XIX. DETECTION AND ESTIMATION THEORY*

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RESEARCH OBJECTIVES AND SUMMARY OF RESEARCH

The work of our group is focused upon the development of efficient methods for the processing of analog and digital signals over channels of interest. We shall divide the discussion of our work into three areas.

1. Sonar and Oceanographic Research

a. Current echo sounders for oceanographic survey ships have a quoted accuracy of 2-3° which implies a topographic resolution in excess of 700-1000 ft at deep-water depths of 20,000 ft. The random characteristics of the ocean channel caused by effects, such as turbulence, internal waves, scattering, and ambient noise, in addition to the random errors in measuring platform orientation, will limit further improvements in accuracy. Work has commenced for determining the following.

(i) The point at which these effects become significant for conventional echo-sounding methods with a large array.

(ii) Potential improvements that could be obtained by signal processing which exploits the statistical characteristics of the signal.

(iii) Data processing methods of multibeam signals that would permit surveillance of large geographical regions with enhanced accuracy.

b. There are many situations in which the propagation effects of the channel have a significant effect upon the signals observed. These effects are particularly interesting when the signal propagates through a layered medium where appreciable distortion and multiple travel paths are often encountered. Current analyses are deterministic and are confined to methods such as ray tracing or mode analysis. While there is a limited number of results available, at present, which involve the characteristics of random processes propagated over these channels, we are investigating several topics in this area of process propagation for the sonar, or underwater, channel where random effects and receiver structure implementation are issues in the analysis.

2. Array and Space/Time Processing

We are concerned with effective space-time processing techniques. Specific topics include the following.

a. Development of suboptimum process procedures that are simpler to implement than the optimum receiver, but do not lead to a significant performance degradation.

b. Investigations into analysis and synthesis procedures in frequency wave-numbers space.

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c. Applications of distributed state variables to space-time processing.

d. A study of the effects of hard-clipping sensor outputs.

e. Analysis of various array-tracking problems.

3. Communication Systems

a. A method of calculating performance bounds that is applicable to a large class of systems. Further applications of this technique are being studied.

b. A detailed analysis of various forms of delta modulation systems is being conducted.

H. L. Van Trees

A. STATISTICAL MICROSTRUCTURE OF RANDOM FIELDS GENERATED BY MULTIPLE-SCATTER MEDIA

1. Introduction

This report presents the results of some present research aimed at obtaining a better understanding of the microstructure of random fields generated by a multiple-scatter medium. The results are based on Maxwell's equations and allow for arbitrary incident radiation fields. The mean or coherent field is obtained, and we show by example that the field decays exponentially as the wave propagates down into the cloud for a plane wave excitation. Such a result agrees with Twersky.[1] Similar results are also obtained for arbitrary incident illumination. We also obtain an integral equation for the correlation matrix of the field. All of these results are for a frozen set of scatterers. The results for a moving set of scatterers will not be presented in this report.

2. Problem Definition

Let an arbitrary nonrandom monochromatic electromagnetic vector field $\underline{E}_o(\underline{r})$ be incident on a cloud at $z = 0$, where $\underline{r}$ is a spatial vector (x, y, z). Let the cloud be of depth d extending from $z = 0$ to $z = d$. The cloud is composed of spherical scatterers with a radius δ and dielectric constant ϵ. There are assumed to be N of these scatterers and they are uniformly and independently distributed over the volume of the cloud. The dielectric constant is $\epsilon = \epsilon_o(1+n')$. The excess index of refraction, n', is such that

$$n'(r) = \begin{cases} n' & r \in B(\delta, \underline{r}_i) \\ 0 & r \notin B(\delta, \underline{r}_i) \end{cases} \tag{1}$$

and $B(\delta, \underline{r}_i)$ is a ball of radius δ centered about the point $\underline{r}_i$ which is the center of the i^{th} scatterer. For such a medium the dielectric constant ϵ is a function of the spatial coordinate r as given by (1). It can be shown that the electric field E(r) satisfies the equation (Tatarski[2])

$$\nabla(\underline{E}(\underline{r}) \cdot \nabla \ln(1+n')) + \nabla^2 \underline{E}(\underline{r}) + \omega^2 \mu_o \epsilon_o (1+n') \underline{E}(\underline{r}) = 0, \tag{2}$$

wnere the spatio-temporal field was assumed to be of the form $\underline{E}(\underline{r}) \exp(-j\omega t)$. If we define k_o^2 as $\omega^2 \mu_o \epsilon_o$, then k_o^2 represents the free-space wave number. Clearly, the first term in this expression is of an impulsive nature so that it represents multipole sources that reradiate in such a fashion that they depend on the field itself.

To avoid the difficulties of discontinuities in (2) we introduce a wave-number transform, $\underline{E}(\underline{k})$, of $\underline{E}(\underline{r})$:

$$\underline{E}(\underline{k}) = \int \underline{E}(\underline{r}) \exp(-j\underline{k} \cdot \underline{r})\, d\underline{r}. \tag{3}$$

The inverse transform is given by Jackson[3]:

$$\underline{E}(\underline{r}) = \frac{1}{(2\pi)^3} \int \underline{E}(\underline{k}) \exp(j\underline{k} \cdot \underline{r})\, d\underline{k}. \tag{4}$$

For those cases of interest, namely clouds, n' = 0.33 so that we may approximate ln (1+n') by n'. This is not necessary, but is done solely to make the analysis simpler. Taking the transform of (2), we obtain

$$\int \underline{K}(\underline{k}, \underline{k}')\, \underline{E}(\underline{k}')\, d\underline{k}' + \left(|\underline{k}|^2 - k_o^2\right) \underline{E}(\underline{k}) = \underline{S}(\underline{k}), \tag{5}$$

where $\underline{K}(\underline{k}, \underline{k}')$ is the 3×3 matrix given by

$$\underline{K}(\underline{k}, \underline{k}') = n' \underline{A}(\underline{k}, \underline{k}')\, I(|\underline{k}-\underline{k}'|) \sum_{i=1}^{N} \exp(j(\underline{k}-\underline{k}') \cdot \underline{r}_i) \tag{6}$$

and

$$A(\underline{k}, \underline{k}') = \begin{bmatrix} k_1(k_1-k_1') - k_o^2 & k_1(k_2-k_2') & k_1(k_3-k_3') \\ k_2(k_1-k_1') & k_2(k_2-k_2') - k_o^2 & k_2(k_3-k_3') \\ k_3(k_1-k_1') & k_3(k_2-k_2') & k_3(k_3-k_3') - k_o^2 \end{bmatrix}, \tag{7}$$

where k_1 represents the wave number associated with x, k_2 with y, and k_3 with z. Also $|\underline{k}|^2 = k_1^2 + k_2^2 + k_3^2$. Furthermore, $I(|\underline{k}-\underline{k}'|)$ is the scattering function of the spherical scatterer and can be shown to be

$$I(|\underline{k}|) = \frac{4\pi\delta}{|\underline{k}|^2}\left[\frac{\sin \delta|\underline{k}|}{\delta|\underline{k}|} - \cos \delta|\underline{k}|\right] \tag{8}$$

Thus the randomness of the field is imparted by the random kernel which depends on the scatterers $\underline{r}_i$. $\underline{S}(\underline{k})$ represents boundary conditions at $z = 0$, obtained by using one-sided transforms. The existence and uniqueness properties of equations like (5) have been discussed by Bharucha-Reid.[4, 5] We shall be interested in obtaining its statistical properties.

3. Field Statistics

We now want to find $\overline{\underline{E}}(\underline{k})$ which is the field averaged over all positions of the scatterers $\{\underline{r}_i\}$. We shall present the results in definition and theorem format.

DEFINITION 1. Let $\{\underline{r}_i\}$ represent the position of the scatterers. Let $p(\underline{r}_1, \dots \underline{r}_N)$ be the joint probability density of these centers. Let the scatterers be identically and uniformly distributed over a volume ∇. Let $\underline{E}(\underline{k})$ be the solution of (5). Then $\overline{\underline{E}}(\underline{k})$, the average field, is given by

$$\overline{\underline{E}}(\underline{k}) = \int \underline{E}(\underline{k})\, p(\underline{r}_1, \dots \underline{r}_N)\, d\underline{r}_1 \dots d\underline{r}_N. \tag{9}$$

Furthermore, let $\overline{\underline{E}}'(\underline{k})$ be the average of $\underline{E}(\underline{k})$ over all but the particle at $\underline{r}'$. This is given by

$$\overline{\underline{E}}'(\underline{k}) = \int \underline{E}(\underline{k})\, p(\underline{r}_1, \dots \underline{r}_N/\underline{r}')\, d\underline{r}_1 \dots d\underline{r}_N. \tag{10}$$

Lemma 1

The average random field generated by the scattering medium is a solution to the following integral equation

$$\rho V n' \int p(\underline{r}')\, d\underline{r}' \int \underline{A}(\underline{k},\underline{k}')\, I(|\underline{k}-\underline{k}'|) \exp(-j(\underline{k}-\underline{k}')\cdot\underline{r}') \tag{11}$$

$$\overline{\underline{E}}'(\underline{k}')\, d\underline{k} + \overline{\underline{E}}(\underline{k})\left(|\underline{k}|^2 - \underline{k}_o^2\right) = \underline{S}(\underline{k}),$$

where ρ is the density of scatterers (N/V), and V is the volume occupied by the scatterers. Here $p(\underline{r}')$ is the probability that there is a scatterer at $\underline{r}'$.

Proof: This is easily obtained by averaging over (5), except in each of the N terms taking the conditional expectation. There are N of these and we assume that they are identically distributed. So we can add them up to obtain Eq. 11. Q.E.D.

Lemma 2

The field $\overline{\underline{E}}'(\underline{k})$ which is the average field with a single particle fixed at $\underline{r}'$ is given by

$$\overline{\underline{E}}'(\underline{k}) = \underline{E}_{inc}(\underline{k}) + \underline{E}_{sc}(\underline{k}), \tag{12}$$

where $\underline{E}_{inc}(\underline{k})$ is the field incident on the scatterer at $\underline{r}'$ in k space, and $\underline{E}_{sc}(\underline{k})$ is the field scattered by the scatterer located at $\underline{r}'$ with incident field $\underline{E}_{inc}(\underline{k})$.

Proof: This follows directly from classical scattering theory when one takes the wave-number transform (see Jackson[6]). Q.E.D.

We now assume that the field incident on the i^{th} scatterer averaged over all (N-1) scatter positions is the average field. That is, we assume that $\underline{E}_{inc}(\underline{k})$ equals $\overline{\underline{E}}(\underline{k})$. Thus Lemma 2 can be stated as

$$\overline{\underline{E}}'(\underline{k}) = \overline{\underline{E}}(\underline{k}) + \overline{\underline{E}}'_{sc}(\underline{k}), \tag{13}$$

where $\overline{\underline{E}}'_{sc}(\underline{k})$ is completely determined by $\underline{r}'$ and $\overline{\underline{E}}(\underline{k})$. This assumption was made by Lax,[7] Foldy,[8] and Twersky[9-12] and is usually justified for scattering volumes of large N but of density small enough to retain the concept of individual scattering centers.

Lemma 3

Let a single spherical scatterer be located at $\underline{r}'$. Let a plane wave $\underline{E}(\underline{r}')$ be incident on the scatterer at $\underline{r}'$. The scattered field at $\underline{r}$, where $\underline{r} \neq \underline{r}'$, and $|\underline{r}-\underline{r}'|$ is greater than many particle diameters and wavelengths, is given by

$$\underline{E}_{sc}(\underline{r}) = \underline{T}^{-1}(\underline{k}_2)\, \underline{S}_o(\underline{k}_2, \underline{k}_1)\, \underline{T}(\underline{k}_1)\, \underline{E}(\underline{r}'), \tag{14}$$

where $\underline{k}_1$ is the direction of propagation of $\underline{E}(\underline{r}')$, and $\underline{k}_2$ is the vector between $\underline{r}$ and $\underline{r}'$. $\underline{T}(\underline{k})$ is a rotation matrix. Also

$$\underline{S}_o(\underline{k}_2, \underline{k}_1) = \frac{\exp(jk_o|\underline{r}'-\underline{r}|)}{jk_o|\underline{r}'-\underline{r}|} \begin{bmatrix} S_2(\theta) & 0 & 0 \\ 0 & S_1(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

with θ the angle between $\underline{k}_1$ and $\underline{k}_2$. The functions $S_2(\theta)$ and $S_1(\theta)$ are given in Goody.[13]

Proof: From Goody we know that a plane wave propagating in some direction $\underline{k}_1$ can be decomposed into two components in the plane to which $\underline{k}_1$ is normal. Likewise, the components in the scattered field can be decomposed into two components in a plane of which $\underline{k}_2$ is the normal. These incident and scattered components are related by the scattering matrix in Goody.[14] The radial components, that is, those along the $\underline{k}_1$ and $\underline{k}_2$ directions can also be related and they can be shown to be of order $1/|\underline{r}-\underline{r}'|^2$; that is, they are vanishing small compared with other terms (see Born and Wolf[15]). Now the vector $\underline{E}(\underline{r})$ is defined with respect to the (x, y, z) coordinate system. Thus, using a rotational transformation $\underline{T}(\underline{k}_1)$, we can reference it to the propagating system. This is done by Chandrasekhar[16] for Stokes parameter scattering. Q.E.D.

Let us define $\underline{S}(\underline{r}, \underline{r}', \underline{k}_1)$ as

$$\underline{S}(\underline{r}, \underline{r}', \underline{k}_1) = \underline{T}^{-1}(\underline{k}_2) \begin{bmatrix} S_2(\theta) & 0 & 0 \\ 0 & S_1(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{T}(\underline{k}_1), \tag{16}$$

where $\underline{k}_2$ is defined as the wave number going in direction $\underline{r}-\underline{r}'/|\underline{r}-\underline{r}'|$ and having magnitude $|\underline{k}_2|$. Clearly, for free space scattering, $|\underline{k}_2| = k_o$. Then the following lemma follows.

Lemma 4

The field scattered to position $\underline{r}$ from a spherical scatterer at $\underline{r}'$ with an incident field $\overline{\underline{E}}(\underline{r}')$ is given by

$$\overline{\underline{E}}_{sc}(\underline{r}) = \int \underline{S}(\underline{r}, \underline{r}', \underline{k}) \frac{\exp(jk_o|\underline{r}-\underline{r}'|)}{j|\underline{r}-\underline{r}'|\, k_o} \overline{\underline{E}}(\underline{k})\, e^{j\underline{k}\cdot\underline{r}'}\, d\underline{k}. \tag{17}$$

Furthermore,

$$\overline{\underline{E}}_{sc}(\underline{k}') = \int e^{-j\underline{k}\cdot\underline{r}'+j\underline{k}'\cdot\underline{r}'}\, d\underline{k} \int \underline{S}(\underline{r}'', \underline{k}) \frac{\exp(j\underline{k}_o|\underline{r}''|)}{jk_o|\underline{r}''|} \underline{E}(\underline{k})\, e^{-j\underline{k}'\cdot\underline{r}''}\, d\underline{r}''. \tag{18}$$

Proof: From Lemma 4 we know that we can find the field at $\underline{r}$ when a plane wave from direction $\underline{k}$ is incident upon a scatterer at $\underline{r}'$. Now, by means of the wave-number transform, $\underline{E}(\underline{k}) \exp(j\underline{k}\cdot\underline{r}')\, d\underline{k}$ represents a plane wave at $\underline{r}'$ coming from directions $\underline{k}$, $\underline{k} + d\underline{k}$. Thus adding the contributions from plane waves coming from all directions, we obtain (17). Note that if we had a plane wave incident, then $\overline{\underline{E}}(\underline{k})$ would be $\delta(\underline{k}-\underline{k}_1)$ and (17) would reduce to the result of Lemma 3. Now the second part of Lemma 4 follows directly from an application of the wave-number transform along with a change in variables. Note also that in (18) $\underline{r}'' = \underline{r} - \underline{r}'$, and will take on the limits of $\underline{r} - \underline{r}'$. Here, $r''_z \geq 0$, but $0 \leq r'_z \leq d$ so that $-d \leq r''_z \leq 0$. Thus, since d is the depth of the cloud, the integral over $\underline{r}''$ takes note of this fact. Q.E.D.

Theorem 1

If the volume of the scatterers is large compared with the wave number, then the average field in wave-number space is given by the solution to the following algebraic equation:

$$\left\{\rho n' \int \underline{A}(\underline{k},\underline{k}')\, I(|\underline{k}-\underline{k}'|)\, \underline{\Lambda}(\underline{k},\underline{k}')\, d\underline{k}'\right\} \overline{\underline{E}}(\underline{k}) + \left(-|\underline{k}|^2+k_1^2\right) \overline{\underline{E}}(\underline{k}) = \underline{S}(\underline{k}), \tag{19}$$

where

$$k_1^2 = k_o^2\left(1+n' \frac{4}{3} \pi\delta^3 \rho\right) \tag{20}$$

and

$$\underline{\Lambda}(\underline{k},\underline{k}') = \int \underline{S}(\underline{r}'',\underline{k}) \frac{\exp(jk_o|\underline{r}''|)}{jk_o|\underline{r}''|} e^{-j\underline{k}'\cdot\underline{r}''} d\underline{r}''. \tag{21}$$

Proof of Theorem 1

Using the approximation of (13) in (11), together with (18), we obtain

$$\rho V n' \int p(\underline{r}')\, d\underline{r}' \int \underline{A}(\underline{k},\underline{k}')\, I(|\underline{k}-\underline{k}'|) \exp(-j(\underline{k}-\underline{k}')\cdot\underline{r}')$$

$$\left[\overline{\underline{E}}(\underline{k}') + \iint \exp(-j\underline{k}''\cdot\underline{r}'+j\underline{k}'\cdot\underline{r}')\, \underline{S}(\underline{r}'',\underline{k}'') \frac{\exp(jk_o|\underline{r}''|)}{jk_o|\underline{r}''|}\right.$$

$$\left.\overline{\underline{E}}(\underline{k}'') \exp(-j\underline{k}'\cdot\underline{r}'')\, d\underline{k}''d\underline{r}''\right] + \left(|\underline{k}|^2-k_o^2\right) \overline{\underline{E}}(\underline{k}) = \underline{S}(\underline{k}). \tag{22}$$

Now if we use the assumption that the scatterers are uniformly distributed over V, this implies that $p(\underline{r}')$ is $1/V$. Then integrating over $\underline{r}'$ yields

$$\int \exp(-j(\underline{k}-\underline{k}')\cdot\underline{r}')\, d\underline{r}' = \delta(\underline{k}-\underline{k}') \tag{23}$$

and

$$\int \exp(-j(\underline{k}-\underline{k}')\cdot\underline{r}') \exp(-j\underline{k}''\cdot\underline{r}'+j\underline{k}'\cdot\underline{r}')\, d\underline{r}' = \delta(\underline{k}-\underline{k}''). \tag{24}$$

Equations 23 and 24 follow from the assumption that the scatterers are uniformly distributed over distances large compared with k_1, k_2 and k_3 (see Papoulis[17]). Then using these in (21), noting what $\underline{A}(\underline{k},\underline{k}')$ is, and showing that $I(0)$ is $(4/3)\pi\delta^3$, we obtain (19).

Q.E.D.

Lemma 5

The matrix $\underline{\Lambda}(\underline{k},\underline{k}')$ is equal to

$$\underline{\Lambda}(\underline{k}, \underline{k}') = \frac{2\pi}{k_o^2} \int_{-d}^{\infty} \exp(j(|\underline{k}|-k_3')r_z')\, dr_z'\, \underline{\Sigma}(\underline{k})\, T(\underline{k}), \tag{25}$$

where

$$\underline{\Sigma}(\underline{k}) = \begin{bmatrix} S_2(\theta) & 0 & 0 \\ 0 & S_1(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{26}$$

Proof: Using the method of stationary phase on (21) (see Born and Wolf,[18] Erydeli[19] or Papoulis[20]), we obtain (25). Note that the θ that appears in $\underline{\Sigma}(\underline{k})$ is the angle that the $\underline{k}$ vector makes with the (x, y, z) coordinate system. This can easily be defined in terms of the components of k_1, k_2 and k_3. Q.E.D.

For a thick cloud, $\underline{\Lambda}(\underline{k}, \underline{k}')$ reduces to a simple form,

$$\underline{\Lambda}(\underline{k}, \underline{k}') = \frac{2\pi}{k_o^2}\, \delta(k_3'-|\underline{k}|)\, \underline{\Sigma}(\underline{k})\, \underline{T}(\underline{k}). \tag{27}$$

We can apply this to Eq. 19. Note that $I(|\underline{k}-\underline{k}'|)$ and the impulse are even functions, while part of $\underline{A}(\underline{k}, \underline{k}')$ is odd. Using a Cauchy limit interpretation of the integral, we obtain

$$\rho n' \int \underline{A}(\underline{k}, \underline{k}')\, I(|\underline{k}-\underline{k}'|)\, \underline{\Lambda}(\underline{k}, \underline{k}')\, d\underline{k}' \tag{28}$$

$$= -\rho n' 2\pi \int I(\underline{k}')\, dk_1 dk_2\, \underline{\Sigma}(k)\, \underline{T}(\underline{k})$$

which can be integrated by using (8) to yield

$$= -\rho n' \delta 8\pi^2.$$

Thus we can restate Theorem 1 as a corollary.

COROLLARY. The average field in a thick cloud of large volume is given by the solution to

$$\underline{F}(\underline{k})\, \overline{\underline{E}}(\underline{k}) = \underline{S}(\underline{k}), \tag{29}$$

where

$$\underline{F}(\underline{k}) = |\underline{k}|^2\, I - k_o^2 \left[\left(1 + \frac{4}{3}\pi n' \delta^3\right) I + \frac{\rho n' \delta}{k_o^2}\, 8\pi^2 \underline{\Sigma}(\underline{k})\, \underline{T}(\underline{k}) \right] \tag{30}$$

with I a 3×3 identity matrix. In general the solution of (29) is not trivial and may require computer inversion. It can be shown that (29) possesses simpler solutions when the irradiating field is a plane wave. In that case it can be shown that $\overline{\underline{E}(\underline{r})}$ is exponentially decreasing in depth. The damping factor depends on ρ, n', δ and the imaginary part of $\underline{\Sigma}(\underline{k})$. Such a result has been previously obtained by Twersky.[21, 22, 1]

We can now also consider the field correlation matrix. If $\underline{E}(\underline{r})$ is the field at $\underline{r}$, and $\underline{E}^+(\underline{r})$ is the complex conjugate transpose, then we can give the following definition.

DEFINITION 2. Let $\underline{R}(\underline{r}_1, \underline{r}_2)$ be the two-point spatial correlation matrix of the random field $\underline{E}(\underline{r})$ given by

$$\underline{R}(\underline{r}_1, \underline{r}_2) = \overline{\underline{E}(\underline{r}_1)\, \underline{E}^+(\underline{r}_2)}. \tag{31}$$

Then the two-point spectral correlation matrix of the random field is $\underline{R}(\underline{r}_1, \underline{r}_2)$, and is given by

$$\underline{R}(\underline{r}_1, \underline{r}_2) = \int \exp(-j\underline{k}_1 \cdot \underline{r}_1 + j\underline{k}_2 \cdot \underline{r}_2)\, R(\underline{k}_1, \underline{k}_2)\, d\underline{k}_1 d\underline{k}_2. \tag{32}$$

This can be seen to be well defined (see Doob[23] or Papoulis[24]).

Now $\underline{R}(\underline{k}_1, \underline{k}_2)$ can be shown to satisfy a certain integral equation.

Theorem 2

The two-point spectral correlation matrix is given by the solution to the following matrix integral equation:

$$\int \underline{C}(\underline{k}, \underline{k}')\, \underline{R}(\underline{k}', \underline{k}'+\underline{k}''-\underline{k})\, \underline{D}(\underline{k}''+\underline{k}'\underline{k}, \underline{k}'')\, d\underline{k}'$$

$$+ \underline{F}(\underline{k})\, \underline{R}(\underline{k}, \underline{k}'')\, \underline{F}^+(\underline{k}'') = \underline{S}(\underline{k})\, \underline{S}^+(\underline{k}''), \tag{33}$$

where

$$\underline{C}(\underline{k}, \underline{k}') = \rho n' \underline{A}(\underline{k}, \underline{k}')\, I(|\underline{k}-\underline{k}'|) \tag{34}$$

$$\underline{D}(\underline{k}''+\underline{k}'-\underline{k}, \underline{k}'') = \underline{\Lambda}(\underline{k}''+\underline{k}'-\underline{k}, \underline{k}'')\, \underline{F}^+(\underline{k}''), \tag{35}$$

and $\underline{F}(\underline{k})$ is given by (30).

Proof of Theorem 2

This can be obtained by multiplying Eq. 5 by $\underline{E}^+(\underline{k}'')$ and then averaging, as was done with the mean field. Q.E.D.

Under certain circumstances a series solution is possible (see Courant and Hilbert[25]) and such a solution yields $\underline{R}(\underline{k}_1, \underline{k}_2)$ as a sum of the product of the mean field and a convolution of that field. It is this convolution term that is associated with what we term the incoherent field. For plane-wave excitation the incoherent field is the total field deep in the cloud, since the coherent field decays to zero. For nonabsorbing scatterers the total flux is constant so that this allows us to obtain the incoherent field directly (see Chandrasekhar[26]).

4. Conclusions

This report briefly sketches the first set of important results that have been obtained in this research. There are several areas that will be reported on later but are worth mentioning now. First, the term S(k) represents a boundary condition. It is the sum of all fields at the boundary, both forward and reverse scattered. This follows directly from Helmholtz's theorem where we assume Sommerfield radiation condition (see Sneddon[27] or Courant and Hilbert[28]). It is possible to include and evaluate backscattered by another approach. Obviously the average backscattered field is zero but the incoherent intensity is not. Fritz[29,30] has evaluated this for clouds in terms of albedo so that in using his results a general calculation is feasible. Second, the inclusion of motion of the scatterers is possible. This leads to kernels that are also a function of ω, the frequency transform variable. Data on these motions have been presented by Warner.[31-33] Third, the evaluation of field moments is possible by using the wave-number transforms, as well as generalized "time constants." Finally, coherence volumes or regions of finite nonzero correlations may be obtained. These results extend the present understanding of these random fields by allowing their microstructure to be obtained.

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