COMMUNICATION SCIENCES

AND

ENGINEERING

VII. PROCESSING AND TRANSMISSION OF INFORMATION*

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A. CHANNEL MEASUREMENT RECEIVERS FOR SLOWLY FADING NONDISPERSIVE MEDIA

1. Introduction

The concept of channel measurement as an optimal decision problem is developed in this report. The resulting model is applied to digital communications over a slowly fading nondispersive diversity medium, for which both the optimum receiver and error performance bounds are derived in the case of M-ary orthogonal signalling. These results are applicable to the study of heterodyne receivers for optical communication through a turbulent atmosphere.

2. Channel Measurement as a Decision Process

It is known that the reliability of communication through a randomly varying medium may be increased by performing some kind of channel measurement at the receiver.[1] Channel measurement receivers are conventionally dichotomized into an estimation section and a decision unit that functions parametrically on the estimates to reach decisions. This approach requires that an estimation criterion be assigned, and often the assignment is not directly related to the over-all communication objective. For digital communication with a minimum per-baud probability of error criterion, the channel-measurement problem may be formulated directly as an optimal decision problem operating on both present and past received data. The resulting receiver computes the likelihood function of the data (present and past), conditioned upon a particular hypothesis, for each of the hypotheses.

3. Measurement Receiver for a Slowly Fading Nondispersive Channel

The assumption of slow fading is that the received process differs from the transmitted signal in these two respects: the signal suffers a constant (random) gain and

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phase shift, and is added to an independent noise process. In the remainder the gain and phase processes are taken to be slow relative to several baud times. Assuming L statistically independent diversity paths and M equi-energetic orthogonal signals, a sufficient statistic for the n^{th} baud is the set of complex envelope samples

$$r_{m\ell}(n) = Za_\ell e^{j\theta_\ell} \delta_{m,m(n)} + w_{m\ell}(n): \qquad \begin{matrix} 1 \leq m \leq M \\ 1 \leq \ell \leq L \end{matrix} \tag{1}$$

where

$1 \leq m \leq M$ indexes the signals

$1 \leq \ell \leq L$ indexes the diversity paths

Z = rms received signal-to-noise ratio/diversity path

m(n) = transmitted message at the n^{th} baud

a_ℓ = fading amplitude on path ℓ

θ_ℓ = fading phase on path ℓ

δ = Kronecker delta function

w = N(0, 1) complex Gaussian random variable having $\overline{w_{m\ell}(n)\, w^*_{m'\ell'}(n')} = \delta_{mm'}\delta_{\ell\ell'}\delta_{nn'}$.

Let n be an age index so that successively higher integral values represent correspondingly older data (n = 0 represents the present). Then the variables in (1) taken for n = 0, 1, ..., N constitute a sufficient statistic for the decision at baud 0, given the data on bauds 0 through N.

The assumptions about the fading processes are the following.

1. The θ_ℓ are uniform and statistically independent.
2. The a_ℓ each have density $p(\cdot)$ and are statistically independent.
3. Gain and phase (a_ℓ and θ_ℓ) are all statistically independent.
4. $\overline{a_\ell^2} = 1$.

Occasionally, (1) will be required in quadrature form:

$$r_{m\ell}(n) = x_{m\ell}(n) + jy_{m\ell}(n). \tag{2}$$

Let an underscored variable denote the set of all variables by that name. For example,

$$\underline{r} = \{r_{m\ell}(n)\}_{m,\ell,n}, \quad \underline{m} = \{m(n)\}_n, \text{ etc.}$$

Let H_k be the hypothesis that m(o) = k, k = 1, ..., M and let V_k be the set of all

$\underline{m}$ such that m(o) = k. The likelihood functions for the resulting M-ary hypothesis testing problem are

$$\Lambda_k = p(\underline{r}|H_k); \qquad k = 1, \ldots, M. \tag{3}$$

Expressions for the Λ_k are derived next.

Since the $w_{m\ell}(n)$ are all mutually independent, it follows that

$$p(\underline{r}|\underline{a}, \underline{\theta}, \underline{m}) = \prod_{\ell, m, n} p(r_{m\ell}(n)|a_\ell, \theta_\ell, m(n))$$

$$= (2\pi)^{-LM(N+1)} \prod_{\ell, m, n} \exp\left\{-\frac{1}{2}\left|r_{m\ell}(n) - Za_\ell e^{j\theta_\ell} \delta_{m, m(n)}\right|^2\right\}. \tag{4}$$

The result of the average over phase ($\underline{\theta}$) may be expressed in terms of the modified Bessel function $I_o(\cdot)$ and a function, $K(\underline{r})$, of the data.

$$K(\underline{r}) = (2\pi)^{-LM(N+1)} \prod_{\ell, m, n} e^{-\frac{1}{2}|r_{m\ell}(n)|^2} \tag{5}$$

$$p(\underline{r}|\underline{a}, \underline{m}) = \overline{p(\underline{r}|\underline{a}, \underline{\theta}, \underline{m})}^{\underline{\theta}}$$

$$= K(\underline{r}) \prod_{\ell} e^{-\frac{1}{2}(N+1)Z^2a_\ell^2} I_o\left[a_\ell Z \sqrt{\left[\sum_n x_{m(n)\ell}(n)\right]^2 + \left[\sum_n y_{m(n)\ell}(n)\right]^2}\right]. \tag{6}$$

Note that the argument of $I_o(\cdot)$ in (6) reflects coherent addition of the complex received samples along the assumed message sequence $\underline{m}$. This makes it reasonable to define

$$r_{\underline{m}\ell} = \sum_n \left[x_{m(n)\ell}(n)+jy_{m(n)\ell}(n)\right], \tag{7}$$

so that (6) may be written

$$p(\underline{r}|\underline{a}, \underline{m}) = K(\underline{r}) \prod_{\ell} e^{-\frac{1}{2}(N+1)Z^2a_\ell^2} I_o(a_\ell Z|r_{\underline{m}\ell}|). \tag{8}$$

The average over $\underline{a}$ may be expressed in terms of the "generalized frustration function."[2-4]

$$F_p(\alpha, \beta) = \int_0^\infty p(u)\, I_o(2\beta\sqrt{\alpha}\, u)\, e^{-\alpha u^2}\, du, \tag{9}$$

where p(·) is the probability density of a positive random variable, in this case, the fading amplitude a_ℓ. Thus

$$p(\underline{r}|\underline{m}) = \overline{p(\underline{r}|\underline{a}, \underline{m})}^{\underline{a}}$$

$$= K(\underline{r}) \prod_\ell F_p\left(\frac{N+1}{2} Z^2, \frac{|r_{\underline{m}\ell}|}{\sqrt{2(N+1)}}\right). \tag{10}$$

Now, according to (3),

$$\Lambda_k = \overline{p(\underline{r}|\underline{m})}^{\underline{m}|V_k}. \tag{11}$$

If all messages have a priori probability $\frac{1}{M}$ for each n, and are chosen independently at each baud, then the probability of the event $(\underline{m}|V_k)$ is M^{-N}. Hence

$$\Lambda_k = M^{-N}K(\underline{r}) \sum_{\underline{m} \in V_k} \prod_\ell F_p\left(\frac{N+1}{2} Z^2, \frac{|r_{\underline{m}\ell}|}{\sqrt{2(N+1)}}\right). \tag{12}$$

Discarding hypothesis-independent terms and taking the logarithm yields a sufficient statistic

$$q_k(\underline{r}) = \ln\left[\sum_{\underline{m} \in V_k} \prod_\ell F_p\left(\frac{N+1}{2} Z^2, \frac{|r_{\underline{m}\ell}|}{\sqrt{2(N+1)}}\right)\right]. \tag{13}$$

The optimum receiver chooses H_k when

$$q_k(\underline{r}) > q_i(\underline{r}) \qquad \forall\, i \neq k. \tag{14}$$

4. Performance Bounds

We shall now find an upper bound to P(e). The bound is left in a doubly parametric form, since optimization over the parameters is analytically intractable.

First of all, note that

$$P(e|H_k, q_k) = 1 - \Pr\{q_i < q_k | H_k, q_k \;\; \forall\, i \neq k\}$$

$$= 1 - [1 - \Pr\{q_i > q_k | H_k, q_k\}]^{M-1}; \qquad i \neq k. \tag{15}$$

For convenience the explicit dependence of q_i and q_k upon the data $\underline{r}$ has been dropped in (15) and is reinstated later where needed. A well-known form of bound for (15) is[5]

$$P(e|H_k, q_k) \leq M^{\rho}[\Pr\{q_i > q_k | q_k, H_k\}]^{\rho}; \qquad 0 \leq \rho \leq 1, \tag{16}$$

and consequently

$$P(e|H_k) \leq \overline{M^{\rho}[\Pr\{q_i > q_k | q_k, H_k\}]^{\rho}}^{\,q_k|H_k}. \tag{17}$$

Define

$$Q(a) \triangleq \Pr\{q_i > a | a, H_k\}.$$

$$= \overline{u_{-1}(q_i - a)}^{\,q_i|H_k}. \tag{18}$$

Application of a Chernov bound yields

$$Q(a) \leq \overline{\exp[t(q_i - a)]}^{\,q_i|H_k}; \qquad 0 \leq t \leq \infty. \tag{19}$$

Equation 19 can equally well be written as an average over $\underline{r}|H_k$. First define

$$L_i(\underline{r}) = e^{q_i(\underline{r})}; \qquad 1 \leq i \leq M. \tag{20}$$

Then

$$Q(a) \leq \overline{e^{-ta}\, e^{tq_i(\underline{r})}}^{\,\underline{r}|H_k}$$

$$= e^{-ta}\, \overline{[L_i(\underline{r})]^t}^{\,\underline{r}|H_k}$$

$$= e^{-ta} \int \dots \int d\underline{r}\, M^{-N} K(\underline{r})\, L_k(r)[L_i(\underline{r})]^t$$

$$= e^{-ta}\, e^{\gamma_1(t)}, \tag{21}$$

where

$$\gamma_1(t) = \ln \int \ldots \int d\underline{r}\; M^{-N} K(\underline{r})\; L_k(\underline{r})[L_i(\underline{r})]^t. \tag{22}$$

Now (17) can be written

$$P(e|H_k) \leq M^\rho \overline{[Q(q_k)]^\rho}^{\,q_k|H_k} \tag{23}$$

$$\leq M^\rho\, \overline{e^{-\rho t q_k(\underline{r})}}^{\,\underline{r}|H_k}\; e^{\rho \gamma_1(t)}. \tag{24}$$

The average in (24) is of the form

$$\overline{e^{s q_k(\underline{r})}}^{\,\underline{r}|H_k} = \overline{[L_k(\underline{r})]^s}^{\,\underline{r}|H_k}$$

$$= \int \ldots \int d\underline{r}\; M^{-N} K(\underline{r})\; L_k(\underline{r})[L_k(\underline{r})]^s$$

$$= \int \ldots \int d\underline{r}\; M^{-N} K(\underline{r})[L_k(\underline{r})]^{s+1} \tag{25}$$

which is abbreviated

$$\overline{e^{s q_k(\underline{r})}}^{\,\underline{r}|H_k} \triangleq e^{\gamma_o(s)}. \tag{26}$$

Equations 24 and 26 combine to yield

$$P(e) \leq M^\rho\, e^{\gamma_o(-\rho t) + \rho \gamma_1(t)}; \qquad 0 \leq \rho \leq 1, \quad t \geq 0. \tag{27}$$

Because of the assumptions about a priori probabilities, (27) is also the bound to the unconditional error probability P(e).

Kennedy and Hoversten[6] have shown (for the "no measurement" case, N = 0) that

$$\gamma_o(s) = \gamma_1(s+1) \tag{28}$$

and additionally that the choice

$$t = \frac{1}{1+\rho} \tag{29}$$

is optimum, which results in the single-parameter bound

$$P(e) \leq M^{\rho} \exp\left\{(1+\rho)\, \gamma_1\left(\frac{1}{1+\rho}\right)\right\}. \tag{30}$$

At present, it is not known whether the choice (29) optimizes the bound in (27).

5. Applications

The optimum receiver, (13) and (14), and the error bound (27) are applicable to heterodyne reception of optical communication signals transmitted through the turbulent atmosphere. Amplitude and phase coherence time of the order of milliseconds have been determined for this channel,[7] so that essentially constant fading over large numbers of adjacent bauds is a reasonable prospect. For the atmosphere the amplitude density appearing in the frustration function is log-normal with parameter σ; that is,

$$p(u) = \frac{1}{\sqrt{2\pi}\,\sigma u} \exp\left[-\frac{(\sigma^2 + \ln u)^2}{2\sigma^2}\right]; \qquad u \geq 0. \tag{31}$$

The results may be extended to include correlated baud-to-baud fades, and nonuniform phase densities.

R. S. Orr

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B. VARIABLE-RATE OPTICAL COMMUNICATION THROUGH THE TURBULENT ATMOSPHERE

1. Introduction

The prospect of communicating at optical frequencies through the Earth's turbulent atmosphere is often discounted: atmospheric fading tends to degrade system performance below acceptable levels. In this report, we shall examine two adaptive laser communication schemes over an Earth-to-Space link which circumvent the effects of atmospheric turbulence. These heterodyne communication systems exploit atmospheric reciprocity and the relatively long coherence time of the turbulence to monitor the transient state of the Earth-to-Space channel by using a satellite beacon signal and making appropriate measurements at the ground terminal. Optimal variable-rate strategy based on this channel-state information results in significantly improved performance over nonadaptive optical communication systems.

2. Channel Measurement

Consider an optical communication link between the Earth and a synchronous satellite as shown in Fig. VII-1. The antennas in the ground and satellite terminals are represented by the parallel planar apertures R_1 and R_3, whose axes are assumed to

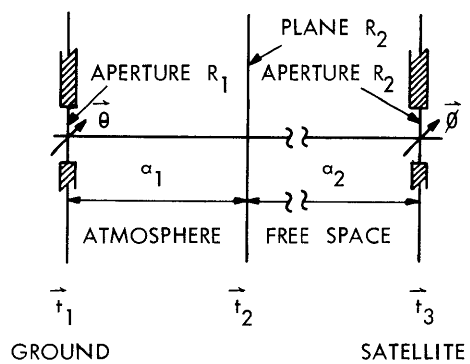

Fig. VII-1. Channel model for optical Earth-to-Space communication link.

be in line. The infinite plane R_2 is parallel to the other planes and tangent to the top of the atmosphere. Propagation between planes R_1 and R_2 occurs through the clear turbulent atmosphere, while propagation between planes R_2 and R_3 is through free space.

We want to measure the atmospheric fading over the Earth-to-Space link by transmitting a pilot-tone from the satellite to the ground terminal and exploiting atmospheric reciprocity. This technique is feasible because the width d_1 of the atmospheric layer around the Earth is of the order of a kilometer, and the coherence time of the turbulence is often of the order of a millisecond or more.[1] Consequently, the round-trip atmospheric propagation time, from R_2 to R_1 to R_2, is substantially less than the coherence time of the turbulence; we shall therefore consider only a single transient atmospheric state here, suppressing the time dependence of our equations. We shall be concerned only with the complex envelopes of the fields, and for convenience we shall use arrows pointing to the right under fields propagating from the ground to the satellite, and conversely for satellite-to-ground transmissions.

Suppose a laser in the ground terminal is used to transmit a collimated plane wave through aperture R_1 in the direction $\vec{\theta}$:

$$\underrightarrow{U}_1(\vec{r}_1) = K\, e^{jk\vec{\theta}\cdot\vec{r}_1}; \qquad k = \frac{2\pi}{\lambda}, \qquad \vec{r}_1 \in R_1. \tag{1}$$

Let $\underrightarrow{h}_a(\vec{r}_2, \vec{r}_1)$ denote the impulse response characterizing field propagation through the atmosphere from R_1 to R_2. Similarly, define $\underrightarrow{h}_f(\vec{r}_3, \vec{r}_2)$ for free-space propagation from R_2 to R_3. Then the field incident on satellite aperture R_3 is

$$\underrightarrow{U}_3(\vec{r}_3) = K \int_{R_1} \int_{R_2} \underrightarrow{h}_a(\vec{r}_2, \vec{r}_1)\, \underrightarrow{h}_f(\vec{r}_3, \vec{r}_2)\, e^{jk\vec{\theta}\cdot\vec{r}_1}\, d\vec{r}_1 d\vec{r}_2; \qquad \vec{r}_3 \in R_3. \tag{2}$$

Assume that an optical heterodyne detector in the satellite extracts the single spatial mode $\underrightarrow{U}_4(\vec{\phi})$ of the field received at the satellite[2]:

$$\underrightarrow{U}_4(\vec{\phi}) = \int_{R_3} \underrightarrow{U}_3(\vec{r}_3)\, e^{-jk\vec{\phi}\cdot\vec{r}_3}\, d\vec{r}_3$$

$$= K \int_{R_1} \int_{R_2} \int_{R_3} \underrightarrow{h}_a(\vec{r}_2, \vec{r}_1)\, \underrightarrow{h}_f(\vec{r}_3, \vec{r}_2)\, e^{jk(\vec{\theta}\cdot\vec{r}_1 - \vec{\phi}\cdot\vec{r}_3)}\, d\vec{r}_1 d\vec{r}_2 d\vec{r}_3. \tag{3}$$

Now suppose a satellite beacon probes the state of the atmosphere by transmitting a collimated plane wave through aperture R_3 in the direction $-\vec{\phi}$:

$$\underleftarrow{U}_3(\vec{r}_3) = K'\, e^{-jk\vec{\phi}\cdot\vec{r}_3}; \qquad \vec{r}_3 \in R_3. \tag{4}$$

Define $\underleftarrow{h}_a(\vec{r}_1, \vec{r}_2)$ and $\underleftarrow{h}_f(\vec{r}_2, \vec{r}_3)$ in a manner analogous to our previous usage, and let an optical heterodyne detector in the ground terminal extract the spatial mode $\underleftarrow{U}_o(-\vec{\theta})$ of the field $\underleftarrow{U}_1(\vec{r}_1)$ incident on aperture R_1:

$$\underleftarrow{U}_o(-\vec{\theta}) = K' \int_{R_1} \int_{R_2} \int_{R_3} \underleftarrow{h}_a(\vec{r}_1, \vec{r}_2)\, \underleftarrow{h}_f(\vec{r}_2, \vec{r}_3)\, e^{jk(\vec{\theta}\cdot\vec{r}_1 - \vec{\phi}\cdot\vec{r}_3)}\, d\vec{r}_1 d\vec{r}_2 d\vec{r}_3. \tag{5}$$

The reciprocal nature of the turbulent channel over an atmospheric coherence interval has been demonstrated theoretically,[3] and free space is known to be reciprocal for optical transmissions. Therefore, the atmospheric and free-space impulse responses satisfy the reciprocity conditions

$$\underset{\leftarrow}{h}_a(\vec{r}_2, \vec{r}_1) = \underset{\leftarrow}{h}_a(\vec{r}_1, \vec{r}_2); \qquad \forall\, \vec{r}_1 \in R_1, \vec{r}_2 \in R_2, \tag{6}$$

$$\underset{\leftarrow}{h}_f(\vec{r}_3, \vec{r}_2) = \underset{\leftarrow}{h}_f(\vec{r}_2, \vec{r}_3); \qquad \forall\, \vec{r}_2 \in R_2, \vec{r}_3 \in R_3. \tag{7}$$

We can therefore conclude that

$$\underset{\leftarrow}{U}_o(-\vec{\theta}) = \frac{K'}{K}\, \underset{\leftarrow}{U}_4(\vec{\phi}). \tag{8}$$

Since $\underset{\leftarrow}{U}_4(\vec{\phi})$ represents all of the effects of atmospheric fading for our optical Earth-to-Space link, Eq. 8 tells us how to interpret the satellite pilot tone received at the ground terminal in order to measure the transient state of this channel.

3. Fixed-Rate Heterodyne System

We now specialize the Earth-to-Space link to the case wherein $\vec{\theta} = \vec{\phi} = \vec{0}$, and introduce time dependence into our equations. Assume that the ground terminal transmits a signal with no time-varying spatial modulation, and that channel multipath can be neglected.[4] Then the complex envelope of the output of an optical heterodyne receiver in the satellite for a single transmission is a random process of the form

$$r(t) = \underset{\leftarrow}{U}_4(\vec{0})\, s(t) + n(t); \qquad t \in (0, T), \tag{9}$$

where s(t) is a narrow-band waveform, and the signal baud time T is much smaller than the channel coherence time. The noise term n(t) is a complex, zero-mean Gaussian random process, whose real and imaginary parts are assumed to be statistically independent, each having spectral height $N_o/2$.[5]

Denote the areas of apertures R_1 and R_3 by A_1 and A_3, respectively. For a synchronous satellite, the separation d_2 of planes R_2 and R_3 is generally great enough relative to the magnitudes of A_1 and A_3 that aperture R_3 subtends a negligible solid angle in comparison with the far-field beamwidth of aperture R_1 in the absence of turbulence. Consequently, by exploiting the atmospheric reciprocity condition of Eq. 6, Eq. 3 becomes

$$\underset{\leftarrow}{U}_4(\vec{0}) = \frac{KA_3\, e^{jkd_2}}{j\lambda d_2} \int_{R_1} \left[\int_{R_2} \underset{\leftarrow}{h}_a(\vec{r}_1, \vec{r}_2)\, d\vec{r}_2 \right] d\vec{r}_1, \tag{10}$$

where the term in brackets is the atmospheric perturbation of an infinite plane wave propagating from R_2 to R_1.[6]

It is convenient to introduce the definition

$$u\, e^{j\psi} = \frac{1}{A_1} \int_{R_1} \left[\int_{R_2} \underset{\leftarrow}{h}_a(\vec{r}_1, \vec{r}_2)\, d\vec{r}_2 \right] d\vec{r}_1. \tag{11}$$

When A_1 is small relative to a spatial coherence area of $\int_{R_2} \underset{\leftarrow}{h}_a(\vec{r}_1, \vec{r}_2)\, d\vec{r}_2$ in plane R_1, it can be shown that u is a log-normal random variable; that is, $u = \exp \chi$, where χ is a Gaussian random variable. On the other hand, if A_1 is large relative to the spatial coherence area above, we can demonstrate theoretically that u is essentially a Rayleigh random variable.[7] In both cases, the phase term ψ tends to be uniformly distributed over $(0, 2\pi)$.

Restricting ourselves to binary, equi-energy orthogonal signalling, and using incoherent detection on the received signal r(t), the probability of error on a single transmission is[8]

$$\epsilon_1 = \frac{1}{2} \overline{\exp\left(-\frac{E_s}{2N_o} u^2\right)}^{\,u}, \tag{12}$$

where

$$E_s = \frac{K^2 A_1^2 A_3^2}{\lambda^2 d_2^2} \int_0^T |s(t)|^2\, dt = P_s/R_F. \tag{13}$$

In Eq. 13, we denote the fixed bit rate for continuous signalling by $R_F = 1/T$, and the average signal power received at the satellite in the absence of turbulence by P_s. Performing the expectation in Eq. 12 and solving for R_F, we can show that

$$R_F = \begin{cases} \dfrac{P_s}{2N_o} f_{\sigma_\chi}(\epsilon_1); & \text{log-normal } u = e^{\chi}, \\[2ex] \dfrac{P_s}{2N_o} \overline{u^2} \left(\dfrac{1}{2\epsilon_1} - 1\right)^{-1}; & \text{Rayleigh } u, \end{cases} \tag{14}$$

where σ_χ^2 is the variance of χ, and $f_{\sigma_\chi}(\epsilon_1)$ can be determined from computer-generated curves of ϵ_1 as a function of $E_s/2N_o$ for the case wherein u is log-normal[9] and the energy-conservation condition $\overline{u^2} = 1$ is satisfied.[5]

4. Optimal Variable-Rate Techniques

From Eqs. 8, 10, and 11, we find that

$$u = \frac{\lambda d_2}{K'A_1A_3} \left|\underset{\leftarrow}{U}_o(\vec{0})\right|. \tag{15}$$

Now that we know how to use a satellite beacon to track the Earth-to-Space channel fading parameter u, we want to devise an adaptive variable-rate scheme to optimize the performance of our communication link. As in the fixed-rate system, we shall confine our attention to the continuous transmission of binary, equi-energy orthogonal signals. The signal baud time T will now be varied, however, for each transmission according to some mapping T(u) of the transient value of u, while the average transmitted power is kept constant. We assume that T(u) is always much less than the coherence time of the fading channel.

Denote the transient bit rate when the channel fading parameter is u by

$$R(u) = \frac{1}{T(u)} \text{ bits/sec,} \tag{16}$$

and assume that the fading process is ergodic. Then the average signalling rate is

$$R_{avg} = \overline{R(u)}^{\,u} \text{ bits/sec.} \tag{17}$$

For incoherent detection, the probability of error on a single transmission conditioned on the corresponding channel state depends only on the transient value of u, and is given by

$$P[\epsilon|u] = \frac{1}{2} \exp\left[-\frac{P_s u^2}{2N_o R(u)}\right]. \tag{18}$$

Since we are signalling continuously at a variable information rate, the bit error rate may be expressed as

$$\epsilon_2 = \overline{\frac{1}{2} R(u) \exp\left[-\frac{P_s u^2}{2N_o R(u)}\right]}^{\,u} \text{ bit errors/sec,} \tag{19}$$

which means that the fraction ϵ_3 of bit errors to total bits received by the satellite is given by

$$\epsilon_3 = \epsilon_2/R_{avg} \text{ bit errors/received bit.} \tag{20}$$

Our design objective is to choose R(u) to maximize R_{avg} for any given ϵ_2, keeping P_s and N_o fixed. Using the Lagrange multiplier technique, we can show that the optimal solution is

$$R(u) = C(\epsilon_2)\, u^2; \qquad \forall\, p(u), \tag{21}$$

where $C(\epsilon_2)$ depends only on the desired bit error rate. Note that this result is independent of the actual probability density $p(u)$. Denoting R_{avg} for the optimal variable-rate system by R_V, we have

$$R_V = \frac{P_s}{2N_o}\, \overline{u^2} \Big/ \ln\left(\frac{1}{2\epsilon_3}\right); \qquad \forall\, p(u). \tag{22}$$

By comparison, since $\epsilon_3 = \epsilon_1$ for the fixed-rate system, clearly

$$\frac{R_V}{R_F} = \begin{cases} \dfrac{1}{\mu^2} \Big/ \ln\left(\dfrac{1}{2\epsilon_3}\right) f_{\sigma_X}(\epsilon_3); & \text{log-normal } u = e^X, \\ \left(\dfrac{1}{2\epsilon_3} - 1\right) \Big/ \ln\left(\dfrac{1}{2\epsilon_3}\right); & \text{Rayleigh } u. \end{cases} \tag{23}$$

As indicated in Fig. VII-2, the gain in average signalling rate, R_V/R_F, is particularly significant for low bit error rates.

As a final exercise, we can find the optimal burst communication system, which operates as follows. The ground terminal divides its time scale into consecutive,

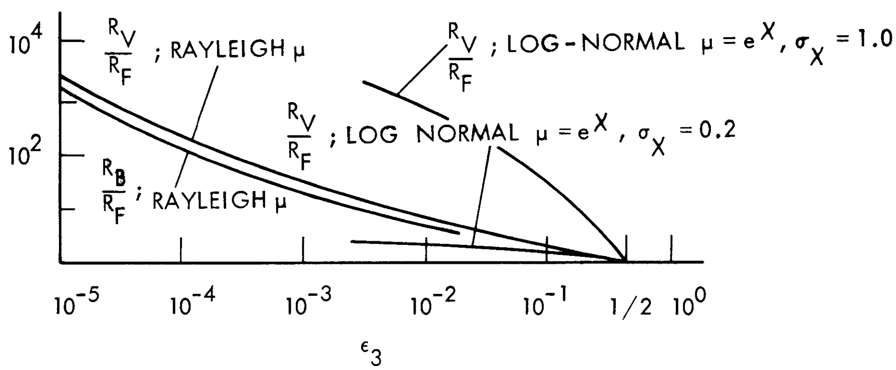

Fig. VII-2. Gain in average signalling rate of adaptive vs fixed-rate optical heterodyne communication link over an Earth-to-Space channel, with turbulent fading parameter u.

nonoverlapping, T-second time slots. A data signal is transmitted in a given time slot if and only if the corresponding value of u exceeds a preselected threshold η; otherwise, no signal is sent in that particular time slot, and the information is stored until the next acceptable transmission interval appears. We must, of course, hope that the associated transmitter buffering problem is not too severe. Because the time slots have a fixed periodicity, the satellite receiver should be able to acquire and maintain bit synchronization quite readily. If the signal-to-noise ratio, P_s/N_o, is sufficiently large, the satellite receiver should be able to decide correctly most of the time whether it is receiving noise or a data signal corrupted by noise in a particular time slot.

In our previous notation, our problem is to optimize R(u) over the class

$$R(u) = \frac{1}{T} u_{-1}(u-\eta), \tag{24}$$

where $u_{-1}(\cdot)$ is a unit step function. When u is Rayleigh, we have the following parametric solution, with parameter β.

$$\begin{aligned} \eta^2 &= \overline{u^2}\beta/(\beta+1) \qquad & R_{avg} &\equiv R_B = \left(\overline{u^2}P_s/2N_o\beta\right) e^{-\beta/(\beta+1)} \\ T &= 2N_o\beta/\overline{u^2}P_s \qquad & \epsilon_3 &= [1/2(\beta+1)]\, e^{-\beta^2/(\beta+1)}. \end{aligned} \tag{25}$$

As is evident from Fig. VII-2, the optimal burst communication system performs almost as well as the optimal variable-rate system.

5. Conclusions

We have demonstrated that a satellite beacon can be used to measure the atmospheric fading over an optical Earth-to-Space communication link. We have also shown theoretically that an adaptive variable-rate laser communication system will perform favorably over this channel. Similar results are available for the more general case wherein the ground terminal makes a noisy estimate of the channel state.[10]

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C. POISSON PROCESS AS A STATISTICAL MODEL FOR PHOTODETECTORS EXCITED BY GAUSSIAN LIGHT

In the literature of optical communication, it has frequently been assumed that the output statistics of an ideal photodetector, excited by a signal plus independent background noise, can be modelled by a Poisson process.[1-6] The arguments used to support this assumption have been less than precise, usually hinging on a qualitative statement about the strength and bandwidth of the background noise. In this report we present a set of criteria through which a quantitative measure of the "Poisson-ness" of the detector output can be obtained, for Gaussian background noise.

In an idealized quantum photodetector, the conditional probability of detecting k events, or "counts," in the time interval $(0, T]$, given the incident radiation, can be shown to obey a Poisson law, with the rate function proportional to the intensity of the field.[6,7] When the incident radiation is a Gaussian process, the photocount probability distribution, conditioned only on the mean of the radiation process, can be obtained, although in general it is in the cumbersome form of an infinite convolution of Laguerre distributions.[6] The counting distribution can be described, however, by its cumulants, which are simple, closed-form expressions in general. The cumulant representation for the photocount distribution is very suggestive of comparisons with a pure Poisson distribution because the cumulants of the latter are all the same. If a set of conditions can be found under which the cumulants of the general photocount distribution are equal, then it can be claimed that the distribution is Poisson. This is the essence of our approach.

First, we introduce some notation by briefly reviewing the Poisson model for an ideal quantum photodetector. The results and terminology are taken from Karp and Clark,[6] in which detailed proofs and discussions can be found. According to the Poisson model, the detector counting statistic N_T at time T, conditioned on a complex function $[E_o(t, \vec{r});\ 0^{\bullet} \leq t < T,\ \vec{r} \in A]$, obeys a Poisson law,

$$\Pr\{N_T = k \mid [E_o(t, r);\ 0 \le t < T,\ \vec{r} \in A]\}$$

$$= \frac{m_T^k}{k!} e^{-m_T} \tag{1}$$

$$m_T = \alpha \int_0^T \int_A |E_o(t, \vec{r})|^2 \, d\vec{r}dt. \tag{2}$$

$E_o(t, \vec{r})$ is the complex envelope of the incident, scalar, narrow-band Gaussian field, and is assumed to have a real covariance function, and to be normalized to a medium with a characteristic impedance of unity. $\alpha = \eta/h\nu$, where η is the quantum efficiency, h is Planck's constant, and ν is the frequency. The symbol A will be used interchangeably to denote the detector surface, and its area.

The probability of k counts in $(0, T]$ is given formally by

$$p_{N_T}(k) = \frac{1}{k!} E\left\{e^{-m_T} m_T^k\right\}, \tag{3}$$

where the expectation is taken over the random variable m_T. In terms of characteristic functions,

$$M_{N_T}(jv) = E\left\{\exp\left[m_T(e^{jv}-1)\right]\right\}. \tag{4}$$

Clearly, $M_{N_T}(jv)$ is simply the moment-generating function of m_T,

$$M_{m_T}(u) = E\left\{e^{m_T u}\right\}, \tag{5}$$

evaluated at $u = e^{jv} - 1$. By expanding $E_o(t, \vec{r})$ in a time-space Karhunen-Loève series, $M_{m_T}(u)$ can be evaluated explicitly in terms of the mean and covariance functions of E_o.

It is assumed for simplicity that, for a fixed time t, the field varies negligibly over the detector surface; that is, that only one spatial mode of the field (the lowest order mode) excites the detector. This assumption limits only slightly the generality of our results; the extension to many spatial modes is straightforward. Under this assumption, the expansion for $E_o(t, \vec{r})$ can be written

$$E_o(t, \vec{r}) = \sum_i E_i \phi_i(t, \vec{r})$$

$$= A^{-1/2} \sum_i E_i \psi_i(t),$$

where the $\{\phi_i\}$ are orthonormal over $[0, T]$ and A, the $\{\psi_i\}$ are orthonormal over $[0, T]$, and E_i is given by

$$E_i = \int_0^T \int_A E_o(t, \vec{r}) \phi_i^*(t, \vec{r}) \, d\vec{r}dt$$

$$= A^{1/2} \int_0^T E_o(t, \vec{r}_o) \psi_i^*(t) \, dt, \qquad \vec{r}_o \in A.$$

Thus we have reduced the space-time expansion in the $\{\phi_i\}$ to a simple time expansion in the $\{\psi_i\}$. Defining $a(t) = A^{1/2}E_o(t, \vec{r}_o)$, $\vec{r}_o \in A$, we can easily show that

$$m_T = \alpha \int_0^T |a(t)|^2 \, dt. \tag{6}$$

It is further assumed that a(t) can be written

$$a(t) = s(t) + n(t), \tag{7}$$

where s(t) is a deterministic signal, and n(t) is a zero-mean Gaussian random process with finite average energy, and covariance $K_n(t, \tau) = E\{n(t)n^*(\tau)\}$. It can be shown that the eigenvalues of $K_n(t, \tau)$ are the same as the eigenvalues of $K_{E_o}(t, \tau; \vec{r}, \vec{\rho})$; thus, the trace of $K_n(t, \tau)$ has physical meaning as average noise power, and is independent of the detector area. On the other hand, the signal power $(s, s) = \int_0^T |s(t)|^2 \, dt$ is directly proportional to the area A.

Returning to Eq. 5, a particularly revealing representation for $M_{m_T}(u)$ is in terms of the cumulants $\{\kappa_i\}$ of m_T, defined by

$$\ln M_{m_T}(u) = \sum_{n=1}^{\infty} \frac{\kappa_n}{n!} u^n. \tag{8}$$

It has been shown that[6]

$$\kappa_i = \alpha^i \left[(i-1)! \text{ Tr } K_n^{(i)} + i! \left(s, K_n^{(i-1)} s \right) \right], \tag{9}$$

where $K_n^{(i)}$ is the i^{th} "iterated kernel" (operator product),[6, 8] and

$$(a, Kb) = \int_0^T \int_0^T a(t) K(t, \tau) b^*(\tau) \, d\tau dt$$

for real, symmetric K. With $\{\lambda_i\}$ the eigenvalues of K_n, and $\{s_i\}$ the coefficients in the expansion of s(t) in the Karhunen-Loève $\{\psi_i\}$-basis, we can write κ_i in the alternative form,

$$\kappa_i = a^i\left[(i-1)!\ \sum_\ell \lambda_\ell^i + i!\ \sum_\ell |s_\ell|^2 \lambda_\ell^{i-1}\right]. \tag{10}$$

The cumulants $\{\tilde{\kappa}_i\}$ of N_T are defined by a relation similar to Eq. 8; it is easily shown that[6]

$$\tilde{\kappa}_n = \sum_{i=1}^{n} A(n,i)\ \frac{\kappa_i}{i!}, \tag{11}$$

where

$$A(n,i) = \sum_{k=1}^{i} \binom{i}{k} (-1)^{i-k}\ k^n > 0. \tag{12}$$

Now, since the cumulants of a discrete distribution are identical if and only if it is Poisson, we can gauge the "Poisson-ness" of N_T by examining the degree to which its cumulants $\{\tilde{\kappa}_i\}$ are equal. Rewriting Eq. 11,

$$\tilde{\kappa}_n = \kappa_1 + \sum_{i=2}^{n} A(n,i)\ \frac{\kappa_i}{i!}, \tag{13}$$

we see that a sufficient set of conditions for the approximate equality of all of the $\{\tilde{\kappa}_i\}$ is

$$\sum_{i=2}^{n} \frac{A(n,i)}{i!} \cdot \frac{\kappa_i}{\kappa_1} \ll 1, \qquad \forall\ n \geq 2. \tag{14}$$

It is instructive to write the characteristic function of N_T in terms of the "cumulant error" $\tilde{\kappa}_n - \kappa_1$:

$$M_{N_T}(jv) = \exp \sum_{n=1}^{\infty} \frac{\tilde{\kappa}_n}{n!} (jv)^n$$

$$= \exp \kappa_1(e^{jv}-1) \exp \sum_{n=1}^{\infty} (\tilde{\kappa}_n-\kappa_1)\ \frac{(jv)^n}{n!}. \tag{15}$$

If the conditions (14) are satisfied for n = 2, 3, ..., m, then

$$M_{N_T}(jv) = \exp \kappa_1(e^{jv}-1) \exp \sum_{n=m+1}^{\infty} (\tilde{\kappa}_n-\kappa_1) \frac{(jv)^n}{n!}. \tag{16}$$

This is the characteristic function $\exp \kappa_1(e^{jv}-1)$ of a pure Poisson distribution multiplied by a perturbation factor that approaches unity as $m \to \infty$. Equation 15 is closely related to the Gram-Charlier Type B series[9] for N_T; indeed, if Eq. 15 is written in terms of the central moments $\{\eta_i\}$ of m_T and Fourier-transformed, the result is

$$p_{N_T}(k) = \frac{\kappa_1^k e^{-\kappa_1}}{k!} \left\{1 + \sum_{n=2}^{\infty} \frac{\eta_n}{\kappa_1^n} L_n^{k-n}(\kappa_1)\right\}, \tag{17}$$

where $L_n^{\beta}(x)$ is the Laguerre polynomial of degree n and order β. As the $\{\eta_i\}$ are complicated functionals of s(t) and $K_n(t,\tau)$, it is more convenient to work with the characteristic function $M_{N_T}(jv)$.

By using some well-known operator inequalities, we can obtain a set of sufficient conditions for (14), which are in a form with considerably greater physical significance. The inequalities are

$$\begin{aligned} \left(s, K_n^{(i-1)} s\right) &\leq \lambda_{max}^{i-1}(s,s) \\ \operatorname{Tr} K_n^{(i)} &\leq \lambda_{max}^{i-1} \operatorname{Tr} K_n \end{aligned} \tag{18}$$

with equality (given a particular s(t)) when all of the nonzero eigenvalues of K_n are the same. λ_{max} is the largest eigenvalue of $K_n(t,\tau)$. Combining Eqs. 9 and 14, and using the inequalities (18), we get

$$\sum_{i=2}^{n} \frac{A(n,i)}{i} a^{i-1}\lambda_{max}^{i-1} \left[1 + (i-1) \frac{(s,s)}{\operatorname{Tr} K_n + (s,s)}\right] \ll 1, \qquad \forall\, n \geq 2,$$

which is certainly satisfied for any s(t) and $K_n(t,\tau)$ if

$$\sum_{i=2}^{n} A(n,i)\, a^{i-1}\lambda_{max}^{i-1} \ll 1, \qquad \forall\, n \geq 2. \tag{19}$$

Note that this is a restriction on the noise energy per temporal mode.

If conditions (19) are satisfied for $n = 2, 3, \ldots, m$, then $\tilde{\kappa}_n \cong \kappa_1$, $n \leq m$, and N_T is "approximately Poisson" with mean κ_1; however, additional conditions must be satisfied to ascertain the value of κ_1 to the order of approximation that we have established. In general, $\kappa_1 = a \text{ Tr } K_n + a(s,s)$, but for $n \leq m$ we have neglected terms in Eq. 13 involving $s(t)$ and $K_n(t,\tau)$, so we must ensure that an insignificant term has not been retained in κ_1. Expanding Eq. 13,

$$\tilde{\kappa}_n = a \text{ Tr } K_n + a(s,s) + \sum_{i=2}^{n} \frac{A(n,i)}{i} a^i \text{ Tr } K_n^{(i)} + \sum_{i=2}^{n} A(n,i)\, a^i \left(s, K_n^{(i-1)} s\right), \tag{20}$$

we obtain four relations by comparing each of the first two terms with each of the remaining terms. If κ_1 is to be approximated by the first two terms in Eq. 20, the four relations are inequalities that must be satisfied. Two of these are satisfied if (19) is satisfied for $n \leq m$. The other two are

$$\begin{aligned} a \text{ Tr } K_n &\gg \sum_{i=2}^{n} A(n,i)\, a^i \left(s, K_n^{(i-1)} s\right) \\ a(s,s) &\gg \sum_{i=2}^{n} \frac{A(n,i)}{i} a^i \text{ Tr } K_n^{(i)}. \end{aligned} \tag{21}$$

A sufficient set of conditions for (21) can be obtained by using the inequalities (18); the result is

$$\sum_{i=2}^{n} \frac{A(n,i)}{i} a^{i-1} \lambda_{max}^{i-1} \ll \frac{(s,s)}{\text{Tr } K_n} \ll \left[\sum_{i=2}^{n} A(n,i)\, a^{i-1} \lambda_{max}^{i-1}\right]^{-1}. \tag{22}$$

The upper and lower limits define an interval that shrinks as n increases. Thus (22) is in reality a single inequality, which need be satisfied only for $n = m$ to ensure that it is satisfied for all smaller n. If the "signal-to-noise ratio" $(s,s)/\text{Tr } K_n$ falls within the bounds of (22), it is then valid to write $\kappa_1 \cong a \text{ Tr } K_n + a(s,s)$. Otherwise, κ_1 is better approximated as $a \text{ Tr } K_n$ or $a(s,s)$ according as $(s,s)/\text{Tr } K_n$ is beyond the lower or the upper limit.

An important example that can be worked for the purposes of illustration is the case of bandlimited white noise. By taking the first M eigenvalues of K_n to be the same (N_o), and the rest to be zero, (19) and (22) become

$$\sum_{i=2}^{n} A(n,i)(aN_o)^{i-1} \ll 1, \qquad 2 \leq n \leq m$$

$$\sum_{i=2}^{m} \frac{A(m,i)}{i} (aN_o)^{i-1} \ll \frac{(s,s)}{MN_o} \ll \left[\sum_{i=2}^{m} A(m,i)(aN_o)^{i-1}\right]^{-1}. \tag{23}$$

For m = 2, these are

$$2\,\alpha N_o \ll 1,$$

$$\alpha N_o \ll \frac{(s,s)}{MN_o} \ll (2\,\alpha N_o)^{-1}, \tag{24}$$

which, if satisfied, yield approximately equal mean and variance,

$$\overline{N_T} \cong \text{var}\,(N_T) \cong \alpha(s,s) + \alpha\,M\,N_o. \tag{25}$$

αN_o, the average number of counts per noise mode, is for visible wavelengths typically of the order of $10^{-7} - 10^{-6}$, so conditions (24) are not unreasonably restrictive.

It should be pointed out that, although Eq. 16 gives a quantitative measure of the degree to which N_T is Poisson, it is not in a form convenient for actual calculation. Further work remains to be done in the area of finding useful bounds for the difference between $p_{N_T}(k)$ and a pure Poisson distribution.

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