the quark masses in $D_{u,d}$ and the phase in V_{CKM}), as anticipated in Eq. (3.47).

A way to restore T invariance in this toy theory is to impose some mass degeneracy. For example, in the limit $b = b^*$, the down mass matrix (3.64) is invariant under permutations (S_3 discrete group) and admits two degenerate eigenvalues ($m_d = m_b$). In that limit the matrix is real and, consequently, T conserving. Equivalently, a pseudo-rotation of 45° in the *d*–*b* plane allows us to rotate away the ω phase and to write the CKM unitary matrix (3.62) as a 'tri-bimaximal' orthogonal one:

$$V_{\text{CKM}} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{3.71}$$

which appears to be of some relevance for neutrino physics.

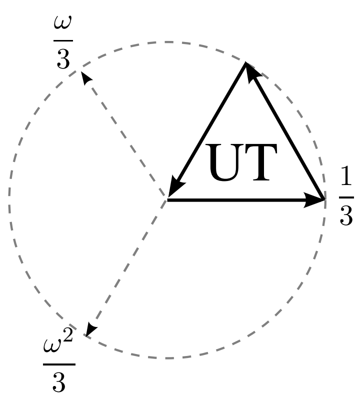
Our simple ansatz defined by (3.64) and (3.65) for the quark mass matrices has revealed a deep connection between mass splitting and T violation. It also provides a rather easy way to understand the concepts of 'unitarity triangles' and 'J-invariant', respectively.

In general, for three generations, six independent unitarity triangles (UTs) in the complex plane are expected from the unitarity constraints

$$\sum_j (V^\dagger_{\text{CKM}})_{ij}(V_{\text{CKM}})_{jk} = 0 \quad \text{if} \quad i \neq k. \tag{3.72}$$

In our toy theory (3.62), they reduce to a single equilateral UT defined by the relation

$$1 + \omega + \omega^2 = 0, \tag{3.73}$$



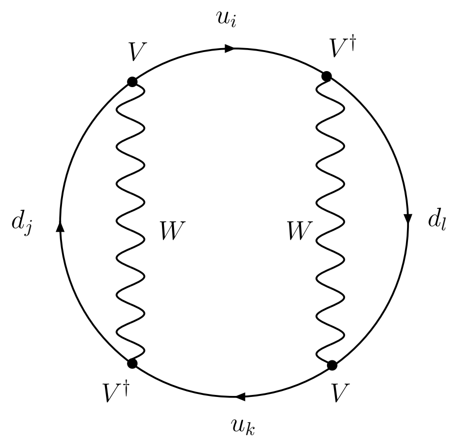
In general, for three generations, the T-violating invariant is defined by

$$\det[M_u, M_d] = 2i(m_t - m_c)(m_t - m_u)(m_c - m_u)(m_b - m_s)(m_b - m_d)(m_s - m_d)J \qquad (3.74)$$

with

$$J \equiv \pm\Im[(V)_{ij}(V^\dagger)_{jk}(V)_{kl}(V^\dagger)_{li}]. \qquad (3.75)$$

As a mnemotechnic, you may consider the flavour structure of a quark loop with two virtual W's:



The imaginary part of any possible 'quartet' $(V)_{ij}(V^\dagger)_{jk}(V)_{kl}(V^\dagger)_{li}$ (no sum over flavour indices) being equal up to a sign, the absolute value of J is unique and, in fact, proportional to the area A of any UT:

$$|J| = 2A_\Delta. \qquad (3.76)$$

In our toy theory (3.62), we indeed obtain

$$J \equiv \Im(V_{12}V_{22}{}^{*}V_{23}V_{13}{}^{*}) = \frac{1}{9}\Im(\omega^2) = \frac{-1}{6\sqrt{3}}. \qquad (3.77)$$

A more realistic CKM matrix has of course to be considered to reproduce the full (K^0, B^0) phenomenology. Expanding in the Cabibbo angle

$$\theta_c = \lambda \approx 0.23, \qquad (3.78)$$

one gets the following (very rough) pattern for the flavour mixings:

$$\begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \overset{C}{\approx} \begin{pmatrix} 1 & \lambda \\ -\lambda & 1 \end{pmatrix} \overset{KM}{\rightarrow} \begin{pmatrix} 1 & \lambda & -\lambda^3\omega \\ -\lambda & 1 & \lambda^2 \\ -\lambda^3\omega & -\lambda^2 & 1 \end{pmatrix}. \qquad (3.79)$$

At this level of approximation, we consider

– one UT directly accessible at $\mathcal{O}(\lambda^3)$ in B_d-physics:

$$(V^\dagger V)_{db} = V_{ud}{}^* V_{ub} + V_{cd}{}^* V_{cb} + V_{td}{}^* V_{tb} \approx -\lambda^3(\omega^2 + 1 + \omega) = 0 \tag{3.80}$$

– the invariant

$$J = \Im(V_{12} V_{22}{}^* V_{23} V_{13}{}^*) \approx \lambda^6 \Im(\omega) \approx 10^{-4}. \tag{3.81}$$

The hierarchy observed in the CKM mixing matrix may suggest that the phenomenon of flavour mixing is intimately related to the quark mass spectrum. Specific textures have indeed been proposed in that context. For illustration, the two-by-two quark mass matrix

$$M_{\text{down}} = \begin{pmatrix} 0 & b \\ b & a \end{pmatrix} \tag{3.82}$$

implies an intriguing relation between the Cabibbo mixing angle and a mass ratio [26],

$$\lambda \approx (\frac{m_d}{m_s})^{\frac{1}{2}}, \tag{3.83}$$

which triggered so many attempts to derive the CKM matrix (3.79) from 'horizontal' symmetries acting in the generation space. The large value conventionally extracted for the CKM phase may rather suggest a 'geometrical' T violation with [27]

$$\delta_{\text{CKM}} = \frac{2\pi}{N_g} \tag{3.84}$$

only depending on the number N_g of generations, not on a mass ratio. However, one should keep in mind that there are in fact nine distinctive parametrizations (P_i) of the CKM matrix,

$$\underset{P_1}{\begin{pmatrix} * & * & \circ \\ * & * & \circ \\ \circ & \circ & \circ \end{pmatrix}} \quad \underset{P_2}{\begin{pmatrix} * & \circ & * \\ \circ & \circ & \circ \\ * & \circ & * \end{pmatrix}} \quad \underset{P_3}{\begin{pmatrix} \circ & \circ & \circ \\ \circ & * & * \\ \circ & * & * \end{pmatrix}}$$

$$\underset{P_4}{\begin{pmatrix} \circ & * & * \\ \circ & * & * \\ \circ & \circ & \circ \end{pmatrix}} \quad \underset{P_5}{\begin{pmatrix} \circ & \circ & \circ \\ * & * & \circ \\ * & * & \circ \end{pmatrix}} \quad \underset{P_6}{\begin{pmatrix} * & \circ & * \\ * & \circ & * \\ \circ & \circ & \circ \end{pmatrix}}$$

$$\underset{P_7}{\begin{pmatrix} \circ & \circ & \circ \\ * & \circ & * \\ * & \circ & * \end{pmatrix}} \quad \underset{P_8}{\begin{pmatrix} * & * & \circ \\ \circ & \circ & \circ \\ * & * & \circ \end{pmatrix}} \quad \underset{P_9}{\begin{pmatrix} \circ & * & * \\ \circ & \circ & \circ \\ \circ & * & * \end{pmatrix}} \quad ,$$

each of them being obtained by imposing that one row and one column be real (see the circles in P_i). The invariant quantity J [28] is indeed expressed in terms of four mixing matrix elements which always form a 'plaquette' (see the asterisks in P_i). So, there are in principle nine independent phase conventions. For illustration, the original KM parametrization corresponds to excluding the first row and the first column (P_3) for the phase, while the standard convention is equivalent to crossing out the first row and the third column (P_5). However, any fundamental theory hidden behind the observed hierarchical quark mass spectrum should privilege one of these nine parametrizations. In this respect, we note that one and only one of them allows a small phase. By crossing out the second row and the second column (P_2), we obtain indeed $V_{ub} \sim \lambda(e^{-i\delta} - 1)$ with a T-violating angle δ of the order of $1°$:

$$\delta(P_2) = (1.1 \pm 0.1)°. \tag{3.85}$$

Within this parametrization P_2, the three angles are roughly equal to λ and the smallness of the J invariant is accounted for by the smallness of the phase ($J \approx \lambda^4\delta$). A natural relation between the CKM phase and a mass ratio is therefore also possible [29]. For example, a 'hierarchical' T violation with

$$\delta_{\text{CKM}} = \frac{m_s}{m_b} \tag{3.86}$$

consistently disappears in the decoupling limit ($m_b \to \infty$) for the third generation.

3.3 Matter–antimatter asymmetry

In the Standard Model of electroweak interactions, Eq. (3.37) implies that all the currently observed T- (or CP-) violating phenomena originate from the complex (CPT-invariant) Yukawa couplings Y of the Higgs field h to the quarks since

$$L_Y^{\text{neutral}} = -\frac{1}{\sqrt{2}}(\bar{q}^0_L Y q^0_R + \bar{q}^0_R Y^\dagger q^0_L)(h+v) \tag{3.87}$$

with

$$M = Y\frac{v}{\sqrt{2}} \xrightarrow{T} M^*. \tag{3.88}$$

In that sense, CP violation is our second compelling argument in favour of a single Higgs field, the first one being the custodial symmetry at the source of a natural zero photon mass. A multi-Higgs-doublet model generically produces a massive photon as well as a large neutron electric dipole moment.

However, there is no natural way to guarantee Hermitian quark mass matrices in this Standard Model. It is a fact that the mass matrices transform as

$$M = Y\frac{v}{\sqrt{2}} \xrightarrow{P} M^\dagger \tag{3.89}$$

under parity which interchanges left-handed and right-handed fields in Eq. (3.87). But we cannot impose this discrete symmetry on the whole theory since the weak gauge interactions are known to violate parity. So we have to consider another invariant involving now the commutator of the $MM^\dagger$ Hermitian matrices:

$$\begin{aligned} \det[M_u M_u^\dagger, M_d M_d^\dagger] &\equiv \det[D^2_{\text{up}}, V D^2_{\text{down}} V^\dagger] \\ &= 2i(m_t^2 - m_c^2)(m_t^2 - m_u^2)(m_c^2 - m_u^2)(m_b^2 - m_s^2)(m_b^2 - m_d^2)(m_s^2 - m_d^2)J. \end{aligned} \tag{3.90}$$

It also contains all the information about T violation (*squared* mass splittings and phase) since

$$\det[M_u M_u^\dagger, M_d M_d^\dagger] \xrightarrow{T} -\det[M_u M_u^\dagger, M_d M_d^\dagger] \tag{3.91}$$

but clearly has no defined parity:

$$\det[M_u M_u^\dagger, M_d M_d^\dagger] \xrightarrow{P} +\det[M_u^\dagger M_u, M_d^\dagger M_d] \tag{3.92}$$

as expected. So, let us once more invoke the polar decomposition, to write

$$M_{u(d)} = H_{u(d)} U_R^{u(d)}. \tag{3.93}$$

A diagonalization in two steps is then necessary to bring both mass matrices into their diagonal, real form defining the physical quark states. First, one exploits the fact that the right-handed quark fields are sterile with respect to the charged weak currents to eliminate U_R through a *chiral* transformation. In this first step, the QCD axial anomaly in Eq. (2.76) just holds back the flavour singlet phase with angle

$$\theta_M = \arg\det(U_R^u U_R^d) \neq 0. \tag{3.94}$$

The second step consists then in a *vectorial* transformation acting equally on the left- and right-handed up (down) quark fields to diagonalize the remaining Hermitian matrices $H_{u(d)}$. The observed mass hierarchy for the up and down quarks then gives

$$2im_t^2 m_c m_b^2 m_s J \approx \det[H_u, H_d] \neq 0. \tag{3.95}$$

In other words, the maximal parity-violation in the quark charged currents (3.44) implies that the θ_M and δ_{CKM} angles are totally unrelated at the tree level. On the one hand, strong CP violation in flavour diagonal transitions occurs through a T-violating quantity which is C-even but P-odd since

$$\arg\det(U_R^u U_R^d) \xrightarrow{P} -\arg\det(U_R^u U_R^d). \tag{3.96}$$

And this is precisely what is required to generate an electric dipole moment for the neutron, as already displayed in Eq. (2.106):

$$d_n \approx \theta \times 10^{-16} \text{ e.cm}. \tag{3.97}$$

On the other hand, weak CP violation in (V–A) flavour-changing transitions occurs through a T-violating quantity which is P-even but C-odd since

$$\det[H_u, H_d] \xrightarrow{C} -\det[H_u, H_d]. \tag{3.98}$$

And this is precisely one of the necessary ingredients to dynamically generate the matter–antimatter asymmetry observed in the present Universe [30]:

$$\left.\frac{(n_B - n_{\overline{B}})}{n_\gamma}\right|_0 = (6.1 \pm 0.2) \times 10^{-10}. \tag{3.99}$$

From the magnitude *and* the quantum number assignment of its two independent sources of T violation:

$$\begin{aligned} |\theta| < 10^{-9} \;\; &; \;\; J^{PC} = 0^{-+} \\ 2m_t^2 m_c m_b^2 m_s \frac{|J|}{(\frac{v}{\sqrt{2}})^6} \approx 10^{-14} \;\; &; \;\; J^{PC} = 0^{+-} \end{aligned} \tag{3.100}$$

we conclude that the Standard Model of strong and electroweak interactions does not seem to be able to produce enough baryon asymmetry. However, both sources are deeply connected to the quark mass spectrum: the former vanishes if one quark is massless (say, $m_u = 0$), while the latter can be rotated away if two quarks with same electric charge are degenerated (say, $m_d = m_s$). So one may conjecture that they have in fact the same magnitude. If such turns out to be the case, $|\theta| \approx 10^{-14}$ and one expects a neutron electric dipole moment around 10^{-30} e.cm.

4 Conclusions

Gauge invariance and time-reversal symmetry provide us with some (modest) steps towards a possible unification of the fundamental interactions. These symmetries explain, for example, why the weakest among the four known basic forces of nature, i.e., gravity, may dominate in the celestial environment (from the spherical shape of planets to the expansion of the Universe).

- Electromagnetic and gravitational interactions indeed obey gauge invariance which requires massless messangers; so they both lead to long-range forces. However, time reversal applied on the corresponding connections ($-iA$ and Γ, respectively) disentangles them: screening only occurs for spin-1-mediated interactions between opposite-sign charges, not for spin 0 or 2 ones which couple positive masses or energies.

– Strong and weak interactions get round gauge invariance through the subtle mechanisms of confinement and spontaneous symmetry breaking, respectively. In this way, T violation is peculiar to short-range nuclear forces.

This striking correlation between gauge invariance and time-reversal symmetry challenges us. Questions at issue are the unexpectedly tiny value of the cosmological constant Λ in the Einstein–Hilbert action and of the angle θ in the QCD action. A direct observation of non-baryonic dark matter and of the neutron electric dipole moment could bring these fundamental questions to a successful conclusion.

It is needless to emphasize that a theoretical understanding of the full fermion mass spectrum or (and) the discovery of the Higgs boson would be a major breakthrough in any case.

– If the Higgs boson turns out to be elementary, it will open the door to other hypothetical scalar fields (quintessence, inflaton, axion etc.) invoked to solve further theoretical puzzles (dark energy, homogeneity, electric dipole moments etc.) in cosmology and particle physics. Moreover, its Yukawa interactions which are genuine sources for T violation would be promoted to the rank of the fifth fundamental interaction and the issue of universal coupling reopened. Our knowledge about the gravitational interactions may help us in that venture. At this point we simply note that in the historical Thomson experiment which led to the discovery of the first elementary particle, only charged particles could feel the electric field, not neutral ones. Similarly, in the early Nordström theory, only massive particles could feel the gravitational field, not massless ones. So, the Higgs boson in its present formulation looks more like a scalar graviton. Could the analogy with a more successful background-independent theory of gravity guide us towards a geometrical interpretation of the Yukawa interactions?
– If the Higgs boson proves not to be elementary, no doubt the strong interactions will continue to inspire us in the quest for our precise weight. After all, less than 5% of the matter–energy content of our Universe is currently understood.

Acknowledgements

I would like to congratulate the organizers of this School for bringing together students from so many different countries in such a lively and stimulating atmosphere.

I am also grateful to Céline Degrande, Fabio Maltoni, and Jacques Weyers for their comments on the manuscript as well as to Cathy Brichard and Vincent Boucher for their help in preparing these lecture notes.

References

[1] A. Einstein, as quoted in A. Pais, *Subtle is the Lord* (Clarendon Press, Oxford, 1982).
[2] K. Nordvedt, Phys. Rev. **170** (1968) 1186.
[3] S.G. Turyshev, *Experimental tests of general relativity*, `arXiv:0806.1731[gr-qc]`.
[4] For an historical account, see F. Ravndal, *Scalar gravitation and extra dimensions*, `arXiv:gr-qc/0405030`.
[5] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (W.H. Freeman, San Francisco, 1973).
[6] For an historical account, see D. Kennefick, *Not only because of theory: Dyson, Eddington and the competing myths of the 1919 eclipse expedition*, `arXiv:0709.0685(physics.hist-ph)`.
[7] A. Einstein, as quoted in A. Pais, *Subtle is the Lord* (Clarendon Press, Oxford, 1982).
[8] J.-M. Gérard, Class. Quantum Grav. **24** (2007) 1867.
[9] P.A.M. Dirac, *General Theory of Relativity* (Princeton University Press, 1996).

[10] R.P. Feynman, *Gauge theories*, in *29th Ecole d'Eté de Physique Théorique: Weak and Electromagnetic Interactions at High Energy*, Les Houches, France, 1976, (eds) R. Balian and C.H. Llewellyn Smith (North Holland, Amsterdam, 1977).

[11] C.M. Will and H.W. Zaglauer, Astrophys. J. **346** (1989) 366.

[12] F. Wilczek, *Asymptotic freedom: from paradox to paradigm*, Rev. Mod. Phys. **77** (2005) 857.

[13] H. Georgi, *Weak Interactions and Modern Particle Theory* (Benjamin-Cummings, Menlo Park, 1984).

[14] S. Weinberg, *The Quantum Theory of Fields*, vol. II: Modern Applications (Cambridge University Press, 1996).

[15] G. 't Hooft, Phys. Rev. Lett. **37** (1976) 8.

[16] J.-M. Gérard and E. Kou, Phys. Lett. B **616** (2005) 85.

[17] R.J. Crewther, P. Di Vecchia, G. Veneziano and E. Witten, Phys. Lett. B **88** (1979) 123.

[18] R.D. Peccei and H.R. Quinn, Phys. Rev. Lett. **38** (1977) 1440.

[19] See for example, G.G. Raffelt, *Axions-motivation, limits and searches*, `arXiv:hep-ph/0611118`.

[20] J.-M. Gérard, Mod. Phys. Lett. A5 (1990) 391; H. Leutwyler, Phys. Lett. B **378** (1996) 313.

[21] M. Veltman, *Reflections on the Higgs system*, CERN Report 97-05.

[22] J.-M. Gérard and M. Herquet, Phys. Rev. Lett. **98** (2007) 251802.

[23] G.C Branco, L. Lavoura and J.P. Silva, *CP Violation* (Oxford University Press, 1999).

[24] I. Dunietz, O.W. Greenberg and D.D. Wu, Phys. Rev. Lett. **55** (1985) 2935;
In the context of a model with three neutrinos, see also N. Cabibbo, Phys. Lett. B **72** (1978) 333;
L. Wolfenstein, Phys. Rev. D **18** (1978) 958.

[25] Y. Koide, Mod. Phys. Lett. A **5** (1990) 2319.

[26] R. Gatto, G. Sartori and M. Tonin, Phys. Lett. B **28** (1968) 128;
N. Cabibbo and L. Maiani, Phys. Lett. B **28** (1968) 131.

[27] G.C. Branco, J.-M. Gérard and W. Grimus, Phys. Lett. B **136** (1984) 383.

[28] C. Jarlskog, Phys. Rev. Lett. **55** (1985) 1039.

[29] J.-M. Gérard and F. Goffinet, in preparation.

[30] A.D. Sakharov, JETP Lett. **5** (1967) 24.