# COMMUNICATION SCIENCES

# AND

# ENGINEERING

# IX. PROCESSING AND TRANSMISSION OF INFORMATION*


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## A. ON THE NUMBER OF DISCRETE MONOTONE FUNCTIONS OF k INTEGER VARIABLES

This report summarizes research undertaken between February and August, 1972 submitted to the Department of Electrical Engineering, M.I.T., August 1972, in partial fulfillment of the requirements for the degree of Master of Science.[1]

Consider the set $(0, n-1)^k = (x_1, x_2, \ldots, x_k)$, $x_i$ integers and $0 \leqslant x_i \leqslant n-1$, together with the partial order $\leqslant$ given by

$$\underline{x} \leqslant \underline{y} \Longleftrightarrow x_i \leqslant y_i \ \forall\, i \qquad 0 \leqslant i \leqslant n-1.$$

This constitutes a lattice which we shall refer to as the $(0, n-1)^k$ lattice. We say an integer-valued function f is monotone on $(0, n-1)^k$ if

$$\underline{x} \leqslant \underline{y} \Longrightarrow f(\underline{x}) \leqslant f(\underline{y}).$$

The problem we are concerned with is to count the number of monotone functions $f: (0, n-1)^k \to (0, 1, 2, \ldots, N)$, to which we refer as N-restricted $n^k$-partitions (of any integer). For instance when $k = 1$, the problem is to find the number of "staircase" functions reaching N or less in n steps, which is $\binom{N+n}{n}$.

A simple inductive proof of Carlitz[2] shows that the number of functions $f: (0, n-1)^2 \to (0, N)$ is

$$L_2(N, n) = \frac{(N+2n-1)!!\ (N-1)!!\ [(n-1)!!]^2}{[(N+n-1)!!]^2\ (2n-1)!!},$$


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where $k!! = k!(k-1)!, \dots, 2!\ 1!$. In higher dimensions, however, an exact count does not exist and therefore we looked for upper and lower bounds on the desired count.

Let $L_k(N, n)$ be the number of monotone functions $f: (0, n-1)^k \to (0, \dots, N)$, then it can be shown[3] that

$$L_k(N, n) \leq [L_2(N, n)]^{n^{k-2}} \tag{1}$$

and, since we know $L_2$ exactly, we take (1) as the upper bound on $L_k(N, n)$.

In particular, we have shown the following.

Case 1: For $N \leq n$

$$\lg L_k(N, n) \leq (1+\epsilon_n) n^k g \lg 4,$$

where $g = N/n$, and $\epsilon_n \to 0$ as $n \to \infty$.

Case 2: For $N \geq n$

$$\lg L_k(N, n) \leq (1+\epsilon_n) n^k \lg (1.12(g+2)),$$

where $g = N/n$, and again $\epsilon_n \to 0$ as $n \to \infty$.

To obtain a lower bound on $L_k$, we consider the $(0, n-1)^k$ lattice as an ordered collection of sets

$$\left\{S_j = \underline{x} \;\middle|\; \sum_1^k x_i = j\right\}.$$

We then consider only those functions that increase from one $S_j$ to the next. We have shown[1] that all of these functions are indeed monotone, and so their count is a lower bound on $L_k$.

This turns out to be quite a powerful method to obtain lower bounds on $L_k$, largely because there are no monotonicity constraints on each $S_j$. Therefore we get

$$\lg L_k(N, n) \geq h(g)\, n^k \lg (1+g/\sqrt{k}),$$

where $1/2 \leq h(g) \leq \lim_{g\to\infty} h(g) = 1$ for all $n$ large enough. We can then show[1] that these bounds are of the same order; that is, they differ at most by a constant factor which can be shown to be $\leq 2\sqrt{k}$ for all $n$ large enough. Moreover, for $N \geq n^{1+\epsilon}$ ($\epsilon > 0$), these bounds are "tight" in the sense that they differ by a factor close to 1 and approach 1 as $n$ tends to infinity.

A. Bolour

References

1. A. Bolour, "Bounds on the Number of Integer Valued Monotone Functions of k Integer Arguments," S.M. Thesis, Department of Electrical Engineering, M.I.T., August 1972.

2. L. Carlitz, "Rectangular Arrays and Plane Partitions," Acta Arithmetica 13, 29-47 (1967).

3. Ibid., Chap. 2.

## B. DERIVATION OF A TESTING METHOD FOR REPEAT REQUEST (RQ) LOGIC

This report summarizes research submitted to the Department of Electrical Engineering, M.I.T., on August 1, 1972, in partial fulfillment of the requirements for the degree of Master of Science.[1]

In digital data communication it is standard practice to recover errors by use of error detection plus retransmission upon request. A finite-automata approach to the error-recovery procedures is developed by defining the events as regular expressions on a two-symbol alphabet. The communication system (source, destination, and channel) is viewed as a single, global, finite-state acceptor. The channel is modeled as a block-erasure channel; that is, the channel transforms the input blocks denoted by $\{I\}$ into the output set $\{I, X\}$. This model is quite appropriate for digital data channels. The regular expression characterizing the global automaton is indicative of the reliability of the error-recovery procedures in question. The thesis also presents a delay-handling procedure.

The regular expression representation of the systems is used to derive the mean throughput rate in terms of the probabilities of the events I and X, when the events are assumed statistically independent. When the independence assumption is ruled out, a weaker assumption, the weak law of large numbers, permits us to derive bounds on the mean throughput rate. Thus finite automata and regular expressions provide a formal and simple way of testing and analyzing the error-recovery procedures.

F. Nourani

References

1. F. Nourani, "Derivation of a Testing Method for Repeat Request (RQ) Logic," S.M. Thesis, Department of Electrical Engineering M.I.T., August 1, 1972.

## C. SEQUENTIAL DETECTION OF SIGNALS TRANSMITTED BY A QUANTUM SYSTEM (EQUIPROBABLE BINARY PURE STATE)

Suppose we want to transmit a binary signal with a quantum system S that is not corrupted by noise. The system is in state $|s_0\rangle$ when digit zero is sent, and in state $|s_1\rangle$

when digit one is sent. Let the a priori probabilities that the digits zero and one are sent each be equal to one-half. The performance of detection is given by the probability of error. We try to consider the performance of a sequential detection scheme by bringing an apparatus A to interact with the system S and then performing a measurement on S and then on A, or vice versa. The structure of the second measurement is optimized as a consequence of the outcome of the first measurement. Previously[1] we considered the case in which the joint state of S and A can be factored into the tensor product of a state in S and a state in A. In general, the joint state of S and A does not factor, and we now wish to treat this general case.

Let the initial state of A before interaction be $|a_o\rangle$. If digit zero is sent, the joint state of S+A before interaction is $|s_o\rangle|a_o\rangle$. If digit one is sent, the state is $|s_1\rangle|a_o\rangle$.

The interaction between S and A can be characterized by a unitary transformation U on the joint state of S+A.

$$|s_o^f+a_o^f\rangle\rangle = U|s_o\rangle|a_o\rangle$$

$$|s_1^f+a_1^f\rangle\rangle = U|s_1\rangle|a_o\rangle.$$

By symmetry of the equiprobability of digits one and zero, we select a measurement on A characterized by the self-adjoint operator $O_A$ such that the probability that it will decide a zero, given that zero is sent, is equal to the probability that it will decide on one, given one is sent. Let $|\phi_o\rangle$ and $|\phi_1\rangle$ be its eigenstates. Then $\{|\phi_i\rangle\}_{i=1,2}$ spans the Hilbert space, $\mathcal{H}_A$. Let $\{|\psi_j\rangle\}_{j=1,2}$ be an arbitrary orthonormal basis in the Hilbert space, $\mathcal{H}_S$. Then the orthonormal set $\{|\phi_i\rangle|\psi_j\rangle\}_{\substack{i=1,2\\j=1,2}}$ is a complete orthonormal basis for the tensor product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_S$.

Then

$$|s_o^f+a_o^f\rangle\rangle = \sum_{\substack{i=1,2\\j=1,2}} a_{ij}|\phi_i\rangle|\psi_j\rangle$$

$$|s_1^f+a_1^f\rangle\rangle = \sum_{\substack{i=1,2\\j=1,2}} b_{ij}|\phi_i\rangle|\psi_j\rangle,$$

where $a_{ij}$ and $b_{ij}$ are complex numbers. Since unitary transformations preserve inner products,

$$\langle\langle s_1^f + a_1^f | a_o^f + s_o^f \rangle\rangle = \sum_{\substack{i=1,2\\j=1,2}} b_{ij}^* a_{ij}$$

$$= \langle s_1 | s_o \rangle.$$

If we perform the measurement characterized by $O_A$, the probabilities that we shall find A in states $|\phi_o\rangle$ and $|\phi_1\rangle$, given that digit one or digit zero is sent, are

$$\Pr[|\phi_o\rangle | 0] = \sum_{j=1,2} |a_{oj}|^2$$

$$\Pr[|\phi_1\rangle | 0] = \sum_{j=1,2} |a_{1j}|^2$$

$$\Pr[|\phi_o\rangle | 1] = \sum_{j=1,2} |b_{oj}|^2$$

$$\Pr[|\phi_1\rangle | 1] = \sum_{j=1,2} |b_{1j}|^2.$$

But by symmetry we choose $\Pr[|\phi_o\rangle | 0] = \Pr[|\phi_1\rangle | 1]$

$$\Pr[|\phi_1\rangle | 0] = \Pr[|\phi_o\rangle | 1].$$

Given as a result of the measurement that we find system A to be in state $|\phi_o\rangle$, we wish to update the a priori probabilities of digits one and zero. Using Bayes' rule, we obtain

$$\Pr[0 | |\phi_o\rangle] = \frac{\Pr[|\phi_o\rangle | 0] \Pr[0]}{\Pr[|\phi_o\rangle]}$$

$$\Pr[0] = \frac{1}{2}$$

$$\Pr[|\phi_o\rangle] = \Pr[|\phi_o\rangle | 0] \Pr[0] + \Pr[|\phi_o\rangle | 1] \Pr[1]$$

$$= \frac{1}{2}\{\Pr[|\phi_o\rangle | 0] + \Pr[|\phi_1\rangle | 0]\} = \frac{1}{2}$$

$$\therefore\ P[0|\ |\phi_o\rangle] = \Pr[|\phi_o\rangle|0] = \sum_{j=1,2} |a_{oj}|^2$$

$$\Pr[1|\ |\phi_o\rangle] = \sum_{j=1,2} |b_{oj}|^2$$

$$= \sum_{j=1,2} |a_{1j}|^2.$$

Given that the outcome is $|\phi_o\rangle$, the system S is now in well-defined states. If zero is sent,

$$|s_o^f\rangle = \frac{\sum_{j=1,2} a_{oj}|\psi_j\rangle}{\left\{\sum_{j=1,2} |a_{oj}|^2\right\}^{1/2}}.$$

If one is sent,

$$|s_1^f\rangle = \frac{\sum_{j=1,2} b_{oj}|\psi_j\rangle}{\left\{\sum_{j=1,2} |b_{oj}|^2\right\}^{1/2}}$$

After the measurement on A we have a new set of a priori probabilities and a new set of states for system S. We choose a measurement on S characterized by the self-adjoint operator $O_S$ such that the performance is optimum. From previous calculations,[1] the probability of error, given $|\phi_o\rangle$, is the result of the first measurement

$$\Pr[\epsilon|\ |\phi_o\rangle] = \frac{1}{2}\left\{1 - \left[1 - 4\Pr[0|\ |\phi_o\rangle]\Pr[1|\ |\phi_o\rangle]\,|\langle s_1^f|s_o^f\rangle|^2\right]^{1/2}\right\}$$

$$\langle s_1^f|s_o^f\rangle = \frac{\left|\sum_{j=1,2} b_{oj}^* a_{oj}\right|^2}{\left\{\sum_{j=1,2} |a_{oj}|^2\right\}\left\{\sum_{j=1,2} |b_{oj}|^2\right\}}$$

$$\therefore\ \Pr[\epsilon|\ |\phi_o\rangle] = \frac{1}{2}\left\{1 - \left[1 - 4\left|\sum_{j=1,2} b_{oj}^* a_{oj}\right|^2\right]^{1/2}\right\}.$$

By symmetry

$$\Pr[\epsilon|\ |\phi_1\rangle] = \frac{1}{2}\left\{1 - \left[1 - 4\left|\sum_{j=1,2} b_{1j}^* a_{1j}\right|^2\right]^{1/2}\right\}$$

$$\therefore\ \Pr[\epsilon] = \frac{1}{2}\left\{1 - \frac{1}{2}\left[1-4\left|\sum_{j=1,2} b_{oj}^{*}a_{oj}\right|^{2}\right]^{1/2} - \frac{1}{2}\left[1-4\left|\sum_{j=1,2} b_{1j}^{*}a_{1j}\right|^{2}\right]^{1/2}\right\}.$$

Minimizing $\Pr[\epsilon]$, subject to the inner product constraint,

$$\sum_{\substack{i=1,2\\ j=1,2}} b_{ij}^{*}a_{ij} = \langle s_1 | s_o \rangle,$$

yields

$$\Pr[\epsilon]_{opt} = \frac{1}{2}\left[1 - \sqrt{1 - |\langle s_1 | s_o \rangle|^{2}}\right].$$

This is the same result that was derived for the case when the joint state of S+A can be factored into the tensor product of states in S and A.

V. Chan

References

1. V. Chan, "Interaction Formulation of Quantum Communication Theory," Quarterly Progress Report No. 106, Research Laboratory of Electronics, M.I.T., July 15, 1972, pp. 128-132.