IV. ELECTRODYNAMICS OF MEDIA*

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A. LOW-POWER FREQUENCY MIXING EXPERIMENTS USING METAL-METAL POINT-CONTACT DIODES

In order to make precise measurements of laser frequencies in the infrared region, it is generally necessary to run a laser cw at a low power level. This is done to insure frequency stability that is very hard to obtain in high-power, Q-switched lasers. Microwave techniques such as heterodyning can be used for extremely accurate measurements of absolute laser frequencies, provided a radiation detection and mixing element can be found that will operate reliably under the conditions of low laser power and cw operation.

We have completed a frequency mixing experiment to determine the frequency separations of adjacent P-transitions of a 10.6-μ CO_2 laser, using a tungsten-copper point contact diode as the mixing element. Measurements of these separations have been made previously. In one case,[1] a GaAs crystal was the mixing element in an experiment using cw lasers. In another,[2] a metal-metal point contact diode was used; however, the experiment was done with a high-power Q-switched laser and thus is lacking in accuracy. We have successfully duplicated the accurate results of the GaAs measurements, using the metal-metal mixing element under the conditions of low power, cw, and high-frequency stability of the CO_2 lasers. A similar technique has been used recently in measurements of the 28-μm and 78-μm cw water-vapor laser lines.[3] Our experiment is, however, the first to successfully measure the frequency separations that exist between adjacent P-transitions in a CO_2 laser, using a metal-metal diode at low power!

A schematic diagram of our experiment is shown in Fig. IV-1. The two CO_2 lasers that are stabilized at line center of their respective P-transitions are focused on the point-contact diode. The power of each laser, after attenuation, was approximately 80 mW. The transition of both lasers was constantly monitored by using an infrared spectrometer. The difference frequency between adjacent transitions is

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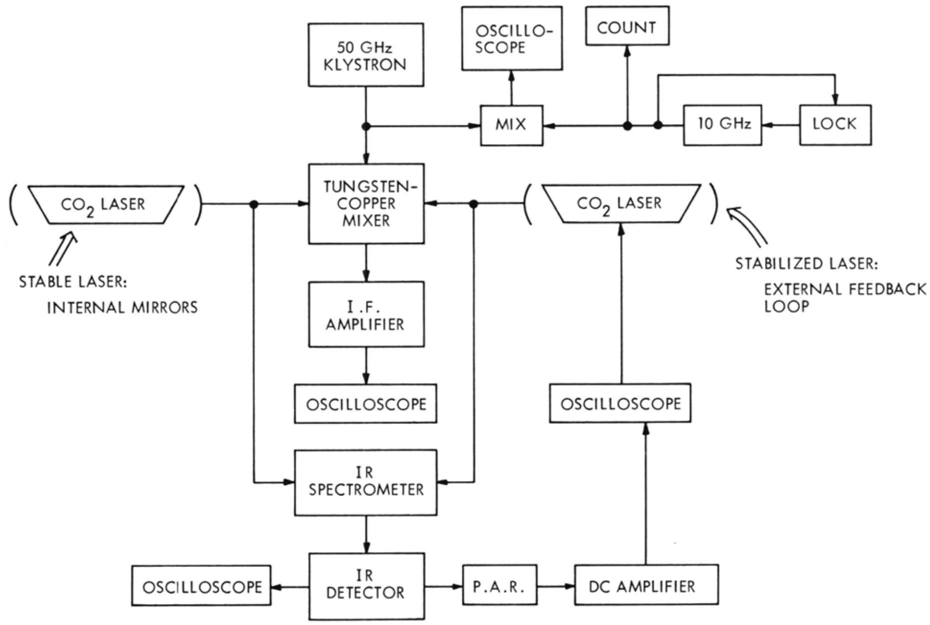

Fig. IV-1. Experimental arrangement.

in the neighborhood of 50 GHz. If we mix the difference signal with a variable 50-GHz klystron signal, it will produce a beat note when passed through an IF amplifier and viewed on an oscilloscope screen. The exact frequency of the 50-GHz klystron is determined by mixing it with a locked, ultrastable 10-GHz klystron whose frequency is precisely known by using a digital counter.

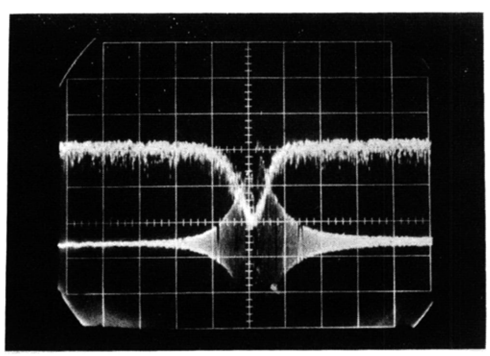

Fig. IV-2. Oscilloscope traces of beat-frequency of lasers and of beat-frequency of 50-GHz klystron with the harmonic of a stable klystron.

Figure IV-2 shows the synchronized oscilloscope traces of the beat between the lasers and the 50-GHz source (upper trace), and the beat between the 50 GHz klystron and the fifth harmonic of the 10-GHz klystron (lower trace).

The measurements thus derived were accurate within ±1 MHz, and were in close agreement with the previously reported frequencies. (Microwave equipment for this experiment was loaned by the Electronics Research Center, NASA.)

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B. QUANTUM THEORY OF ČERENKOV RADIATION IN A UNIAXIAL CRYSTAL

Visible light radiation emitted by pure liquids under the action of fast electrons was first observed experimentally by Čerenkov[1] in 1934. An extensive theoretical treatment based upon the classical macroscopic electrodynamics was launched by Frank and Tamm[2] in 1937. The success of the macroscopic theory in this and other respects greatly stimulated the development of phenomenological quantum electrodynamics. The quantum theory of Čerenkov radiation in isotropic media has been fully investigated.[3-7] In this report we apply a quantum theoretical method to study the Čerenkov radiation of a Dirac particle moving along the optical axis of a uniaxial crystal. An experiment of this nature has been carried out[8] and shown to be consistent with predictions of the classical theory.[9] We shall first investigate radiation photon fields in the crystal and free electron states, and then employ time-dependent perturbation theory to account for their interactions.

Radiation fields in a uniaxial medium are quantized by imposing commutation relations.[10]

$$\left[D_i(\bar{r},t), B_j(\bar{r}',t)\right] = -i\hbar\epsilon_{ijk} \frac{\partial}{\partial x_k} \delta(\bar{r}-\bar{r}') \tag{1}$$

and all other commutators zero to the classical fields. The Hamiltonian takes the form

$$H_e = \frac{1}{2}\int d^3x \left(\frac{1}{\mu} B^2 + \frac{1}{\epsilon}\left(D_x^2+D_y^2\right) + \frac{1}{\epsilon_z} D_z^2\right), \tag{2}$$

where the z direction is chosen to be along the optical axis. Since the presence of the medium introduces inherently a preferred reference frame, we do not attempt a covariant treatment.

Let

$$\bar{B} = \nabla \times \bar{A}. \tag{3}$$

We define annihilation and creation operators a and a^+ by

$$a_j = \left(\frac{1}{\alpha_j} A_j + i\alpha_j D_j\right)/\sqrt{2\hbar} \tag{4}$$

$$a_j^+ = \left(\frac{1}{\alpha_j} A_j - i\alpha_j D_j\right)/\sqrt{2\hbar}. \tag{5}$$

The reality conditions for $\overline{A}(r,t)$ and $\overline{D}(r,t)$ are satisfied, since

$$A_j = \sqrt{\frac{\hbar}{2}}\, \alpha_j\left(a_j^+ + a_j\right) \tag{6}$$

$$D_j = \sqrt{\frac{\hbar}{2}}\, \frac{i}{\alpha_j}\left(a_j^+ - a_j\right), \tag{7}$$

$\overline{A}^+ = \overline{A}$, and $\overline{D}^+ = \overline{D}$, provided $\alpha_j^* = \alpha_j$. Insertion of (3)-(5) in (2) gives the commutation relations

$$\left[A_i(\overline{r},t), D_j(\overline{r}',t)\right] = -i\hbar\delta_{ij}\delta(\overline{r}-\overline{r}') \tag{8}$$

$$\left[a_i, a_j^+\right] = \delta_{ij}. \tag{9}$$

Equation 9 justifies our claim that a and a^+ are the annihilation and creation operators. In (4)-(7) the α_j are unknown constants to be determined.

Longitudinal components of all field operators can be eliminated, and the Hamiltonian can be easily diagonalized if we focus our attention on the propagation vector of a particular photon. To do that, we make a transition to momentum space.

Let

$$\dot{\overline{A}}(\overline{r}) = (2\pi)^{-3/2} \int d^3k\, \overline{A}(\overline{k})\, e^{-i\overline{k}\cdot\overline{r}} \tag{10}$$

and

$$\overline{D}(\overline{r}) = (2\pi)^{-3/2} \int d^3k\, \overline{D}(\overline{k})\, e^{-i\overline{k}\cdot\overline{r}}. \tag{11}$$

Again the reality condition requires $\overline{D}(-\overline{k}) = \overline{D}^+(\overline{k})$ and $\overline{A}(-\overline{k}) = \overline{A}^+(\overline{k})$. By Eqs. 6 and 7, we obtain

$$A_j(\overline{k}) = \sqrt{\frac{\hbar}{2}}\, \alpha_j\left(a_j^+(\overline{k}) + a_j(-\overline{k})\right) \tag{12}$$

and

$$D_j(\bar{k}) = \sqrt{\frac{\hbar}{2}}\frac{i}{\alpha_j}\left(a_j^+(\bar{k})-a_j(-\bar{k})\right). \tag{13}$$

Let us assume that the photon with propagation vector $\bar{k}$ makes an angle with the optical axis. We can choose for each $\bar{k}$ vector an orthogonal coordinate system such that the unit vectors $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ coincide with the spherical coordinate unit vectors

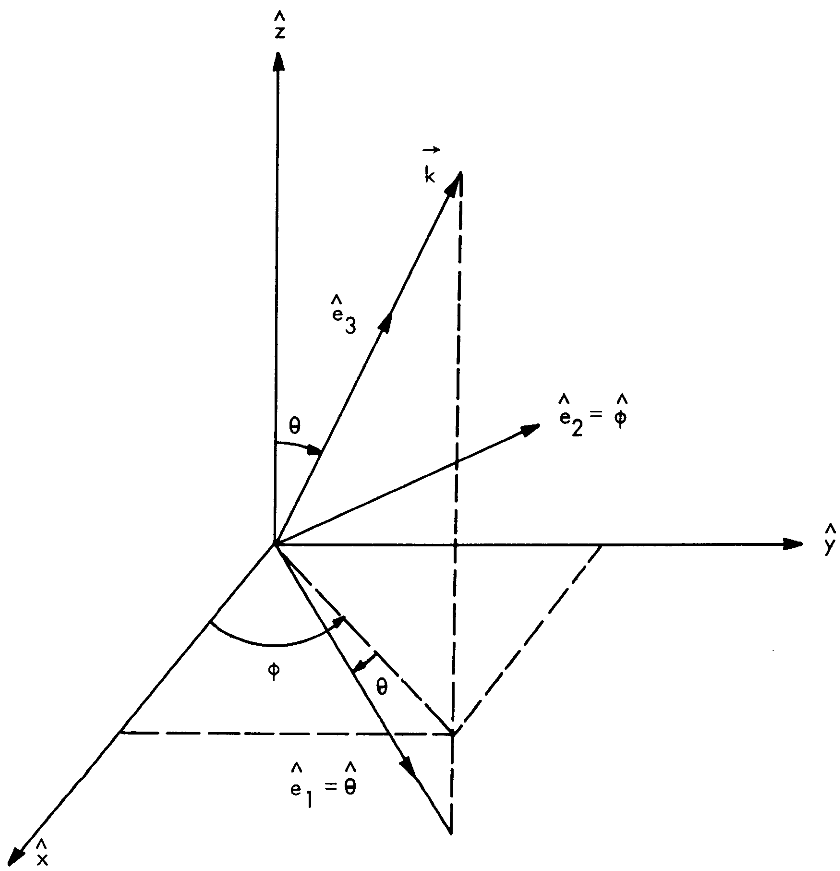

Fig. IV-3. Coordinate system for each $\bar{k}$-vector.

$\hat{\theta}$. $\hat{\phi}$ and $\hat{r}$, respectively (Fig. IV-3). $\bar{k}$ points in the direction of $\hat{e}_3 = \hat{r}$. In such a coordinate system, it is obvious that $D_3(\bar{k}) = 0$ and $B_3(\bar{k}) = 0$. The Hamiltonian then becomes

$$H_e = \frac{1}{2}\int d^3k \left\{\frac{k^2}{\mu}\left(A_1(\bar{k})A_1^+(\bar{k})+A_2(\bar{k})A_2^+(\bar{k})\right) + \frac{1}{\epsilon}\left(1+\left(\frac{\epsilon}{\epsilon_z}-1\right)\sin^2\theta\right) D_1(\bar{k})\, D_1^+(\bar{k}) + \frac{1}{\epsilon} D_2(\bar{k})\, D_2^+(\bar{k})\right\}$$

$$= \frac{c\hbar}{2}\int d^3k\, k \left\{\epsilon_1\left(a_1(\bar{k})a_1^+(\bar{k})+a_1^+(\bar{k})a_1(\bar{k})\right) + \epsilon_2\left(a_2(\bar{k})a_2^+(\bar{k})+a_2^+(\bar{k})a_2(\bar{k})\right)\right\}, \tag{14}$$

provided we choose a_j to be

$$|a_1|^2 = \frac{c\mu}{k}\epsilon_1, \quad \epsilon_1 = \left(1 + \left(\frac{\epsilon}{\epsilon_z} - 1\right)\sin^2\theta\right)^{1/2} \Big/ n \tag{15}$$

$$|a_2|^2 = \frac{c\mu}{k}\epsilon_2, \quad \epsilon_2 = 1/n. \tag{16}$$

We see from Eq. 14 that the Hamiltonian has been diagonalized. Operation of the Hamiltonian on photon states gives the total energy of the ordinary and extraordinary photons. All ordinary photons are linearly polarized perpendicular to the plane defined by the $\bar{k}$ vector and the $\hat{z}$-axis, whereas all extraordinary photons are polarized in the plane. The energy of an ordinary photon with momentum $\hbar\vec{k}$ is $\hbar ck\epsilon_2$, and that of an extraordinary photon is $\hbar ck\epsilon_1$.

The fast-traveling charged particle that causes Čerenkov radiation is assumed to obey the Dirac equation

$$H_p\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{17}$$

where

$$H_p = \vec{a}\cdot\bar{p}c + \beta mc^2 \tag{18}$$

in the absence of electromagnetic fields. The Dirac matrices $\vec{a}$ and β are

$$\vec{a} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \tag{19}$$

$$\beta = \begin{bmatrix} \bar{\bar{I}} & 0 \\ 0 & -\bar{\bar{I}} \end{bmatrix} \tag{20}$$

where the $\vec{\sigma}$ are Pauli spin matrices. The plane-wave solutions to the Dirac equation (17) can be written[11]

$$\psi = u(\bar{p})\exp[i(Et - \bar{p}\cdot\bar{r})]. \tag{21}$$

The four-component spinor $u(\bar{p})$ satisfies

$$(\vec{a}\cdot\bar{p} + \beta)U = EU \tag{22}$$

and can be written in terms of the upper and lower two-component spinors u and v

$$U = \begin{bmatrix} u \\ v \end{bmatrix}. \tag{23}$$

We consider only positive energy solutions of the particle, where $E = (p^2c^2+m^2c^4)^{1/2}$. Any spin state can be written as a superposition of the two linearly independent solutions $\chi^{1/2}$ and $\chi^{-1/2}$. Thus

$$U = \begin{bmatrix} b_+ \\ b_- \end{bmatrix} = b_+\chi^{1/2} + b_-\chi^{-1/2} \tag{24}$$

$$\chi^{1/2} = \sqrt{\frac{E+mc^2}{2E}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{25}$$

$$\chi^{-1/2} = \sqrt{\frac{E+mc^2}{2E}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{26}$$

Normalization leads to $|b_+|^2 + |b_-|^2 = 1$, and the Dirac equations give the lower two-component spinor

$$v = \frac{c\bar{p}\cdot\vec{\sigma}}{E + mc^2} u. \tag{27}$$

The transition of the electron from an initial state with momentum $\bar{p}$ and energy E to a state with momentum $\bar{p}'$ under emission of a photon with momentum $\hbar\bar{k}$ and energy $\hbar\nu$ will now be treated by time-dependent perturbation theory.

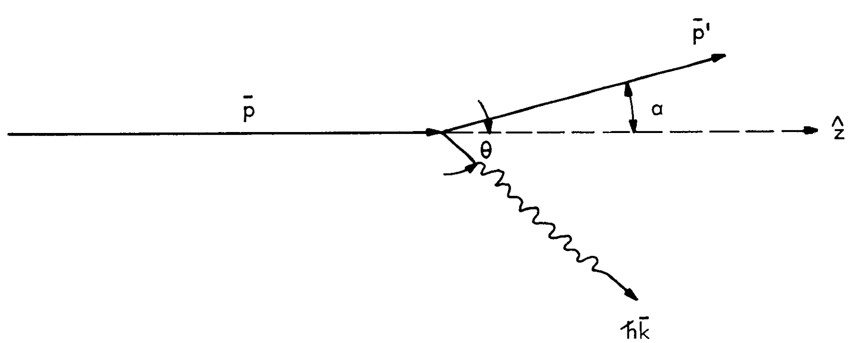

Fig. IV-4. Čerenkov radiation condition in quantum theory.

First, we investigate the radiation condition. As the Dirac particle moves along the optical axis of the crystal, it emits a photon in a direction θ with respect to the $\hat{z}$ axis and itself recoils to the direction of angle α (Fig. IV-4). Conservation of energy and momentum

$$E = E' + \hbar\nu \tag{28}$$

and

$$\bar{p} = \bar{p}' + \hbar\bar{k} \tag{29}$$

yield for angle θ

$$\cos\theta = (2\hbar\nu E+\hbar^2k^2c^2-\hbar^2\nu^2)/2pc\hbar kc. \tag{30}$$

In classical theory only the first term will occur, the last two terms are due to quantum corrections which are proportional to $\hbar$.

By time-dependent perturbation theory, the transition probability per unit time from the initial state of an electron to a final state of an electron and a photon is given by[12]

$$w = \frac{2\pi}{\hbar}\rho(k)\left|\langle f|H_i|i\rangle\right|^2, \tag{31}$$

where $\rho(k)$ is the energy density of final states, and H_i the interaction Hamiltonian. The bra $\langle f|$ and ket $|i\rangle$ are, respectively, the unperturbed initial and final states, and $\rho(k)$ can be determined by considering photons emitted into the solid angle $d\Omega$,

$$\rho(k) = k^2d\Omega(dk/dE_f). \tag{32}$$

Since

$$E_f = \hbar\nu + \left[m^2c^4+\vec{p}c-\hbar\vec{k}_c)^2\right]^{1/2}, \tag{33}$$

by making use of (30), (33) and $d\Omega = \sin\theta\, d\theta d\phi$, we obtain

$$\rho(k) = \frac{E-\hbar\nu}{c^2p\hbar}\, kdkd\phi. \tag{34}$$

Equation 34 is true whenever the energy of a photon is linearly related to the magnitude of its momentum.

The interaction Hamiltonian takes the form

$$H_i = -ce\vec{\alpha}\cdot A. \tag{35}$$

To find the square of the matrix element of H_i averaged over all the spin states of the initial and final states of the electron, we assume that the momentum of the electron is much larger than that of the photon; that is, $p \ll \hbar k$. The result is

$$|\langle f|H_i|i\rangle|^2 = \frac{\hbar c^3 e^2 \mu}{16\pi^3 k} \; |\langle \psi_p' |\bar{e}\cdot\vec{\alpha}|\psi_p\rangle|^2 \tag{36}$$

$$\approx \frac{\hbar c^3 e^2 \mu}{16\pi^3 k E^2} \; (\bar{e}\cdot c\bar{p})^2, \tag{37}$$

where

$$\bar{e} = \hat{e}_1\sqrt{\epsilon_1} + \hat{e}_2\sqrt{\epsilon_2}. \tag{38}$$

From Eq. 37 we know that the radiated photon must be polarized in the $\bar{p} - \bar{k}$ plane. Thus only the extraordinary photons are emitted. The transition probability per unit time can now be determined by inserting (34) and (37) in (31). The total energy radiated per unit time is found to be

$$\frac{dW}{dt} = \int w(\hbar c k \epsilon_1)$$

$$= \frac{c^3 e^2 \mu\beta}{4\pi} \int \epsilon_1^2 \sin^2\theta k\, dk, \tag{39}$$

where $\sin^2\theta$ is determined by Eqs. 30 and 15. In the classical limit as $\hbar \to 0$, it can be shown that (39) reduces to the known results.[9]

These results can be easily extended to the case of electrons moving perpendicular to the optical axis. It is observed that in calculating the transition probability if we had used the exact spinor form of the final states, we would have found ordinary photon radiation as well as the extraordinary photons. Also, in our theoretical derivation, the dispersive nature of the crystal has not been taken account of. Investigation of all of these and other aspects is now in progress.

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