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REGULARIZATION BY NONLOCAL CONDITIONS OF THE INCORRECT PROBLEMS FOR DIFFERENTIAL-OPERATOR EQUATIONS OF THE FIRST ORDER

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1. SECTION 1

We consider the differential-operator equation

$$\frac{du(t)}{dt} + Au(t) = 0 \quad (1)$$

in the interval $(0, T)$, where u is the function of a variable $t \in [0, T]$ with values in the Hilbertian space H , and A is a non-bounded self-adjoint operator in H .

It is assumed that the operator A does not have a fixed sign and has a bounded reciprocal operator A^{-1} within H . Without any loss in generality, for simplicity we assume that the operator A has a point spectrum. An example of this type of operator is the operator in $L_2(\Omega)$ generated by the operator Δ^2 (Δ is the Laplacian given in a bounded domain $\Omega \subset \mathbb{R}^2$) and boundary conditions at the boundary $\partial\Omega$, giving together with the operator Δ^2 a self-adjoint noncoercive problem (see [1], p.397). The Cauchy problem for the eq. (1) with conditions of the form

$$u(0) = \chi \quad \text{or} \quad u(T) = \chi \quad (2)$$

is not correct in sense of Hadamar - Petrovski.

In the present work we have shown that the conditions (2) may be understood as follows. We introduce nonlocal boundary conditions to eq. (1) in the

following way :

$$\alpha u(0) + (1 - \alpha)u(T) = \chi, \quad \chi \in H, \quad 0 < \alpha < 1. \quad (3)$$

Then, the problem (1), (3) has a strongly generalized solution $U(t, \alpha)$ having the following properties :

$$\lim_{\alpha \rightarrow 0} \|u(T, \alpha) - \chi\| = 0; \quad \lim_{\alpha \rightarrow 1} \|u(0, \alpha) - \chi\| = 0, \quad (4)$$

where $\|\cdot\|$ is norm in space H .

2. SECTION 2

Now we demonstrate that the problem (1), (3) has a strongly generalized solution.

Let $\{\lambda_n\}_{n \geq 1}$ be positive eigenvalues of the operator A , and $\{\mu_n\}_{n \geq 1}$ be negative eigenvalues of that operator. Let us denote the eigenvectors of the operator A corresponding to the eigenvalues $\{\lambda_n\}_{n \geq 1}$ by $\{v_n\}_{n \geq 1}$ and those corresponding to the eigenvalues $\{\mu_n\}_{n \geq 1}$ - $\{w_n\}_{n \geq 1}$.

Eigenvectors v_n and w_n of the operator A are forming in H a complete orthogonal system. Without loss in generality, we assume that $\|v_n\| = 1$ and $\|w_n\| = 1$.

Let the vector $\chi \in H$ be represented as

$$\chi = \sum_{n=1}^{\infty} c_n v_n + \sum_{n=1}^{\infty} d_n w_n, \quad (5)$$

where

$$c_n = (v_n, \chi), \quad d_n = (w_n, \chi), \quad (6)$$

and (\cdot, \cdot) is a scalar product in H .

The solution $u(t, \alpha)$ of eq. (1) is sought in the form

$$u(t, \alpha) = \sum_{n=1}^{\infty} \varphi_n e^{-\lambda_n t} v_n + \sum_{n=1}^{\infty} \psi_n e^{\mu_n (T-t)} w_n. \quad (7)$$

substituting this expression into the conditions of (3) and using the representation of (5), we obtain the following equality :

$$\sum_{n=1}^{\infty} [\alpha + (1 - \alpha)e^{-\lambda_n T}] \varphi_n v_n + \sum_{n=1}^{\infty} [\alpha e^{\mu_n T} + (1 - \alpha)] \psi_n w_n =$$

$$\sum_{n=1}^{\infty} c_n v_n + \sum_{n=1}^{\infty} d_n w_n. \quad (8)$$

From this it is inferred that

$$\varphi_n = [\alpha + (1 - \alpha)e^{-\lambda_n T}]^{-1} c_n, \quad (9)$$

$$\psi_n = [\alpha e^{\mu_n T} + (1 - \alpha)]^{-1} d_n. \quad (10)$$

Substituting the values φ_n and ψ_n into (7), we have

$$\begin{aligned} u(t, \alpha) &= \sum_{n=1}^{\infty} e^{-\lambda_n t} [\alpha + (1 - \alpha)e^{-\lambda_n T}]^{-1} (v_n, \chi) v_n \\ &\quad + \sum_{n=1}^{\infty} e^{\mu_n (T-t)} [\alpha e^{\mu_n T} + (1 - \alpha)]^{-1} (w_n, \chi) w_n. \end{aligned} \quad (11)$$

Since

$$\left| e^{-\lambda_n t} [\alpha + (1 - \alpha)e^{-\lambda_n T}]^{-1} \right| \leq \frac{1}{\alpha} \quad (12)$$

$$\left| e^{\mu_n (T-t)} [\alpha e^{\mu_n T} + (1 - \alpha)]^{-1} \right| \leq \frac{1}{1 - \alpha}, \quad (13)$$

then

$$\begin{aligned} &\sup_{0 \leq t \leq T} \left\| \sum_{n=m}^{m+p} e^{-\lambda_n t} [\alpha + (1 - \alpha)e^{-\lambda_n T}]^{-1} (v_n, \chi) v_n \right\|^2 \\ &\quad + \sup_{0 \leq t \leq T} \left\| \sum_{n=m}^{m+p} e^{\mu_n (T-t)} [\alpha e^{\mu_n T} + (1 - \alpha)]^{-1} (w_n, \chi) w_n \right\|^2 \\ &\leq \frac{1}{\alpha^2} \sum_{n=m}^{m+p} (v_n, \chi)^2 + \frac{1}{(1 - \alpha)^2} \sum_{n=m}^{m+p} (w_n, \chi)^2 \end{aligned} \quad (14)$$

and consequently the series (11) converge to the space norm $C([0, T], H)$ for the functions continuous on $[0, T]$ and possessing the values in H .

Thus, for any $\chi \in H$ the problem (1), (3) has a solution strongly generalised in $C([0, T], H)$, that is representable as a series (11), and for this solution the

following estimate is correct:

$$\sup_{0 \leq t \leq T} \|u(t, \alpha)\|^2 \leq \frac{1}{\alpha^2} \sum_{n=1}^{\infty} (v_n, \chi)^2 + \frac{1}{(1-\alpha)^2} \sum_{n=1}^{\infty} (w_n, \chi)^2.$$

3. SECTION 3

In this paragraph it is demonstrated that the solution $u(t, \alpha)$ possesses the properties of (4). Firstly, it is shown that

$$\lim_{\alpha \rightarrow 0} \|\chi - u(T, \alpha)\| = 0, \quad \forall \chi \in H. \tag{15}$$

Let us consider the difference

$$\begin{aligned} \chi - u(T, \alpha) = & \sum_{n=1}^{\infty} \frac{\alpha + (1-\alpha)e^{-\lambda_n T} - e^{-\lambda_n T}}{\alpha + (1-\alpha)e^{-\lambda_n T}} (v_n, \chi) v_n \\ & + \sum_{n=1}^{\infty} \frac{\alpha e^{\mu_n T} + (1-\alpha) - 1}{\alpha e^{\mu_n T} + (1-\alpha)} (w_n, \chi) w_n. \end{aligned} \tag{16}$$

For all $\alpha \in [0, 1]$ the value

$$\frac{\alpha + (1-\alpha)e^{-\lambda_n T} - e^{-\lambda_n T}}{\alpha + (1-\alpha)e^{-\lambda_n T}} = \frac{\alpha(1 - e^{-\lambda_n T})}{\alpha + (1-\alpha)e^{-\lambda_n T}} \tag{17}$$

is nonnegative and nondecreasing with α since its derivative with respect to α is as follows:

$$\frac{(1 - e^{-\lambda_n T})e^{-\lambda_n T}}{(\alpha + (1-\alpha)e^{-\lambda_n T})^2} \geq 0.$$

Therefore, the value (17) takes the greatest value at $\alpha = 1$, i.e.

$$0 \leq \frac{\alpha(1 - e^{-\lambda_n T})}{\alpha + (1-\alpha)e^{-\lambda_n T}} \leq 1 - e^{-\lambda_n T} \leq 1. \tag{18}$$

For all $\alpha \in [0, 1/2]$ the value

$$\frac{\alpha e^{\mu_n T} + (1-\alpha) - 1}{\alpha e^{\mu_n T} + (1-\alpha)} = \frac{\alpha(e^{\mu_n T} - 1)}{\alpha e^{\mu_n T} + (1-\alpha)} \tag{19}$$

is nonpositive and nonincreasing with α inasmuch as its derivative with respect to α is as follows :

$$\frac{(e^{\mu_n T} - 1)}{(\alpha e^{\mu_n T} + (1-\alpha))^2} \leq 0.$$

Hence the value of (19) on the segment $[0, 1/2]$ is the lowest when $\alpha = 1/2$, i.e.

$$-1 \leq \frac{e^{\mu_n T} - 1}{e^{\mu_n T} + 1} \leq \frac{\alpha e^{\mu_n T} + (1 - \alpha) - 1}{\alpha + (1 - \alpha)e^{-\lambda_n T}} \leq 0. \quad (20)$$

Based on the inequalities (18) and (20), from eq. (16) it is inferred that uniformly in $\alpha \in [0, 1/2]$ the following estimate is true :

$$\begin{aligned} \|\chi - u(t, \alpha)\|^2 &= \sum_{n=1}^{\infty} \frac{(\alpha + (1 - \alpha)e^{-\lambda_n T} - e^{-\lambda_n T})^2}{(\alpha + (1 - \alpha)e^{-\lambda_n T})^2} (v_n, \chi)^2 \quad (21) \\ &\quad + \sum_{n=1}^{\infty} \frac{(\alpha e^{\mu_n T} + (1 - \alpha) - 1)^2}{(\alpha e^{\mu_n T} + (1 - \alpha))^2} (w_n, \chi)^2 \\ &\leq \sum_{n=1}^{\infty} (v_n, \chi)^2 + \sum_{n=1}^{\infty} (w_n, \chi)^2 = \|\chi\|^2. \end{aligned}$$

Now we demonstrate that

$$\lim_{\alpha \rightarrow 0} \|\chi - u(T, \alpha)\|^2 = 0, \quad \forall \chi \in M. \quad (22)$$

where M is some set dense in H .

Then by virtue of the Banach-Steinhaus theorem, (15) is derived from (21) and (22).

As a set M we take all χ of the form [see also (5)] :

$$\chi = \sum_{n=1}^N c_n v_n + \sum_{n=1}^{\infty} d_n w_n, \quad \forall N < \infty.$$

Thereupon,

$$\begin{aligned} \|\chi - u(t, \alpha)\|^2 &= \sum_{n=1}^N \frac{\alpha^2 (1 - e^{-\lambda_n T})^2}{(\alpha + (1 - \alpha)e^{-\lambda_n T})^2} (v_n, \chi)^2 \\ &\quad + \sum_{n=1}^{\infty} \frac{\alpha^2 (e^{\mu_n T} - 1)^2}{(\alpha e^{\mu_n T} + (1 - \alpha))^2} (w_n, \chi)^2 \\ &\leq \frac{\alpha^2 e^{2\lambda_N T}}{(1 - \alpha)^2} \sum_{n=1}^N (v_n, \chi)^2 + \frac{\alpha^2}{(1 - \alpha)^2} \sum_{n=1}^{\infty} (w_n, \chi)^2 \\ &= \frac{\alpha^2 e^{2\lambda_N T}}{(1 - \alpha)^2} \|\chi\|^2 \end{aligned}$$

and for such χ (22) is valid.

Now we prove that

$$\lim_{\alpha \rightarrow 1} \|u(0, \alpha) - \chi\| = 0, \quad \forall \chi \in H. \quad (23)$$

Let us consider the difference

$$\chi - u(0, \alpha) = \sum_{n=1}^{\infty} \frac{\alpha + (1 - \alpha)e^{-\lambda_n T} - 1}{\alpha + (1 - \alpha)e^{-\lambda_n T}} (v_n, \chi) v_n \quad (24)$$

$$+ \sum_{n=1}^{\infty} \frac{\alpha e^{\mu_n T} + (1 - \alpha) - e^{\mu_n T}}{\alpha e^{\mu_n T} + (1 - \alpha)} (w_n, \chi) w_n \quad (25)$$

For all $\alpha \in [1/2, 1]$ the value

$$\frac{\alpha + (1 - \alpha)e^{-\lambda_n T} - 1}{\alpha + (1 - \alpha)e^{-\lambda_n T}} = \frac{(\alpha - 1)(1 - e^{-\lambda_n T})}{\alpha + (1 - \alpha)e^{-\lambda_n T}} \quad (26)$$

is nonnegative and nondecreasing with α since its derivative with respect to α is as follows:

$$\frac{1 - e^{-\lambda_n T}}{(\alpha + (1 - \alpha)e^{-\lambda_n T})^2} \geq 0$$

Consequently, the value (25) on the segment $[1/2, 1]$ takes on the lowest value when $\alpha = 1/2$, i.e.

$$-1 \leq \frac{e^{-\lambda_n T} - 1}{e^{-\lambda_n T} + 1} \leq \frac{(\alpha - 1)(1 - e^{-\lambda_n T})}{\alpha + (1 - \alpha)e^{-\lambda_n T}} \leq 0 \quad (27)$$

For all $\alpha \in [0, 1]$ the value

$$\frac{\alpha e^{\mu_n T} + (1 - \alpha) - e^{\mu_n T}}{\alpha e^{\mu_n T} + (1 - \alpha)} = \frac{(1 - \alpha)(1 - e^{\mu_n T})}{\alpha e^{\mu_n T} + (1 - \alpha)} \quad (28)$$

is nonnegative and nonincreasing with α since its derivative with respect to α is as follows:

$$\frac{e^{\mu_n T}(e^{\mu_n T} - 1)}{(\alpha e^{\mu_n T} + (1 - \alpha))^2} \leq 0$$

Therefore, the value (27) is the greatest for $\alpha = 0$, i.e.

$$0 \leq \frac{(1 - \alpha)(1 - e^{\mu_n T})}{\alpha e^{\mu_n T} + (1 - \alpha)} \leq 1 - e^{\mu_n T} \leq 1 \quad (29)$$

Based on the inequalities (26) and (28), from (24) it is inferred that uniformly in $\alpha \in [1/2, 1]$ the following estimate is true:

$$\begin{aligned} \|\chi - u(0, \alpha)\|^2 &= \sum_{n=1}^{\infty} \frac{(\alpha + (1 - \alpha)e^{-\lambda_n T} - 1)^2}{(\alpha + (1 - \alpha)e^{-\lambda_n T})^2} (v_n, \chi)^2 \\ &\quad + \sum_{n=1}^{\infty} \frac{(\alpha e^{\mu_n T} + (1 - \alpha) - e^{\mu_n T})^2}{(\alpha e^{\mu_n T} + (1 - \alpha))^2} (w_n, \chi)^2 \\ &\leq \sum_{n=1}^{\infty} (v_n, \chi)^2 + \sum_{n=1}^{\infty} (w_n, \chi)^2 = \|\chi\|^2. \end{aligned} \quad (30)$$

Now we show that

$$\lim_{\alpha \rightarrow 1} \|u(0, \alpha) - \chi\| = 0, \quad \forall \chi \in M. \quad (31)$$

where M is a particular set dense in H . Then according to the Banach-Steinhaus theorem, (23) results from (29) and (30). In this case as a set M we take all χ representable in the form given below:

$$\chi = \sum_{n=1}^{\infty} c_n v_n + \sum_{n=1}^N d_n w_n, \quad \forall N < \infty.$$

Therefore,

$$\begin{aligned} \|\chi - u(0, \alpha)\|^2 &= \sum_{n=1}^{\infty} \frac{(\alpha - 1)^2 (1 - e^{-\lambda_n T})^2}{(\alpha + (1 - \alpha)e^{-\lambda_n T})^2} (v_n, \chi)^2 \\ &\quad + \sum_{n=1}^N \frac{(1 - \alpha)^2 (1 - e^{\mu_n T})^2}{(\alpha e^{\mu_n T} + (1 - \alpha))^2} (w_n, \chi)^2 \\ &\leq \frac{(1 - \alpha)^2}{\alpha^2} \sum_{n=1}^{\infty} (v_n, \chi)^2 + \frac{(1 - \alpha)^2}{\alpha^2} e^{-2\mu_N T} \sum_{n=1}^N (w_n, \chi)^2 \\ &\leq \frac{(1 - \alpha)^2}{\alpha^2} e^{-2\mu_N T} \|\chi\|^2 \end{aligned}$$

and for such χ the expressions (30) are true.

REFERENCES

- [1] Brish N.I., Valeshkevich I.N. The Fourier Method for Nonstationary Equations with General Boundary Conditions. *Differential Equations* 1(3), 1965, P. 393–399.