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Generalized Jacobi Reproducing Kernel Method in Hilbert Spaces for Solving the Black-Scholes Option Pricing Problem Arising in Financial Modelling

Mohammadreza Foroutan[a], **Ali Ebadian**[b] **and Hadi Rahmani Fazli**[c]

[a] *Department of Mathematics, Payame Noor University*
P.O.Box 19395-3697, Tehran, Iran
[b] *Department of Mathematics, Faculty of Science, Urmia University*
Urmia, Iran
[c] *Faculty of Economics, Allameh Tabataba'i University*
Tehran, Iran

E-mail(*corresp.*): foroutan_mohammadreza@yahoo.com
E-mail: ebadian.ali@gmail.com
E-mail: hady.rahmani@gmail.com

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Abstract. Based on the reproducing kernel Hilbert space method, a new approach is proposed to approximate the solution of the Black-Scholes equation with Dirichlet boundary conditions and introduce the reproducing kernel properties in which the initial conditions of the problem are satisfied. Based on reproducing kernel theory, reproducing kernel functions with a polynomial form will be constructed in the reproducing kernel spaces spanned by the generalized Jacobi basis polynomials. Some new error estimates for application of the method are established. The convergence analysis is established theoretically. The proposed method is successfully used for solving an option pricing problem arising in financial modelling. The ideas and techniques presented in this paper will be useful for solving many other problems.

Keywords: generalized Jacobi polynomials, reproducing kernel Hilbert space method, Black-Scholes equation, Dirichlet boundary conditions, error estimates.

AMS Subject Classification: 35A24; 33C45; 47B32.

1 Introduction

Boundary value problems of ordinary differential equations play an important role in modelling a wide variety of physical and natural phenomena. They have wide applications due to the fact that many practical problems in mechanics, astronomy, economical theory, chemical physics, and electrostatics may be converted directly to such problems or to ones that are closely related to boundary value problems. There are many approaches numerically available to solving ordinary boundary value problems [14,21,27]. The main idea of this paper is to present a new reproducing kernel Hilbert space method for computing solutions of nonlinear second-order Dirichlet boundary problem of the form:

$$\begin{cases} \eta^3(V^{''})^2(\eta) + p\eta^2 V^{''}(\eta) + q(\eta V^{'}(\eta) - V(\eta)) = 0, \\ V(c) = V_c, \quad V(d) = V_d. \end{cases} \tag{1.1}$$

In [19], it is assumed that p and q are positive constants and $V_c < V_d$. These problems are related to financial option pricing since they address the existence of stationary solutions of a class of generalizations of the classical Black-Scholes model, introduced in 1973 [9], with equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + r(S\frac{\partial V}{\partial S} - V) = 0. \tag{1.2}$$

Equation (1.2) is a partial differential equation with variable coefficients. The variables in equation (1.2) have the following meaning: $V(S,t)$ is the value of a call or put option, depending on an underlying asset S and time t, r is the interest rate, and σ represents the volatility function of underlying asset. A solution $V = V(S,t)$ represents the price of an option if the price of the underlying asset is $S > 0$ at the time $t \in [0,T]$. If we include transaction costs in model (1.2) then we can obtain (see [2,12,18]) the following nonlinear version of (1.2):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\tilde{\sigma}^2 S^2 \frac{\partial^2 V}{\partial^2 S} + b\sigma^2 S^3 (\frac{\partial^2 V}{\partial^2 S})^2 + r(S\frac{\partial V}{\partial S} - V) = 0, \tag{1.3}$$

where $\tilde{\sigma}$ is an adjusted volatility. Now, if we consider the stationary version of (1.3), we obtain the above ordinary differential equation (1.1) where $p = \frac{\tilde{\sigma}^2}{2b\sigma^2}$ and $q = \frac{r}{b\sigma^2}$ are constants.

In recent years, the reproducing kernel Hilbert space method has been used for obtaining approximate solutions in a wide class of ordinary differential, partial differential and integral equations. Please refer to [4,6,7,8,22,24,28,29]. Among plethora of studies addressing the reproduction of kernel Hilbert space method for solving various problems and even among a bunch of extensive works related to reproducing kernel Hilbert spaces for solving ordinary equation, we just mention a number of more interesting problems. The method has the many advantages such as being supported by a rigorous theory, requiring a simple process, and easy to implement on computer. It is obvious that constructing a suitable reproducing kernel spaces and effectively calculating the reproducing kernel expression become the key to apply reproducing kernel Hilbert space method.

Recently, based on the reproducing kernel theory, Cui and Geng [10, 15, 16, 17] have made much effort to solve some special boundary value problems. Furthermore, using the reproducing kernel method, some authors have proposed solutions to some two-point boundary value problems [11, 20]. For instance, Foroutan et al. [13] proposed a method based on reproducing kernel Hilbert spaces to obtain approximate solutions of linear and nonlinear four-point boundary value problems. Arqub [5] has investigated a computational iterative method, the reproducing kernel Hilbert space method, in finding approximate solutions for various certain classes of Neumann time-fractional PDEs with parameters derivative in the sense of Riemann-Liouville and Caputo fractional derivatives. In [3], he employed the reproducing kernel algorithm for handling differential algebraic systems of ordinary differential equations and represented the numerical solutions in the form of series through the functions value at the right-hand side of the corresponding differential and algebraic equations. Furthermore, Arqub and Rashaideh applied the reproducing kernel method to obtain approximate solutions of integro-differential algebraic systems of temporal two-point boundary value problems in [8].

In this paper, we apply the reproducing kernel method for solving nonlinear third order differential equations that have been extracted from some Black-Scholes option pricing problem. To this end, we introduce a new technique based on reproducing kernel Hilbert space method with generalized Jacobi functions in polynomial space. We also produce a set of orthonormal basis functions for space solutions by using the kernel function, a boundary operator, a dense sequence of nodal points in the domain of space solution, and Gram-Schmidt orthogonalization process.

This paper is structured as follows: Section 2 shows, that generalized Jacobi functions with the given properties can be applied for the approximation of second order nonlinear differential equations. According to our method, a brief introduction of the reproducing kernel spaces is represented in Section 3. Section 4, discusses the reproducing kernel method for linear operators. It also addresses the application of reproducing kernel Hilbert space method and convergence analysis of reproducing kernel method. We provide the main results and the exact and approximate solutions in this section. Finally, the error estimation is presented in Section 5.

2 Generalized Jacobi polynomials

An important class of orthogonal are the so called Jacobi polynomials, which are denoted by $J_n^{\alpha,\beta}(x)$, $\alpha,\beta > -1$, $n \geq 0$, [26] and can be determined with the aid of the following recurrence formula:

$$\begin{cases} J_0^{\alpha,\beta}(x) = 1, \\ J_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta), \\ J_{n+1}^{\alpha,\beta}(x) = (a_n^{\alpha,\beta}x - b_n^{\alpha,\beta})J_n^{\alpha,\beta}(x) - c_n^{\alpha,\beta}J_{n-1}^{\alpha,\beta}(x), \;\; n \geq 1, \end{cases} \tag{2.1}$$

where

$$a_n^{\alpha,\beta} = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \tag{2.2}$$

$$b_n^{\alpha,\beta} = \frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \tag{2.3}$$

$$c_n^{\alpha,\beta} = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \tag{2.4}$$

The polynomials $J_n^{\alpha,\beta}(x)$ are orthogonal on $[-1,1]$ with respect to the weight function $W^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, in the sense that

$$\int_{-1}^{1} J_n^{\alpha,\beta}(x) J_m^{\alpha,\beta}(x) W^{\alpha,\beta}(x)dx = \gamma_n^{\alpha,\beta}\delta_{m,n},$$

where $\delta_{m,n}$ is the Kronecker function and

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.$$

Now, we define the generalized Jacobi polynomials of degree n with $\alpha,\beta \in \mathbb{Z}$ on interval $[c,d]$ by

$$G_n^{k,l}(x) = \begin{cases} (d-x)^{-k}(x-c)^{-l}J_{n+k+l}^{-k,-l}(x), & \text{if} \quad k,l \le -1,\ n \ge 2, \\ (d-x)^{-k}J_{n+k}^{-k,l}(x), & \text{if} \quad k \le -1, l > -1,\ n \ge 1, \\ (x-c)^{-l}J_{n+l}^{k,-l}(x), & \text{if} \quad k > -1, l \le -1,\ n \ge 1, \\ J_n^{k,l}(x), & \text{if} \quad k > -1, l > -1,\ n \ge 0. \end{cases} \tag{2.5}$$

Let

$$\bar{k} = \begin{cases} -k, & k \le -1 \\ k, & k > -1 \end{cases}, \quad \hat{k} = \begin{cases} -k, & k \le -1 \\ 0, & k > -1 \end{cases},$$

(likewise for $\bar{l}$, $\hat{l}$), and $n_0 := n_0^{k,l} = \hat{k}+\hat{l}$, $n_1 := n_1^{k,l} = n - n_0^{k,l}$. The set of generalized Jacobi functions are orthogonal in the interval $[c,d]$ in terms of weight function $\chi^{k,l}(x) = (d-x)^k(x-c)^l$ i.e.,

$$\int_c^d G_n^{k,l}(x) G_m^{k,l}(x)\chi^{k,l}(x)dx = \xi_n^{k,l}\delta_{m,n}, \quad m,n \ge n_0, \tag{2.6}$$

where

$$\xi_n^{k,l} = \frac{(d-c)^{\bar{k}+\bar{l}+1}\Gamma(n_1+\bar{k}+1)\Gamma(n_1+\bar{l}+1)}{(2n_1+\bar{k}+\bar{l}+1)\Gamma(n_1+1)\Gamma(n_1+\bar{k}+\bar{l}+1)}. \tag{2.7}$$

Lemma 1. *Let $k,l \ge 1$ and $k,l \in \mathbb{Z}$. There exists a set of constants $\{a_j\}$ such that*

$$G_n^{-k,-l}(x) = \sum_{j=n-k-l}^{n} a_j L_j(x), \quad n \ge k+l,$$

where, $L_j(x)$ is the standard Legendre polynomial of degree n. As an important special case, one can verify that

$$G_n^{-1,-1}(x) = \frac{2(n-1)}{2n-1}(L_{n-2}(x) - L_n(x)). \tag{2.8}$$

Proof. For the proof of Lemma 1 (see [12], Lemma 1.4.3). □

3 Reproducing kernel function

In this section, by using the generalized Jacobi basis function, we will introduce a reproducing kernel Hilbert space method for solving the desired boundary value problem with the recognition that this basis function is useful in constructing reproducing kernel method, because of honestly in the boundary conditions

$$V(c) = V(d) = 0.$$

Since $L_n(1) = 1$ and $L_n(-1) = (-1)^n$, from the equation (2.8), we have

$$G_n^{-1,-1}(c) = G_n^{-1,-1}(d) = 0.$$

According to the equations (2.1) and (2.5), we get

$$G_{n+1}^{-1,-1}(x) = (a_n^{-1,-1}x - b_n^{-1,-1})G_n^{-1,-1}(x) - c_n^{-1,-1}G_{n-1}^{-1,-1}(x), \quad n \geq 3,$$

where, a_n, b_n, c_n defined in (2.2), (2.3) and (2.4) with $\alpha = -1$, $\beta = -1$, respectively. Now by using equations (2.6) and (2.7), we define

$$u_n(x) = \sqrt{\frac{(n+2)(2n+3)}{(d-c)^3(n+1)}}G_{n+2}^{-1,-1}(x), \quad n = 0, 1, 2, \ldots .$$

Definition 1. ([13]) For a nonempty set E, let H be a Hilbert space of real value functions on some set E. A function $K : E \times E \longrightarrow \mathbb{R}$ is said to be the reproducing kernel function of H if and only if

(i) For every $y \in E$, $K(\cdot, y) \in H$.

(ii) For every $y \in E$ and $f \in H$, $\langle f(\cdot), K(\cdot, y)\rangle = f(y)$.

Also, a Hilbert space of function H that possesses a reproducing kernel K is a reproducing kernel Hilbert space; we represent the reproducing kernel Hilbert space and it's kernel by $H_K(E)$ and K_y respectively.

Theorem 1. *([25] Theorem 1.24) For the orthonormal system $\{u_n\}_{n=1}^{\infty}$ and for*

$$K_n(x, y) = \sum_{j=0}^{n} u_j(x)u_j(y), \quad x, y \in I = [c, d]. \tag{3.1}$$

We have

$$K_n(x, y) = \frac{k_n(u_{n+1}(x)u_n(y) - u_n(x)u_{n+1}(y))}{k_{n+1}(x - y)}. \tag{3.2}$$

Here, $k_n > 0$ is the coefficient of x^n in $u_n(x)$. We also have

$$K_n(x,x) = \frac{k_n}{k_{n+1}}(u'_{n+1}(x)u_n(x) - u'_n(x)u_{n+1}(x)).$$

Define $K_n : I \times I \to R$ by (3.2). Then, we have an expression of the reproducing kernel Hilbert space $H_{K_n}(I)$:

$$H_{K_n}(I) = H_{K_n}([c,d]) = \{f : \ f \in L^2(I), \ f(c) = f(d) = 0\}.$$

Equation (3.1) shows that the polynomial reproducing kernel function $K_n(x,y)$ and the associated reproducing kernel Hilbert space $H_{K_n}(I)$ can be updated by increasing n.

4 The generalized Jacobi reproducing kernel method

In this section, the formulation of approximate solution of equation (1.1) together with the implementation method is given in the reproducing kernel space $H_{K_n}(I)$. In proving the convergence of the solution of the problem (1.1), we discuss a convergence analysis with the method suggested by [23]. We recall the theorem established by Grossinho and Morais in [19], which state an existence and uniqueness result for problem (1.1).

Theorem 2. *([19] Theorem 2) Consider the nonlinear Dirichlet boundary value problem* (1.1). *The following assertions hold:*

1. The function $V(x) = \frac{V_c}{c}x$ is a solution of the problem (1.1) *if and only if $V_d/d = V_c/c$.*

2. If $V_d/d < V_c/c$, then the problem (1.1) *has a convex solution V such that*

$$\frac{V_d}{d}x \le V(x) \le \frac{V_d - V_c}{d-c}x + \frac{dV_c - cV_d}{d-c}.$$

3. Moreover, V is the unique convex solution of (1.1) *in any of the above cases.*

Notice that uniqueness of solutions cannot be guaranteed when we include variable coefficients p and q and functional boundary conditions.
If we assume that

$$W(\eta) = V(\eta) - \frac{\eta - c}{d-c}V_d - \frac{d-\eta}{d-c}V_c.$$

Then, the problem (1.1) changes into the following problem:

$$\begin{cases} p\eta^2 W^{''}(\eta) + q\eta W^{'}(\eta) - qV(\eta) = G(\eta, W''(\eta)), \\ W(c) = 0, \quad W(d) = 0, \end{cases}$$

where $G(\eta, W''(\eta)) = \eta^3(W^{''})^2(\eta) + \big(q(cV_d - dV_c)\big)/(d-c)$. By defining the linear operator $L : H_{K_n}(I) \to L^2(I)$ as

$$LW(\eta) = p\eta^2 W^{''}(\eta) + q\eta W^{'}(\eta) - qV(\eta).$$

Equation (1.1) changes to

$$\begin{cases} LW(\eta) = G(\eta, W''(\eta)), \quad \eta \in [c,d], \\ W(c) = 0, \quad W(d) = 0. \end{cases} \tag{4.1}$$

Since $W(\eta)$ is sufficiently smooth. It is easy to show that $L : H_{K_n}(I) \to L^2(I)$ is a bounded linear operator [13]. We choose a countable dense subset $\{\eta_i\}_{i=1}^{\infty}$ in the domain $[c,d]$, and for any fixed $\eta_i \in [c,d]$ we define

$$\psi_i(\eta) = \psi_{\eta_i}(\eta) = L^* K_n(\eta, \eta_i),$$

where L^* is the adjoint operator of L.

Theorem 3. *Let $\{\eta_i\}_{i=1}^{\infty}$ is dense on $[c,d]$, then*

$$ImL^* = H_{K_n}([c,d]), \quad (kerL^*)^{\perp} = ImL = L^2([c,d]).$$

Proof. Clearly $\psi_i(\eta) \in H_{K_n}([c,d])$. For any $W \in (ImL^*)^{\perp}$, since $\psi_i(\eta) = L^* K_n(\eta, \eta_i)$, we have

$$\langle W(\eta), \psi_i(\eta) \rangle_{H_{K_n}([c,d])} = 0,$$

which means that

$$\begin{aligned} 0 &= \langle W(\eta), \psi_i(\eta) \rangle_{H_{K_n}([c,d])} = \langle W(\eta), L^* K_n(\eta, \eta_i) \rangle_{H_{K_n}([c,d])} \\ &= \langle LW(\eta), K_n(\eta, \eta_i) \rangle_{L^2([c,d])} = (LW)(\eta_i). \end{aligned}$$

Note that $\{\eta_i\}_{i=1}^{\infty}$ is dense on $[c,d]$. Hence $(LW)(\eta)$=0. So from the existence L^{-1}, we have $W(\eta) = 0$. That is $(ImL^*)^{\perp} = 0$. Therefore $ImL^* = H_{K_n}([c,d])$. Similarly, we can prove $(kerL^*)^{\perp} = ImL = L^2([c,d])$. □

By Gram-Schmidt process, we obtain an orthogonal basis $\{\overline{\psi}_i(\eta)\}_{i=1}^{\infty}$ of $H_{K_n}([c,d])$, such that

$$\overline{\psi}_i(\eta) = \sum_{k=1}^{i} \alpha_{ik} \psi_k(\eta),$$

where α_{ik} represents orthogonal coefficients ($\alpha_{ii} > 0, \ i = 1, 2, ..., n$), which are given by the following relations [1]:

$$\alpha_{11} = \frac{1}{\|\psi_1\|}, \quad \alpha_{ii} = \frac{1}{a_{ik}}, \ i \neq 1, \quad \alpha_{ij} = -\frac{1}{a_{ik}} \sum_{k=j}^{i-1} c_{ik} \alpha_{jk}, \ i > j,$$

such that $a_{ik} = \sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}$, $\quad c_{ik} = \langle \psi_i, \overline{\psi}_k \rangle_{H_{K_n}([c,d])}$ and $\{\psi_i(\eta)\}_{i=1}^{\infty}$ are the orthonormal system in $H_{K_n}([c,d])$.

Theorem 4. *If $\{\eta_i\}_{i=1}^{\infty}$ is dense on $[c,d]$ and W is the exact solution of equation* (4.1)*, then*

$$W(\eta) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \alpha_{ik} G(\eta_k, W''(\eta_k)) \overline{\psi}_i(\eta). \tag{4.2}$$

Proof. Let $W(\eta)$ be solution (4.1) in the space $H_{K_n}([c,d])$. Since $W(\eta) \in H_{K_n}([c,d])$ and $\{\eta_i\}_{i=1}^{\infty}$ is dense on $[c,d]$, then the series $\sum_{i=1}^{\infty}\langle W, \overline{\psi}_i\rangle\overline{\psi}_i$ is convergent in the sense of $\|.\|_{H_{K_n}([c,d])}$. On the other hand,

$$W(\eta) = \sum_{i=1}^{\infty}\langle W(\eta), \overline{\psi}_i(\eta)\rangle_{H_{K_n}}\overline{\psi}_i(\eta)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}\langle W(\eta), L^*K(\eta,\eta_k)\rangle_{H_{K_n}}\overline{\psi}_i(\eta)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}\langle LW(\eta), K(\eta,\eta_k)\rangle_{H_{K_n}}\overline{\psi}_i(\eta)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}\langle G(\eta, W''(\eta)), K(\eta,\eta_k)\rangle_{H_{K_n}}\overline{\psi}_i(\eta)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k, W''(\eta_k))\overline{\psi}_i(\eta).$$

This completes the proof. □

Let $P_n : H_{K_n}([c,d]) \to Span\{\overline{\psi}_1, \overline{\psi}_2, ..., \overline{\psi}_n\}$ be an orthogonal projection operator. Put

$$W_n(\eta) = \sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k, (P_{k-1}W)''(\eta_k))\overline{\psi}_i(\eta). \tag{4.3}$$

Here, $P_0W(\eta)$ is any fixed function in $H_{K_n}([c,d])$.

Theorem 5. *Suppose that the problem* (4.1) *has a unique solution. If* $\{\eta_i\}_{i=1}^{\infty}$ *is dense on* $[c,d]$*, then* $W_n(\eta)$ *in* (4.3) *is convergence to the* $W(\eta)$ *and for any fixed* $W_0(\eta) \in H_{K_n}([c,d])$, $W_n(\eta)$ *is also represented by*

$$W_n(\eta) = \sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k, W''(\eta_k))\overline{\psi}_i(\eta).$$

Proof. From the definition of $W_n(\eta)$ in (4.3), for $j \le n$, we have

$$LW_n(\eta_j) = \sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k, (P_{k-1}W)''(\eta_k))L\overline{\psi}_i(\eta)$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k, (P_{k-1}W)''(\eta_k))\langle\overline{\psi}_i(\eta), LK(\eta,\eta_j)\rangle_{H_{K_n}}$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k, (P_{k-1}W)''(\eta_k))\langle\overline{\psi}_i(\eta), \psi_i(\eta))\rangle_{H_{K_n}}.$$

Therefore

$$\begin{aligned}\sum_{t=1}^{j}\alpha_{jt}LW_n(\eta_j) &= \sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k,(P_{k-1}W)''(\eta_k))\langle\overline{\psi}_i(\eta),\sum_{t=1}^{j}\alpha_{jt}\psi_t\rangle_{H_{K_n}}\\ &= \sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k,(P_{k-1}W)''(\eta_k))\langle\overline{\psi}_i(\eta),\overline{\psi}_j(\eta)\rangle_{H_{K_n}}\\ &= \sum_{k=1}^{j}\alpha_{jk}G(\eta_k,(P_{k-1}W)''(\eta_k)).\end{aligned}$$

If $j=1$, we get

$$(LW_n)(\eta_1) = G(\eta_1,(P_0W)''(\eta_1)).$$

For $j=2$, we have

$$\begin{aligned}\alpha_{21}(Lw_n)(\eta_1)+\alpha_{22}(LW_n)(\eta_2) &= \alpha_{21}G(\eta_1,(P_0W)''(\eta_1))\\ &\quad+\alpha_{22}G(\eta_2,(P_1W)''(\eta_2)).\end{aligned}$$

It follows that

$$(LW_n)(\eta_2) = G(\eta_2,(P_1W)''(\eta_2)).$$

By induction, we conclude that

$$(LW_n)(\eta_j) = G(\eta_j,(P_{j-1}W)''(\eta_j)).$$

Now, we can prove $(LP_nW)(\eta_j) = LW(\eta_j)$ holds for $j\le n$. It follows that

$$\begin{aligned}(LP_nW)(\eta_j) &= \langle LP_nW(\eta),K_n(\eta,\eta_j)\rangle_{L^2} = \langle P_nW(\eta),L^*K_n(\eta,\eta_j)\rangle_{H_{K_n}}\\ &= \langle P_nW(\eta),\psi_j(\eta)\rangle_{H_{K_n}} = \langle W(\eta),P_n\psi_j(\eta)\rangle_{H_{K_n}}\\ &= \langle W(\eta),\psi_j(\eta)\rangle_{H_{K_n}} = \langle LW(\eta),K_n(\eta,\eta_j)\rangle_{H_{K_n}} = LW(\eta_j).\end{aligned}$$

This together with the boundedness of L^{-1} implies that

$$P_nW(\eta_j) = L^{-1}(LP_nW(\eta_j)) = L^{-1}(LW(\eta_j)) = W(\eta_j). \tag{4.4}$$

On the other hand,

$$\begin{aligned}&|(P_nW)''(\eta)-W''(\eta)| = |\partial_\eta^2\langle P_nW(\zeta)-W(\zeta),K_n(\zeta,\eta)\rangle_{H_{K_n}}|\\ &\quad= |\langle P_nW(\zeta)-W(\zeta),\partial_\eta^2K_n(\zeta,\eta)\rangle_{H_{K_n}}|\le\|P_nW-W\|_{H_{K_n}}\|\partial_\eta^2K_n(\zeta,\eta)\|_{H_{K_n}}.\end{aligned}$$

By the boundedness of $\|\partial_\eta^2K_n(\zeta,\eta)\|_{H_{K_n}}$ and equation (4.4), we get

$$(P_nW)''(\eta_j) = W''(\eta_j),\quad j\le n.$$

Since $\{\eta_j\}_{j=1}^{\infty}$ is dense on $[c,d]$, for any $\eta\in[c,d]$, there exists a subsequence $\{\eta_{n_j}\}_{j=1}^{\infty}$ such that $\eta_{n_j}\to\eta$ as $j\to+\infty$. Therefore, from the continuity of G, we have

$$\begin{aligned}\lim_{j\to+\infty}(LW_n)(\eta_{n_j}) &= \lim_{j\to+\infty}G(\eta_{n_j},(P_{n_j-1}W)''(\eta_{n_j}))\\ &= G(\eta,(W)''(\eta)) = LW(\eta). \end{aligned}\tag{4.5}$$

Since

$$\lim_{j\to+\infty}(LW_n)(\eta_{n_j}) = \lim_{j\to+\infty}\sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k,(P_{k-1}W)''(\eta_k))L\overline{\psi}_i(\eta_{n_j})$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k,(P_{k-1}W)''(\eta_k))L\overline{\psi}_i(\eta)$$

$$= \lim_{n\to+\infty} L\sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k,(P_{k-1}W)''(\eta_k))L\overline{\psi}_i(\eta) = \lim_{n\to+\infty} LW_n(\eta). \quad (4.6)$$

So, from equations (4.5) and (4.6), we conclude that

$$\lim_{n\to+\infty} LW_n(\eta) = LW(\eta).$$

Therefore,

$$\lim_{n\to+\infty} W_n(\eta) = LW(\eta),$$

and consequently, by Theorem 4 and equation (4.3), we get

$$W_n(\eta) = \sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k,(W)''(\eta_k))\overline{\psi}_i(\eta), \quad (4.7)$$

where $W_0(\eta) = P_0W(\eta) \in H_{K_n}([c,d])$. □

5 Error estimation

The estimation of the truncation errors is stated as follows.

Theorem 6. *Let $W_n(\eta)$ be the approximate solution of* (4.1) *in space $H_{K_n}([c,d])$ and $W(\eta)$ be the exact solution of* (4.1)*, then*

$$|W(\eta) - W_n(\eta)|^2 \le \|W\|^2\Big(K_n(\eta,\eta) - \sum_{i=1}^{n}|\overline{\psi}_i(\eta)|^2\Big).$$

Proof. From equations (4.2) and (4.7), we have

$$W_n(\eta) = \sum_{i=1}^{n}\langle W,\overline{\psi}_i\rangle\overline{\psi}_i = \langle\sum_{i=1}^{n}\overline{\psi}_i(\eta)\overline{\psi}_i, W\rangle.$$

It follows that,

$$|W(\eta) - W_n(\eta)|^2 = |\langle K_n(.,\eta) - \sum_{i=1}^{n}\overline{\psi}_i(\eta)\overline{\psi}_i, W\rangle|^2$$

$$\le \|W\|^2\|K_n(.,\eta) - \sum_{i=1}^{n}\overline{\psi}_i(\eta)\overline{\psi}_i\|^2. \quad (5.1)$$

Since $\sum_{i=1}^{n}\overline{\psi}_i(\eta)\overline{\psi}_i=\sum_{i=1}^{n}\langle K_n(.,\eta),\overline{\psi}_i\rangle\overline{\psi}_i$, then, we get

$$K_n(.,\eta)-\sum_{i=1}^{n}\overline{\psi}_i(\eta)\overline{\psi}_i\perp\sum_{i=1}^{n}\overline{\psi}_i(\eta)\overline{\psi}_i.$$

Therefore,

$$\begin{aligned}\|K_n(.,\eta)-\sum_{i=1}^{n}\overline{\psi}_i(\eta)\overline{\psi}_i\|^2&=\langle K_n(.,\eta),K_n(.,\eta)\rangle-\sum_{i=1}^{n}\langle K_n(.,\eta),\overline{\psi}_i\rangle\overline{\psi}_i(\eta)\\&=K_n(\eta,\eta)-\sum_{i=1}^{n}|\overline{\psi}_i(\eta)|^2. \qquad (5.2)\end{aligned}$$

By combining the equations (5.1) and (5.2), the proof of the Theorem is complete. □

Theorem 7. *Let $W_n(\eta)$ be the approximate solution of* (4.1) *in space $H_{K_n}([c,d])$ and $W(\eta)$ be the exact solution of* (4.1)*, then*

$$\|W-W_n\|^2_{H_{K_n}([c,d])}\to 0\quad as\quad n\to+\infty.$$

Moreover, the sequence $\|W-W_n\|^2_{H_{K_n}([c,d])}$ is monotonically decreasing in n.

Proof. From equations (4.2) and (4.7), we have

$$\|W-W_n\|_{H_{K_n}([c,d])}=\|\sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k,W''(\eta_k))\overline{\psi}_i\|_{H_{K_n}([c,d])}.$$

Thus

$$\|W-W_n\|^2_{H_{K_n}([c,d])}\to 0\quad as\quad n\to+\infty.$$

Furthermore,

$$\begin{aligned}\|W-W_n\|^2_{H_{K_n}([c,d])}&=\|\sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}G(\eta_k,W''(\eta_k))\overline{\psi}_i\|^2_{H_{K_n}([c,d])}\\&=\sum_{i=n+1}^{\infty}(\sum_{k=1}^{i}\alpha_{ik}G(\eta_k,W''(\eta_k))\overline{\psi}_i)^2.\end{aligned}$$

Clearly, $\|W-W_n\|^2_{H_{K_n}([c,d])}$ is monotonically decreasing in n. □

Theorem 8. *Let $W_n(\eta)$ be the approximate solution of* (4.1) *in space $H_{K_n}([c,d])$ and $W(\eta)$ be the exact solution of* (4.1)*. If $c=\eta_1<\eta_2<\ldots<\eta_n=d$, and if $G(\eta,W''(\eta))\in C^6[c,d]$, then*

$$\|W-W_n\|_{L^2}\le\alpha h^6,$$

where α is a constant, $h=\max_{1\le i\le n-1}|\eta_{i+1}-\eta_i|$.

Proof. From equations (4.2) and (4.7), we have

$$LW_n(\eta_j) = G(\eta_j, W''(\eta_j)), \quad j = 1, 2, ..., n.$$

Put $R_n(\eta) = G(\eta_j, W''(\eta_j)) - LW_n(\eta_j)$. Obviously,

$$R_n(\eta_j) = 0, \quad j = 1, 2, ..., n.$$

On interval $[\eta_i, \eta_{i+1}]$, the application of Roll's theorem to $R_n(\eta)$ yields

$$R'_n(\alpha_i) = 0, \quad \alpha_i \in (\eta_i, \eta_{i+1}), \quad j = 1, 2, ..., n-1.$$

On interval $[\alpha_i, \alpha_{i+1}]$, the application of Roll's theorem to $R'_n(\eta)$ yields

$$R''_n(\beta_i) = 0, \quad \beta_i \in (\alpha_i, \alpha_{i+1}), \quad j = 1, 2, ..., n-2.$$

On interval $[\beta_i, \beta_{i+1}]$, the application of Roll's theorem to $R''_n(\eta)$ yields

$$R^{(3)}_n(\gamma_i) = 0, \quad \gamma_i \in (\beta_i, \beta_{i+1}), \quad j = 1, 2, ..., n-3.$$

On interval $[\gamma_i, \gamma_{i+1}]$, the application of Roll's theorem to $R'''_n(\eta)$ yields

$$R^{(4)}_n(\delta_i) = 0, \quad \delta_i \in (\gamma_i, \gamma_{i+1}), \quad j = 1, 2, ..., n-4.$$

Now set,

$$\begin{aligned} h &= \max_{1\le i\le n-1} |\eta_{i+1} - \eta_i|, \quad h_\alpha = \max_{1\le i\le n-2} |\alpha_{i+1} - \alpha_i|, \\ h_\beta &= \max_{1\le i\le n-3} |\beta_{i+1} - \beta_i|, \quad h_\gamma = \max_{1\le i\le n-4} |\gamma_{i+1} - \gamma_i| \\ h_\delta &= \max_{1\le i\le n-5} |\delta_{i+1} - \delta_i|. \end{aligned}$$

Therefore

$$h_\alpha \le 2h, \quad h_\beta \le 2h_\alpha \le 4h, \quad h_\gamma \le 2h_\beta \le 8h, \quad h_\delta \le 2h_\gamma \le 16h.$$

Suppose that $l(\eta)$ is a polynomial of degree=1 that interpolates the function $R'''_n(\eta)$ at $\delta_1,\ \delta_2$. It is clear that $l(\eta) = 0$, because

$$l(\eta) = R^{(4)}_n(\delta_1) + \frac{R^{(4)}_n(\delta_2) - R^{(4)}_n(\delta_1)}{\delta_2 - \delta_1}(\eta - \delta_1),$$

where, $R^{(4)}_n(\delta_i),\ i - 1, 2, ..., n-4$. So, we have $l(\eta) \equiv 0$. On the other hand, for $\forall \eta \in [\eta_1, \delta_2]$, there exists $\epsilon_1 \in [\eta_1, \delta_2]$ and a constant a_1 such that

$$R^{(4)}_n(\eta) = R^{(4)}_n(\eta) - l(\eta) = \frac{R^{(6)}_n(\epsilon_1)}{2}(\eta - \delta_1)(\eta - \delta_2) \le a_1 h^2.$$

In a similar way, there exists a constant b_i, a_2 such that

$$\begin{aligned} R^{(4)}_n(\eta) &\le b_i h^2, \quad \eta \in [\delta_i, \delta_{i+1}], \quad i = 2, 3, ..., n-5, \\ R^{(4)}_n(\eta) &\le a_2 h^2, \quad \eta \in [\delta_{n-4}, \eta_n]. \end{aligned}$$

So, there exists a constant c_2 such that

$$\|R_n^{(4)}(\eta)\|_\infty \leq c_2 h^2.$$

On interval $[\beta_i, \beta_{i+1}],\ i = 1, 2, ..., n-3$ there exists a constant $\bar{c}_i$ such that

$$\begin{aligned} |R_n^{(3)}(\eta)| &\leq \int_{\gamma_i}^{\eta} |R_n^{(4)}(t)|dt \leq \int_{\gamma_i}^{\eta} \max|R_n^{(4)}(t)|dt \\ &= \int_{\gamma_i}^{\eta} \|R_n^{(4)}(\eta)\|_\infty dt = \|R_n^{(4)}(\eta)\| |\eta - \gamma_i|. \end{aligned}$$

It turns out that

$$\|R_n^{(3)}(\eta)\|_\infty = \max\|R_n^{(3)}(\eta)\| \leq \max \bar{c}_i h^2 |\eta - \gamma_i| \leq c_3 h^3, \quad \eta \in [c, d],$$

where c_3 is a constant. In the same way, on interval $[\alpha_i, \alpha_{i+1}],\ i = 1, 2, ..., n-2$ there exists a constant $\bar{d}_i$ such that

$$\|R_n^{(2)}(\eta)\|_\infty = \max\|R_n^{(2)}(\eta)\| \leq \max \bar{d}_i h^3 |\eta - \beta_i| \leq c_4 h^4, \quad \eta \in [c, d],$$

where c_4 is a constant. On interval $[\eta_i, \eta_{i+1}],\ i = 1, 2, ..., n-1$ there exists a constant $\bar{e}_i$ such that

$$\|R_n'(\eta)\|_\infty = \max\|R_n'(\eta)\| \leq \max \bar{e}_i h^3 |\eta - \alpha_i| \leq c_5 h^5, \quad \eta \in [c, d],$$

Obviously,

$$|R_n(\eta)| \leq \max\|R_n(\eta)\| = \|R_n(\eta)\|_\infty \leq c_6 h^6. \tag{5.3}$$

According to equation (5.3), we have

$$\|R_n(\eta)\|_{L^2} = \Big(\int_c^d |R_n(\eta)|^2\Big)^{\frac{1}{2}} \leq ch^6.$$

Note that $W(\eta) - W_n(\eta) = L^{-1}R_n(\eta)$. So, there exists a constant α such that

$$\|W(\eta) - W_n(\eta)\|_{H_{K_n}([c,d])} = \|L^{-1}R_n(\eta)\|_{H_{K_n}([c,d])} \leq \|L^{-1}\|\|R_n(\eta)\|_{L^2} \leq \alpha h^6.$$

□

6 Numerical examples

In this section, two numerical examples are provided to show the accuracy of the present method. All computations are performed by Maple 16. Results obtained by the method are compared with the exact solution for each example and they are found to be in good agreement with each other.

Example 1. Consider problem (1.1) in the interval $[c, d] = [1, 5]$, with $p = q = 1$, and boundary conditions $V(1) = 2$, $V(5) = 10$. Using the present method, the numerical results are given in as Figure 1 and Table 1.

Example 2. Consider problem (1.1) in the interval $[c, d] = [1, 5]$, with $p = 7$, $q = 2$, and boundary conditions $V(1) = 1$, $V(5) = 3$. Using the present method, the numerical results are given in as Figure 2 and Table 1.

The tables and figures show the effectiveness of Theorem 2. Therefore, it can be said that the results obtained in the present study are accurate.

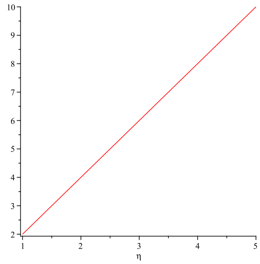

Figure 1. A plot of the price $V(\eta)$ with boundary conditions $V(1) = 2$ and $V(5) = 10$.

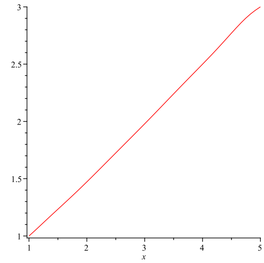

Figure 2. A plot of the price $V(\eta)$ with boundary conditions $V(1) = 1$ and $V(5) = 3$.

Table 1. Values of $V(\eta)$ for different values of V_c, V_d with $c = 1$, $d = 5$.

	$V(1) = 2$, $V(5) = 10$	$V(1) = 1$, $V(5) = 3$
η	$V(\eta)$	$V(\eta)$
1.00	1.99999999999	0.99999999999
1.30	2.60000000319	1.14007775816
1.70	3.40000000327	1.32713494311
2.50	5.00000000445	1.72471006038
3.75	7.50000000601	2.37059796047
4.40	8.80000000750	2.71468357183
5.00	9.99999999999	2.99999999999

7 Conclusions

In this study, a new method for finding a solution in the reproducing kernel Hilbert space is proposed. Each function satisfies boundary conditions of considered problem. In the reproducing kernel space, the construction of the orthogonal basis is described for the first time. Using this method, we constructed the sequence and proved its uniform convergence to the exact solution. We also obtained the truncation error estimate of the series solution. These error estimates can be extended to more general linear and nonlinear boundary value problems.

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