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NUMERICAL SOLUTIONS AND THEIR SUPERCONVERGENCE FOR WEAKLY SINGULAR INTEGRAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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ABSTRACT

The piecewise polynomial collocation method is discussed to solve second kind Fredholm integral equations with weakly singular kernels $K(t, s)$ which may be discontinuous at $s = d$, $d = \text{const}$. The main result is given in Theorem 4.1. Using special collocation points, error estimates at the collocation points are derived showing a more rapid convergence than the global uniform convergence in the interval of integration available by piecewise polynomials.

1. INTEGRAL EQUATION

Consider the linear integral equation

$$u(t) = \int_0^b K(t, s)u(s)ds + f(t), \quad 0 \leq t \leq b, \quad (1.1)$$

where $b \in \mathbb{R}$ and $f: [0, b] \rightarrow \mathbb{R}$ is a given continuous function. Throughout this paper we shall suppose that the kernel K has the form

$$K(t, s) = a(t, s)\kappa(t - s) \quad (1.2)$$

where

(A1) the function $\kappa(\tau)$ is $m - 1$ times ($m \geq 1$) continuously differentiable with respect to τ for $\tau \in [-b, b] \setminus \{0\}$ and such that the estimates

$$|\kappa^{(k)}(\tau)| \leq b_k |\tau|^{-\alpha-k}, \quad k = 0, 1, \dots, m - 1, \quad (1.3)$$

hold with $0 < \alpha < 1$ and some positive constants b_0, b_1, \dots, b_{m-1} ;

(A2) the function $a(t, s)$ is m times continuously differentiable on $[0, b] \times [0, d]$ and $[0, b] \times [d, b]$ independently, where d is a fixed point in the interval $(0, b)$.

Let $C^m(X)$, where $X \subset \mathbb{R}$, denote the space of m times continuously differentiable functions $x: X \rightarrow \mathbb{R}$. For $0 < \alpha < 1$, $m \in \mathbb{N}$, $0 < d < b$, define

$$E^{\alpha, m} \equiv \left\{ u \in C[0, b] \cap C^m(0, d) \cap C^m(d, b) : \right.$$

$$\left. \sup_{\substack{0 < t < b \\ t \neq d}} \frac{|u^{(m)}(t)|}{t^{-(\alpha+m-1)} + |t-d|^{-(\alpha+m-1)} + (b-t)^{-(\alpha+m-1)}} < \infty \right\};$$

$E^{\alpha, m}$ is a Banach space under the norm

$$\|u\|_{E^{\alpha, m}} = \max_{0 \leq t \leq b} |u(t)| + \sup_{\substack{0 < t < b \\ t \neq d}} \frac{|u^{(m)}(t)|}{t^{-(\alpha+m-1)} + |t-d|^{-(\alpha+m-1)} + (b-t)^{-(\alpha+m-1)}}.$$

It follows from $u \in E^{\alpha, m}$ that $u \in C[0, b] \cap C^m(0, d) \cap C^m(d, b)$ and for $0 < t < d$ and $d < t < b$ the following estimates hold:

$$|u^{(k)}(t)| \leq c_k [t^{-(\alpha+k-1)} + |t-d|^{-(\alpha+k-1)} + (b-t)^{-(\alpha+k-1)}], \quad k = 1, \dots, m, \tag{1.4}$$

where c_1, \dots, c_k are some positive constants. Note also that $C^m[0, b] \subset E^{\alpha, m}$.

The following result (see [4; 6; 2]) states the regularity properties of solutions of equation (1.1).

LEMMA 1.1. *Let the assumptions (A1) and (A2) hold, and let $f \in E^{\alpha, m}$. If integral equation (1.1) has a solution $u \in L^1(0, b)$ then $u \in E^{\alpha, m}$.*

REMARK 1. If the function $a(t, s)$ is continuous on $[0, b] \times [0, b]$ then the estimates (1.4) for the derivatives of the solution $u(t)$ of equation (1.1) can be specified (see [4]).

2. PIECEWISE POLYNOMIAL APPROXIMATION

Let $N \in \mathbb{N}$, $r \in \mathbb{R}$, $r \geq 1$. We introduce in the interval $[0, d]$ the following $2N$ grid points

$$\begin{aligned} t_j^{(N)} &= \left(\frac{j}{N}\right)^r \frac{d}{2}, \quad j = 0, 1, \dots, N; \\ t_{N+j}^{(N)} &= d - t_{N-j}^{(N)}, \quad j = 1, \dots, N-1, \end{aligned} \tag{2.1}$$

and in the interval $[d, b]$ $2N + 1$ grid points

$$\begin{aligned} t_{2N+j}^{(N)} &= d + \left(\frac{j}{N}\right)^r \frac{b-d}{2}, \quad j = 0, 1, \dots, N; \\ t_{3N+j}^{(N)} &= b - t_{3N-j}^{(N)}, \quad j = 1, \dots, N-1; \quad t_{4N}^{(N)} = b. \end{aligned} \quad (2.2)$$

Here $r \geq 1$ characterizes the degree of the nonuniformity of the grid. If $r = 1$ then the grid points (2.1) and (2.2) are uniformly located in the intervals $[0, d]$ and $[d, b]$ respectively (and in $[0, b]$ if $d = b/2$). If $r > 1$ then the grid points $\{(2.1), (2.2)\}$ are more densely located towards the end points of the intervals $[0, d]$ and $[d, b]$.

We determine the collocation points in the following way. We choose m points η_1, \dots, η_m in the interval $[-1, 1]$:

$$-1 \leq \eta_1 < \eta_2 < \dots < \eta_m \leq 1. \quad (2.3)$$

By affine transformations we transfer them into the interval $[t_{j-1}^{(N)}, t_j^{(N)}]$:

$$\xi_{j,q}^{(N)} = t_{j-1}^{(N)} + \frac{\eta_q + 1}{2} (t_j^{(N)} - t_{j-1}^{(N)}), \quad q = 1, \dots, m; \quad j = 1, \dots, 4N. \quad (2.4)$$

Note that $\xi_{j,m}^{(N)} = \xi_{j+1,1}^{(N)} = t_j^{(N)}$, if $\eta_1 = -1, \eta_m = 1$ ($j = 1, \dots, 4N - 1$).

For a continuous function $u: [0, b] \rightarrow \mathbb{R}$ we construct a piecewise polynomial interpolation function $P_N u: [0, b] \rightarrow \mathbb{R}$ as follows: on every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ ($j = 1, \dots, 4N$), $P_N u$ is a polynomial of degree not exceeding $m - 1$ and

$$(P_N u)(\xi_{j,q}^{(N)}) = u(\xi_{j,q}^{(N)}), \quad q = 1, \dots, m; \quad j = 1, \dots, 4N.$$

Thus the interpolation function $(P_N u)(t)$ is uniquely defined in every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ ($j = 1, \dots, 4N$) separately and may have jumps if $t = t_j^{(N)}$, $j = 1, \dots, 4N - 1$. If $\eta_1 = -1, \eta_m = 1$, then $P_N u$ is a continuous function on the interval $[0, b]$. We can define $(P_N u)(t)$ by the formula

$$(P_N u)(t) = \sum_{q=1}^m u(\xi_{j,q}^{(N)}) \varphi_{j,q}^{(N)}(t), \quad t \in [t_{j-1}^{(N)}, t_j^{(N)}], \quad j = 1, \dots, 4N, \quad (2.5)$$

where $\varphi_{j,q}^{(N)}(t)$, $t \in [t_{j-1}^{(N)}, t_j^{(N)}]$, $q = 1, \dots, m$, are the polynomials of degree $m - 1$ such that

$$\varphi_{j,q}^{(N)}(\xi_{j,p}^{(N)}) = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases}, \quad p = 1, \dots, m. \quad (2.6)$$

Let us denote by E_N the range of the operator $P_N \equiv P_N^{(m)}$. This is the space of piecewise polynomial functions u_N on $[0, b]$ which on every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ ($j = 1, \dots, 4N$) are polynomials of the degree not exceeding $m - 1$.

The approximation properties of $P_N u$ on grid $\{(2.1), (2.2)\}$ are considered in [5] (cf. also [6; 7; 8]). These results can be summarized as follows.

LEMMA 2.1. *Assume that $u \in E^{\alpha, m}$. Then*

$$\|u - P_N u\|_{L^\infty(0, b)} \leq \text{const} \begin{cases} h_N^{r(1-\alpha)} & \text{for } 1 \leq r \leq \frac{m}{1-\alpha}, \\ h_N^m & \text{for } r \geq \frac{m}{1-\alpha}, \end{cases} \quad (2.7)$$

where

$$h_N = \max \left\{ \frac{d}{2N}, \frac{b-d}{2N} \right\}. \quad (2.8)$$

3. COLLOCATION METHOD

We look for an approximate solution $u_N \in E_N$ to integral equation (1.1). We require that u_N should satisfy the equation (1.1) at the collocation points (2.4):

$$\left[u_N(t) - \int_0^b K(t, s) u_N(s) ds - f(t) \right]_{t=\xi_{i,p}^{(N)}} = 0, \quad (3.1)$$

$$p = 1, \dots, m, \quad i = 1, \dots, 4N.$$

By the representation (2.5), we can find $u_N \in E_N$ in the form

$$u_N(t) = \sum_{q=1}^m c_{j,q}^{(N)} \varphi_{j,q}^{(N)}(t), \quad t \in [t_{j-1}^{(N)}, t_j^{(N)}], \quad j = 1, \dots, 4N,$$

where, as it follows from (2.6),

$$c_{j,q}^{(N)} = u_N(\xi_{j,q}^{(N)}), \quad q = 1, \dots, m; \quad j = 1, \dots, 4N.$$

Now the collocation conditions (3.1) will take the following form of a system which determines the coefficients $c_{i,p}^{(N)} = u_N(\xi_{i,p}^{(N)})$:

$$c_{i,p}^{(N)} = \sum_{j=1}^{4N} \sum_{q=1}^m a_{i,p,j,q}^{(N)} c_{j,q}^{(N)} + f(\xi_{i,p}^{(N)}), \quad p = 1, \dots, m; \quad i = 1, \dots, 4N, \quad (3.2)$$

where

$$a_{i,p,j,q}^{(N)} = \int_0^b K(\xi_{i,p}^{(N)}, s) \varphi_{j,q}^{(N)}(s) ds.$$

If $\eta_1 > -1$ or $\eta_m < 1$, then all collocation points $\xi_{j,q}^{(N)}$ ($q = 1, \dots, m$, $j = 1, \dots, 4N$) are different and there are $4mN$ collocation points. In this

case the system (3.2) (system (3.1)) has $4mN = \dim E_N$ equations and the same number of unknowns. If $\eta_1 = -1$ and $\eta_m = 1$, then part of the collocation points will coincide. The number of different collocation points is $[4N(m-1) + 1] = \dim E_N$ and the system (3.2) (system (3.1)) has the same number of equations and unknowns.

THEOREM 3.1. (cf. [5]). *Assume that the following conditions are fulfilled: 1) the kernel (1.2) satisfies the assumptions (A1) it and (A2); 2) $f \in E^{\alpha,m}$; 3) the homogeneous integral equation*

$$u(t) = \int_0^b K(t,s)u(s)ds \quad (3.3)$$

has only the trivial solution $u = 0$; 4) the collocation points (2.4) are used. Then the equation (1.1) has a unique solution u^ and there exists N_0 such that, for $N \geq N_0$, the collocation conditions (3.1) define a unique approximation $u_N^* \in E_N$ to u^* . The following error estimates hold:*

$$\|u_N^* - u^*\|_{L^\infty(0,b)} \leq c \begin{cases} h_N^{r(1-\alpha)} & \text{for } 1 \leq r \leq \frac{m}{1-\alpha}, \\ h_N^m & \text{for } r \geq \frac{m}{1-\alpha}, \end{cases} \quad (3.4)$$

where r is the scaling parameter of the grid $\{(2.1), (2.2)\}$, h_N is defined in (2.8) and c is a positive constant independent of h_N .

Proof. We write the integral equation (1.1) in the form $u = Tu + f$ where

$$(Tu)(t) = \int_0^b K(t,s)u(s)ds, \quad t \in [0, b]. \quad (3.5)$$

It follows from (A1) and (A2) (see [1]) that $T: L^\infty(0, b) \rightarrow C[0, b]$, moreover, $T: L^\infty(0, b) \rightarrow L^\infty(0, b)$, is compact. As the homogeneous equation $u = Tu$ has only the trivial solution $u = 0$, then the equation $u = Tu + f$ has a unique solution $u^* \in L^\infty(0, b)$. Due to Lemma 1.1, $u^* \in E^{\alpha,m}$. The collocation conditions (3.1) can be written in the form

$$u_N = P_N T u_N + P_N f \quad (3.6)$$

where P_N is defined in Section 2. If $N \rightarrow \infty$ then $\|P_N u - u\|_{L^\infty(0,b)} \rightarrow 0$ for every $u \in C[0, b]$. Therefore $\|P_N T - T\|_{L^\infty(0,b) \rightarrow L^\infty(0,b)} \rightarrow 0$, $N \rightarrow \infty$. From this and from the boundedness of $(I - T)^{-1}$ in $L^\infty(0, b)$ we obtain that $I - P_N T$ is invertible for sufficiently large $N \geq N_0$ and uniformly bounded in N :

$$\|(I - P_N T)^{-1}\|_{L^\infty(0,b) \rightarrow L^\infty(0,b)} \leq c. \quad (3.7)$$

Let $N \geq N_0$ and $u_N^* = (I - P_N T)^{-1} P_N f$ be the solution of the equation (3.6). Then $u_N^* - u^* = (I - P_N T)^{-1} (P_N u^* - u^*)$ and

$$\|u_N^* - u^*\|_{L^\infty(0,b)} \leq c \|P_N u^* - u^*\|_{L^\infty(0,b)}.$$

Due to (2.7) we obtain the estimate (3.4). \square

4. SUPERCONVERGENCE AT COLLOCATION POINTS

Now we assume that the points (2.3) are the nodes of a quadrature formula

$$\int_{-1}^1 g(s) ds = \sum_{q=1}^m A_q g(\eta_q) + R_m(g), \quad -1 \leq \eta_1 < \dots < \eta_m \leq 1, \quad (4.1)$$

which is exact for all polynomials of degree $m + 1$ ($m \geq 2$).

THEOREM 4.1. *Let $m \in \mathbb{N}$, $m \geq 2$. Assume that the following conditions are fulfilled:*

(i) *the kernel K has the form (1.2) where*

1) *the function $\kappa(\tau)$ is m times continuously differentiable with respect to τ for $\tau \in [-b, b] \setminus \{0\}$ and such that the estimates*

$$|\kappa^{(k)}(\tau)| \leq b_k |\tau|^{-\alpha-k}, \quad k = 0, 1, \dots, m,$$

hold with $0 < \alpha < 1$ and some positive constants b_0, b_1, \dots, b_m ;

2) *the function $a(t, s)$ is $m+1$ times continuously differentiable on $[0, b] \times [0, d]$ and $[0, b] \times [d, b]$, where $0 < d < b$;*

(ii) *$f \in E^{\alpha, m+1}$;*

(iii) *homogeneous integral equation (3.3) has only the trivial solution $u = 0$;*

(iv) *the grid $\{(2.1), (2.2)\}$ is used with $r \geq (m+1)/(1-\alpha)$ and the collocation points (2.4) are generated by the nodes (2.3) of a quadrature formula (4.1) which is exact for all polynomials of degree $m + 1$.*

Then there exists $N_0 \in \mathbb{N}$ such that for $N \geq N_0$

$$\max_{q=1, \dots, m; j=1, \dots, 4N} |u_N^*(\xi_{j,q}^{(N)}) - u^*(\xi_{j,q}^{(N)})| \leq c h_N^m h_N^{1-\alpha}, \quad (4.2)$$

where u^ is the solution of equation (1.1), $u_N^* \in E_N$ is the solution of the system (3.1), h_N is defined in (2.8) and c is a positive constant independent of h_N (of N).*

Proof. Due to Lemma 1.1, $u^* \in E^{\alpha, m+1}$. We have

$$|u_N^*(\xi_{j,q}^{(N)}) - u^*(\xi_{j,q}^{(N)})| \leq \|u_N^* - P_N u^*\|_{L^\infty(0,b)} \quad (q = 1, \dots, m; j = 1, \dots, 4N). \quad (4.3)$$

As

$$u_N^* - P_N u^* = (I - P_N T)^{-1} P_N T (P_N u^* - u^*) \quad N \geq N_0,$$

then with help of (3.7)

$$\|u_N^* - P_N u^*\|_{L^\infty(0,b)} \leq c \|T(P_N u^* - u^*)\|_{L^\infty(0,b)}. \quad (4.4)$$

Let us estimate $\|T(P_N u^* - u^*)\|_{L^\infty(0,b)}$. Fix $t \in [0, b]$ and let

$$\eta(t, h_N) = (t - h_N, t + h_N) \cap [0, b]. \quad (4.5)$$

Then

$$\left| \int_0^b K(t, s) [u^*(s) - (P_N u^*)(s)] ds \right| \leq I_1(t) + I_2(t), \quad (4.6)$$

where

$$I_1(t) = \sum_{j: [t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) \neq \emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |K(t, s) [u^*(s) - (P_N u^*)(s)]| ds,$$

$$I_2(t) = \sum_{j: [t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) = \emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |K(t, s) [u^*(s) - (P_N u^*)(s)]| ds.$$

It follows from the assumption (i) that

$$I_1(t) \leq c \|u^* - P_N u^*\|_{L^\infty(0,b)} \sum_{j: [t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) \neq \emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |t - s|^{-\alpha} ds.$$

By the Lemma 2.1 we obtain $\|u^* - P_N u^*\|_{L^\infty(0,b)} \leq c' h_N^m$. Due to (4.5)

$$\sum_{j: [t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) \neq \emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |t - s|^{-\alpha} ds \leq c'' \int_{t-2h_N}^{t+2h_N} |t - s|^{-\alpha} ds \leq c''' h_N^{m+1-\alpha}.$$

Thus

$$I_1(t) \leq c_1 h_N^m h_N^{1-\alpha}, \quad t \in [0, b], \quad c_1 = \text{const.} \quad (4.7)$$

Consider the term $I_2(t)$, $t \in [0, b]$. Let

$$t_{j, \frac{1}{2}}^{(N)} = \frac{t_{j-1}^{(N)} + t_j^{(N)}}{2}.$$

In addition to the points (2.3) we fix in $[-1, 1]$ a point η_{m+1} ($\eta_{m+1} \neq \eta_i$, $i = 1, \dots, m$). By an affine transformation we transfer η_{m+1} into the point $\xi_{j, m+1}^{(N)} \in [t_{j-1}^{(N)}, t_j^{(N)}]$ so that $\xi_{j, m+1}^{(N)} \neq \xi_{j, i}^{(N)}$, $i = 1, \dots, m$ ($j = 1, \dots, 4N$). Similarly to the definition of P_N (see Section 2) we define for a continuous function $u: [0, b] \rightarrow \mathbb{R}$ a piecewise polynomial function $P_N^{(m+1)}u: [0, b] \rightarrow \mathbb{R}$ as follows: $P_N^{(m+1)}u$ is on every interval $[t_{j-1}^{(N)}, t_j^{(N)}]$ ($j = 1, \dots, 4N$) a polynomial of degree not exceeding m and

$$P_N^{(m+1)}u(\xi_{j, q}^{(N)}) = u(\xi_{j, q}^{(N)}), \quad q = 1, \dots, m+1; j = 1, \dots, 4N.$$

We have

$$I_2(t) \leq I_{21}(t) + I_{22}(t) + I_{23}(t), \quad t \in [0, b],$$

where

$$\begin{aligned} I_{21}(t) &= \sum_{j: [t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) = \emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} \left| K(t, s) - K(t, t_{j, \frac{1}{2}}^{(N)}) \right| |u^*(s) - (P_N u^*)(s)| ds, \\ I_{22}(t) &= \sum_{j: [t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) = \emptyset} \left| K(t, t_{j, \frac{1}{2}}^{(N)}) \right| \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |u^*(s) - (P_N u^*)(s)| ds, \\ I_{23}(t) &= \left| \sum_{j: [t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) = \emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} K(t, t_{j, \frac{1}{2}}^{(N)}) [(P_N^{(m+1)} u^*)(s) - (P_N u^*)(s)] ds \right|. \end{aligned}$$

Let us consider $I_{21}(t)$, $t \in [0, b]$. It follows from Lemma 2.1 that

$$I_{21}(t) \leq c h_N^m \sum_{j: [t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) = \emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} \left| \frac{\partial K(t, s)}{\partial s} \right|_{s=\tau_j} \left| s - t_{j, \frac{1}{2}}^{(N)} \right| ds,$$

where $\tau_j \in (s, t_{j, \frac{1}{2}}^{(N)})$. We have for $s \in [t_{j-1}^{(N)}, t_j^{(N)}]$

$$\left| \frac{\partial K(t, s)}{\partial s} \right|_{s=\tau_j} |s - t_{j, \frac{1}{2}}^{(N)}| \leq c' h_N |t - \tau_j|^{-\alpha-1}.$$

Since $[t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) = \emptyset$, $s \in [t_{j-1}^{(N)}, t_j^{(N)}]$ and $\tau_j \in (s, t_{j, \frac{1}{2}}^{(N)})$, then

$$\tilde{c}_1 \leq \frac{|t - \tau_j|}{|t - s|} \leq \tilde{c}_2,$$

where \tilde{c}_1 and \tilde{c}_2 are some positive constants. Therefore

$$\begin{aligned} I_{21}(t) &\leq c'' h_N^{m+1} \sum_{j: [t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) = \emptyset} \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} |t - s|^{-\alpha-1} ds \leq \\ &\leq c''' h_N^{m+1} \int_{[0, b] \setminus \eta(t, h_N)} |t - s|^{-\alpha-1} ds. \end{aligned}$$

Due to (4.5) $\int_{[0, b] \setminus \eta(t, h_N)} |t - s|^{-\alpha-1} ds \leq c'''' h_N^{-\alpha}$. Thus

$$I_{21}(t) \leq c_2 h_N^m h_N^{1-\alpha}, \quad t \in [0, b], \quad c_2 = \text{const.}$$

Let us turn to $I_{22}(t)$, $t \in [0, b]$. It follows from $[t_{j-1}^{(N)}, t_j^{(N)}] \cap \eta(t, h_N) = \emptyset$ that $|K(t, t_{j, \frac{1}{2}}^{(N)})| \leq c |t - t_{j, \frac{1}{2}}^{(N)}|^{-\alpha} \leq c' h_N^{-\alpha}$. Due to Lemma 2.1 $\|u^* - P_N^{(m+1)} u^*\|_{L^\infty(0, b)} \leq c'' h_N^{m+1}$. Therefore

$$I_{22}(t) \leq c_3 h_N^m h_N^{1-\alpha}, \quad t \in [0, b], \quad c_3 = \text{const.}$$

Consider $I_{23}(t)$, $t \in [0, b]$. Due to the assumption (iv) we obtain that the quadrature formula

$$\int_{t_{j-1}^{(N)}}^{t_j^{(N)}} g(s) ds = \frac{t_j^{(N)} - t_{j-1}^{(N)}}{2} \sum_{q=1}^m A_q g(\xi_{j,q}^{(N)}) + \frac{t_j^{(N)} - t_{j-1}^{(N)}}{2} R_m(g)$$

remains to be exact for polynomials of degree $m+1$. Using this we have

$$\int_{t_{j-1}}^{t_j} [(P_N^{(m+1)} u^*)(s) - (P_N u^*)(s)] ds = 0,$$

and therefore $I_{23}(t) = 0$, $t \in [0, b]$. Thus

$$I_2(t) \leq c_4 h_N^m h_N^{1-\alpha}, \quad t \in [0, b], \quad c_4 = \text{const.} \quad (4.8)$$

Now the estimate (4.2) follows from (4.3), (4.4), (4.6), (4.7) and (4.8). \square

REMARK 2. For $a \in C^{m+1}([0, b] \times [0, b])$ the estimate (4.2) follows from the corresponding results in [6; 3].

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