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# Numerical Analysis of an Implicit Fully Discrete Local Discontinuous Galerkin Method for the Fractional Zakharov–Kuznetsov Equation

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**Abstract.** In this paper we develop and analyze an implicit fully discrete local discontinuous Galerkin (LDG) finite element method for a time-fractional Zakharov–Kuznetsov equation. The method is based on a finite difference scheme in time and local discontinuous Galerkin methods in space. We show that our scheme is unconditional stable and  $L^2$  error estimate for the linear case with the convergence rate  $O(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}})$  through analysis.

**Keywords:** time-fractional partial differential equations, Zakharov–Kuznetsov equation, local discontinuous Galerkin method, stability, error estimates.

**AMS Subject Classification:** 35Q53; 65M60.

## 1 Introduction

During the last few decades the numerical modeling and simulation for fractional calculus have been the focus of many studies, and various fractional order differential equation have been solved including e.g. space-time fractional partial differential equation [10], space and time fractional Fokker–Planck equation [5], fractional order two point boundary value problem [7], the fractional KdV equation [13], fractional convection-diffusion equation [14], fractional partial differential equations fluid mechanics [15], fractional KdV–Burgers–Kuramoto equation [16] and so on. Solving such fractional partial differential using numerical schemes has been stimulated due to their frequent appearance in various applications in physics and engineering.

The Zakharov–Kuznetsov (ZK) equation is a generalization of the Korteweg–de Vries (KdV) equation. It was obtained by Zakharov and Kuznetsov [19] to describe the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform

magnetic field. Several properties of this equation including existence and stability of solitary wave solutions have been extensively studied in the literature [2, 6]. Some methods [1, 17] have been used to handle the integer-order systems, however, to the best of our knowledge, the study of the fractional Zakharov–Kuznetsov (ZK) equation has not been widespread.

In this paper, we consider the following time-fractional ZK equation

$$\begin{aligned} \frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} + g(u)_x + (\nabla^2 u)_x &= 0, \quad (x, y, t) \in \Omega \times [0, T], \\ u(x, y, 0) &= u_0(x, y), \quad x \in \Omega, \end{aligned} \quad (1.1)$$

where  $\nabla^2 = \partial_x^2 + \partial_y^2$  is the isotropic Laplacian.  $0 < \alpha \leq 1$  is a parameter describing the order of the fractional time. In this paper we do not pay attention to boundary conditions, hence the solution is considered to be either periodic or compactly supported.

The time fractional derivative in the equation (1.1), uses the Caputo fractional partial derivative of order  $\alpha$ , defined as [11]

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, y, s)}{\partial s} \frac{ds}{(t-s)^\alpha} & \text{if } 0 < \alpha < 1, \\ \frac{\partial u(x, y, t)}{\partial t} & \text{if } \alpha = 1, \end{cases} \quad (1.2)$$

here  $\Gamma(\cdot)$  is the Gamma function.

There are only a few numerical works in the literature to solve the fractional ZK equation. Molliq et al. [12] presents the approximate analytical solution of a fractional ZK equation using the variational iteration method. Yildirim [18] extended He's homotopy perturbation method to derive explicit and numerical solutions of fractional ZK equations. The discontinuous Galerkin finite element method is a very attractive method for partial differential equations because of its flexibility and efficiency in terms of mesh and shape functions, and the higher order of convergence can be achieved without over many iterations.

In the present paper we propose an implicit fully discrete local discontinuous Galerkin (LDG) finite element method for solving time-fractional ZK equation. Our fully discrete scheme is based on a finite difference scheme in time and local discontinuous Galerkin methods in space. Stability is ensured by a careful choice of interface numerical fluxes. We prove that our scheme is unconditionally stable and  $L^2$  error estimate for the linear case with the convergence rate  $O(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\alpha/2} h^{k+\frac{1}{2}})$ .

This paper is organized as follows. First we introduce some basic notations and mathematical preliminaries, then in Section 3 we discuss the fully discrete LDG scheme for the fractional ZK equation (1.1), and prove that the scheme is unconditionally stable, and the numerical solution is convergent. Finally in Section 4 concluding remarks are provided.

## 2 Notations and Auxiliary Results

### 2.1 Notations and projection

In the domain  $[a, b] \times [c, d]$  we define cells

$$I_{i,j} = \left\{ (x, y) : x_{i-\frac{1}{2}} \leq x \leq x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}} \leq y \leq y_{j+\frac{1}{2}} \right\},$$

for  $i = 1, \dots, N_x, j = 1, \dots, N_y$ , where

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \dots < x_{N_x+\frac{1}{2}} = b, \quad c = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \dots < y_{N_y+\frac{1}{2}} = d,$$

We define  $I_i$  and  $J_j$  as:  $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ ,  $J_j = [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$  for  $i = 1, \dots, N_x, j = 1, \dots, N_y$ . So we have  $I_{i,j} = I_i \times J_j$ .

We denote by  $u_{i+\frac{1}{2},y}^+$  and  $u_{i+\frac{1}{2},y}^-$  the values of  $u$  at  $x_{i+1/2}$ , from the right cell  $I_{i+1} \times J_j$  and from the left cell  $I_i \times J_j$  when  $y \in J_j$ , on all vertical edges, respectively. Similarly, we denote by  $u_{x,j+1/2}^+$  and  $u_{x,j+1/2}^-$  the values of  $u$  at  $y_{j+1/2}$ , from the top cell  $I_i \times J_{j+1}$  and from the bottom cell  $I_i \times J_j$ , when  $x \in I_i$ , on all horizontal edges, respectively.

Define the space  $V_h^k$  as the space of tensor product piecewise polynomials of degree at most  $k$  in each variable on every element, i.e.

$$V_h^k = \{v : v \in Q^k(I_i \times J_j), \forall (x, y) \in I_i \times J_j, i = 1, \dots, N_x, j = 1, \dots, N_y\}.$$

### 2.2 Projection

In order to prove the error estimates for two-dimensional problems in Cartesian meshes, we use the same projections as in [3]. However, for the sake of completeness, we present the definition. First, we will give the projection in one dimension  $[a, b]$ , denoted by  $\mathcal{P}_x$ , i.e., for each  $j$ ,

$$\int_{I_i} (\mathcal{P}_x \omega(x) - \omega(x))v(x) = 0, \quad \forall v \in P^k(I_i),$$

and special projection  $\mathcal{P}_x^\pm$ , i.e., for each  $j$ ,

$$\int_{I_i} (\mathcal{P}_x^+ \omega(x) - \omega(x))v(x) = 0, \quad \forall v \in P^{k-1}(I_i), \quad \mathcal{P}_x^+ \omega(x_{i-\frac{1}{2}}^+) = \omega(x_{i-\frac{1}{2}}),$$

$$\int_{I_i} (\mathcal{P}_x^- \omega(x) - \omega(x))v(x) = 0, \quad \forall v \in P^{k-1}(I_i), \quad \mathcal{P}_x^- \omega(x_{i+\frac{1}{2}}^-) = \omega(x_{i+\frac{1}{2}}).$$

Similarly, we can also define the projection  $\mathcal{P}_y, \mathcal{P}_y^+, \mathcal{P}_y^-$ .

On a rectangle  $[a, b] \times [c, d]$ , for a function  $\omega(x, y)$ , define

$$\mathbb{P}\omega = \mathcal{P}_x \otimes \mathcal{P}_y \omega(x, y), \quad \mathbb{P}^\pm \omega = \mathcal{P}_x^\pm \otimes \mathcal{P}_y^\pm \omega(x, y), \tag{2.1}$$

where the subscripts indicate the application of the one dimensional operators  $\mathcal{P}$  or  $\mathcal{P}^\pm$  with respect to the corresponding variable. We list some properties for the projections  $\mathbb{P}$ :

$$\int_{I_i} \int_{J_j} (\mathbb{P}^\pm \omega(x, y) - \omega(x, y))v(x, y) dy dx = 0, \tag{2.2}$$

for any  $v(x, y) \in (P^{k-1}(I_i) \otimes P^k(J_j)) \cup (P^k(I_i) \otimes P^{k-1}(J_j))$ . Also

$$\begin{aligned} \int_{I_i} (\mathbb{P}^+ \omega(x, y_{j-\frac{1}{2}}^+) - \omega(x, y_{j-\frac{1}{2}})) v(x, y_{j-\frac{1}{2}}^+) &= 0, \quad \forall v \in Q^k(I_i \otimes J_j), \\ \int_{I_i} (\mathbb{P}^- \omega(x, y_{j+\frac{1}{2}}^-) - \omega(x, y_{j+\frac{1}{2}})) v(x, y_{j+\frac{1}{2}}^-) &= 0, \quad \forall v \in Q^k(I_i \otimes J_j) \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} \int_{J_j} (\mathbb{P}^+ \omega(x_{i-\frac{1}{2}}^+, y) - \omega(x_{i-\frac{1}{2}}, y)) v(x_{i-\frac{1}{2}}^+, y) &= 0, \quad \forall v \in Q^k(I_i \otimes J_j), \\ \int_{J_j} (\mathbb{P}^- \omega(x_{i+\frac{1}{2}}^-, y) - \omega(x_{i+\frac{1}{2}}, y)) v(x_{i+\frac{1}{2}}^-, y) &= 0, \quad \forall v \in Q^k(I_i \otimes J_j). \end{aligned} \quad (2.4)$$

For the projections (2.1), the following inequality holds [3, 8]

$$\|\omega^e\| + h\|\omega^e\|_\infty + h^{\frac{1}{2}}\|\omega^e\|_{\tau_h} \leq Ch^{k+1}, \quad (2.5)$$

where  $\omega^e = \mathbb{P}\omega - \omega$  or  $\omega^e = \mathbb{P}^\pm\omega - \omega$ . The positive constant  $C$ , solely depending on  $\omega$ , is independent of  $h$ .  $\|\omega^e\|$  and  $\|\omega^e\|_{\tau_h}$  denote the  $L^2$ -norm of  $\omega^e$  on  $\Omega$  and  $\tau_h$ , which are

$$\begin{aligned} \|\omega^e\| &= \left( \sum_{1 \leq i \leq N_x} \sum_{0 \leq j \leq N_y} \int_{I_i} \int_{J_j} (\omega^e)^2 \right)^{\frac{1}{2}}, \\ \|\omega^e\|_{\tau_h} &= \left( \sum_{0 \leq i \leq N_x} \int_c^d (((\omega^e)_{i+\frac{1}{2}}^+)^2 + ((\omega^e)_{i+\frac{1}{2}}^-)^2) dy \right. \\ &\quad \left. + \sum_{0 \leq j \leq N_y} \int_a^b (((\omega^e)_{x,j+\frac{1}{2}}^+)^2 + ((\omega^e)_{x,j+\frac{1}{2}}^-)^2) dx \right)^{\frac{1}{2}}, \end{aligned}$$

respectively, and  $\|\omega^e\|_\infty = \text{ess sup } |\omega^e|$ . Here and below we use  $C$  to denote a positive constant which may have a different value in each occurrence.

### 2.3 Numerical flux

In this paper we will be using the flux  $\widehat{g}(\phi^-, \phi^+)$  which is related to the discontinuous Galerkin spatial discretization.  $\widehat{g}(\phi^-, \phi^+)$  is a monotone numerical flux, which is dependent on the two values of the function  $\phi$  at the discontinuity point  $x_{j+\frac{1}{2}}$ , and satisfies the following conditions:

- (i) it is locally Lipschitz continuous, so it is bounded when  $\phi^\pm$  are bounded;
- (ii) it is consistent with the flux  $g(\phi)$ , i.e.,  $\widehat{g}(\phi, \phi) = g(\phi)$ ;
- (iii) it is a nondecreasing function of its first argument, and a nonincreasing function of its second argument.

### 3 Fully Discrete LDG Scheme

In this section we introduce the numerical scheme for the solution of equation (1.1). Let  $\Delta t = T/M$  be the time mesh size,  $M$  is a positive integer,  $t_n = n\Delta t$ ,  $n = 0, 1, \dots, M$  be mesh point. An approximation to time fractional derivative (1.2) can be obtained by simple quadrature formula given as [9]

$$\frac{\partial^\alpha u(x, y, t_n)}{\partial t^\alpha} = \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{m=0}^{n-1} b_m \frac{u(x, y, t_{n-m}) - u(x, y, t_{n-m-1})}{\Delta t} + \gamma^n(x, y), \tag{3.1}$$

where  $b_m = (m + 1)^{1-\alpha} - m^{1-\alpha}$ ,  $\gamma^n(x, y) \leq C(\Delta t)^{2-\alpha}$ ,  $C$  is dependent on  $u, T, \alpha$ . We know that

$$1 = b_0 > b_1 > b_2 > \dots > b_n > 0, \quad b_n \rightarrow 0 \quad (n \rightarrow \infty),$$

$$\sum_{m=1}^n (b_{m-1} - b_m) + b_n = 1. \tag{3.2}$$

We rewrite Eq. (1.1) as a system:

$$p = u_x, \quad q = p_x, \quad s = u_y, \quad r = s_x,$$

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} + g(u)_x + q_x + r_y = 0.$$

Let  $u_h^n, p_h^n, q_h^n, r_h^n, s_h^n \in V_h^k$  be the approximations of  $u(\cdot, t_n), p(\cdot, t_n), q(\cdot, t_n), r(\cdot, t_n), s(\cdot, t_n)$ , respectively,  $f^n(x) = f(x, t_n)$ . With (3.1), we define a fully discrete local discontinuous Galerkin scheme as follows: find  $u_h^n, p_h^n, q_h^n, r_h^n, s_h^n \in V_h^k$ , such that for all test functions  $v, \rho, \xi, \eta, \phi \in V_h^k$ ,

$$\begin{aligned} & \int_a^b \int_c^d u_h^n v \, dx \, dy - \beta \left( \int_a^b \int_c^d g(u_h^n) v_x \, dx \, dy - \sum_{i=1}^{N_x} \int_c^d ((\widehat{g}v^-)_{i+\frac{1}{2},y} - (\widehat{g}v^+)_{i-\frac{1}{2},y}) \, dy \right) \\ & - \beta \left( \int_a^b \int_c^d q_h^n v_x \, dx \, dy - \sum_{i=1}^{N_x} \int_c^d ((\widehat{q}_h v^-)_{i+\frac{1}{2},y} - (\widehat{q}_h v^+)_{i-\frac{1}{2},y}) \, dy \right) \\ & - \beta \left( \int_a^b \int_c^d r_h^n v_y \, dx \, dy - \sum_{j=1}^{N_y} \int_a^b ((\widetilde{r}_h v^-)_{x,j+\frac{1}{2}} - (\widetilde{r}_h v^+)_{x,j-\frac{1}{2}}) \, dx \right) \\ & = \sum_{m=1}^{n-1} (b_{m-1} - b_m) \int_a^b \int_c^d u_h^{n-m} v \, dx \, dy + b_{n-1} \int_a^b \int_c^d u_h^0 v \, dx \, dy, \tag{3.3} \\ & \int_\Omega q_h^n \rho \, dx \, dy + \int_a^b \int_c^d p_h^n \rho_x \, dx \, dy - \sum_{i=1}^{N_x} \int_c^d ((\widehat{p}_h^n \rho^-)_{i+\frac{1}{2},y} - (\widehat{p}_h^n \rho^+)_{i-\frac{1}{2},y}) \, dy = 0, \\ & \int_\Omega p_h^n \xi \, dx \, dy + \int_a^b \int_c^d u_h^n \xi_x \, dx \, dy - \sum_{i=1}^{N_x} \int_c^d ((\widehat{u}_h^n \xi^-)_{i+\frac{1}{2},y} - (\widehat{u}_h^n \xi^+)_{i-\frac{1}{2},y}) \, dy = 0, \\ & \int_\Omega r_h^n \eta \, dx \, dy + \int_a^b \int_c^d s_h^n \eta_x \, dx \, dy - \sum_{i=1}^{N_x} \int_c^d ((\widehat{s}_h^n \eta^-)_{i+\frac{1}{2},y} - (\widehat{s}_h^n \eta^+)_{i-\frac{1}{2},y}) \, dy = 0, \end{aligned}$$

$$\int_{\Omega} s_h^n \phi \, dx \, dy + \int_a^b \int_c^d u_h^n \phi_y \, dx \, dy - \sum_{j=1}^{N_y} \int_a^b ((\widetilde{u_h^n \phi^-})_{x,j+\frac{1}{2}} - (\widetilde{u_h^n \phi^+})_{x,j-\frac{1}{2},y}) \, dx = 0,$$

where  $\beta = (\Delta t)^\alpha \Gamma(2 - \alpha)$ . From (3.1) we know the truncation error is  $\gamma^n(x)$ .

The “hat” terms in (3.3) in the cell boundary terms from integration by parts are “numerical fluxes”, in order to ensure stability, we can take the following choices simply

$$\begin{aligned} \widehat{u_h^n} &= (u_h^n)^-, & \widehat{p_h^n} &= (p_h^n)^+, & \widehat{q_h^n} &= (q_h^n)^+, & \widetilde{u_h^n} &= (u_h^n)^-, \\ \widetilde{r_h^n} &= (r_h^n)^+, & \widetilde{r_h^n} &= \widetilde{r_h^n} - \tau [u_h^n], & \widehat{s_h^n} &= (s_h^n)^+, \end{aligned} \tag{3.4}$$

here  $\tau > 0$ . In order to control the boundary terms, a dissipative term is added in the flux  $\widetilde{r_h^n}$ . The flux  $\widehat{g}((u_h^n)^-, (u_h^n)^+)$  is a monotone flux as described in (2.3). Examples of monotone fluxes which are suitable for the local discontinuous Galerkin methods can be found in, e.g., [4]. For example, one could use the Lax–Friedrichs flux, which is given by

$$\widehat{g}^{LF}(w^-, w^+) = \frac{1}{2}(g(w^-) + g(w^+) - \lambda_0(w^+ - w^-)), \quad \lambda_0 = \max_w |g'(w)|.$$

We remark that the choice for the fluxes (3.4) is not unique. In fact the crucial part is taking  $\widehat{u_h^n}$  and  $\widehat{p_h^n}$  from opposite sides,  $\widetilde{u_h^n}$  and  $\widetilde{r_h^n}$  from opposite sides.

Since the problem is nonlinear, we can use an iterative method to solve the problem. Now the definition of the algorithm is complete.

Next we consider the numerical analysis of the scheme (3.3). First we examine the stability property.

**Theorem 1.** *For periodic or compactly supported boundary conditions, fully-discrete LDG scheme (3.3) with flux (3.4) is unconditionally stable, and the numerical solution  $u_h^n$  satisfies*

$$\begin{aligned} \|u_h^n\|^2 + \beta \sum_{i=1}^{N_x} \int_c^d ([p_h^n]^2 + [s_h^n]^2)_{i-\frac{1}{2},y} \, dy + 2\beta\tau \sum_{j=1}^{N_y} \int_a^b [u_h^n]_{x,j-\frac{1}{2}}^2 \, dx \\ \leq \|u_h^0\|^2, \quad n = 0, 1, \dots, M. \end{aligned}$$

*Proof.* Taking the test functions  $v = u_h^n$ ,  $\rho = \beta p_h^n$ ,  $\xi = -\beta q_h^n$ ,  $\eta = \beta s_h^n$ ,

$\phi = -\beta r_h^n$  in scheme (3.3), and with the fluxes choice (3.4) we obtain

$$\begin{aligned} & \|u_h^n\|^2 + \beta \tilde{G}(u_h^n) + \frac{\beta}{2} \sum_{i=1}^{N_x} \int_c^d ([p_h^n]^2 + [s_h^n]^2)_{i-\frac{1}{2},y} dy + \beta \tau \sum_{j=1}^{N_y} \int_a^b [u_h^n]_{x,j-\frac{1}{2}}^2 dx \\ & + \beta \sum_{i=1}^{N_x} \int_c^d (\Psi_1(u_h^n, q_h^n)_{i+\frac{1}{2},y} - \Psi_1(u_h^n, q_h^n)_{i-\frac{1}{2},y} + \Theta_1(u_h^n, q_h^n)_{i-\frac{1}{2},y}) dy \\ & + \beta \sum_{j=1}^{N_y} \int_a^b (\Psi_2(u_h^n, r_h^n)_{x,j+\frac{1}{2}} - \Psi_2(u_h^n, r_h^n)_{x,j-\frac{1}{2}} + \Theta_2(u_h^n, r_h^n)_{x,j-\frac{1}{2}}) dx \\ & = \sum_{m=1}^{n-1} (b_{m-1} - b_m) \int_a^b \int_c^d u_h^{n-m} u_h^n dx dy + b_{n-1} \int_a^b \int_c^d u_h^0 u_h^n dx dy, \end{aligned} \tag{3.5}$$

here

$$\tilde{G}(u_h^n) = - \int_a^b \int_c^d g(u_h^n) v_x dx dy + \sum_{i=1}^{N_x} \int_c^d ((\hat{g}v^-)_{i+\frac{1}{2},y} - (\hat{g}v^+)_{i-\frac{1}{2},y}) dy,$$

$$\begin{aligned} \Psi_1(u_h^n, q_h^n) &= -(q_h^n)^- (u_h^n)^- + \widehat{q}_h^n (u_h^n)^- + \widehat{u}_h^n (q_h^n)^-, \\ \Psi_2(u_h^n, r_h^n) &= -(r_h^n)^- (u_h^n)^- + \widetilde{\widetilde{r}}_h^n (u_h^n)^- + \widetilde{\widetilde{u}}_h^n (r_h^n)^-, \\ \Theta_1(u_h^n, q_h^n) &= -(q_h^n)^- (u_h^n)^- + (q_h^n)^+ (u_h^n)^+ + \widehat{q}_h^n (u_h^n)^- - \widehat{q}_h^n (u_h^n)^+ \\ &\quad + \widehat{u}_h^n (q_h^n)^- - \widehat{u}_h^n (q_h^n)^+, \\ \Theta_2(u_h^n, r_h^n) &= -(r_h^n)^- (u_h^n)^- + (r_h^n)^+ (u_h^n)^+ + \widetilde{\widetilde{r}}_h^n (u_h^n)^- - \widetilde{\widetilde{r}}_h^n (u_h^n)^+ \\ &\quad + \widetilde{\widetilde{u}}_h^n (r_h^n)^- - \widetilde{\widetilde{u}}_h^n (r_h^n)^+. \end{aligned}$$

If we take fluxes (3.4) then after some simple calculation, we easily obtain  $\Theta_1(u_h^n, q_h^n) = \Theta_2(u_h^n, r_h^n) = 0$ . Let us denote  $G(u) = \int_0^u g(u) du$ , and use a mean value theorem, then for the nonlinear term we obtain

$$\tilde{G}(u_h^n) = \sum_{i=1}^{N_x} \int_c^d (G'(\eta) - \hat{g}) [u_h^n]_{i-\frac{1}{2},y} dy \geq 0, \tag{3.6}$$

where  $\eta$  is a value between  $(u_h^n)_{i-\frac{1}{2},y}^-$  and  $(u_h^n)_{i-\frac{1}{2},y}^+$ . By the monotonicity of flux function  $\hat{g}$  we have the inequality (3.6).

Then based on the equation (3.5), and by the property (3.2), we can get

$$\begin{aligned} & \|u_h^n\|^2 + \frac{\beta}{2} \sum_{i=1}^{N_x} \int_c^d ([p_h^n]^2 + [s_h^n]^2)_{i-\frac{1}{2},y} dy + \beta \tau \sum_{j=1}^{N_y} \int_a^b [u_h^n]_{x,j-\frac{1}{2}}^2 dx \\ & = \sum_{m=1}^{n-1} (b_{m-1} - b_m) \int_a^b \int_c^d u_h^{n-m} u_h^n dx dy + b_{n-1} \int_a^b \int_c^d u_h^0 u_h^n dx dy \end{aligned}$$

$$\begin{aligned} &\leq \sum_{m=1}^{n-1} (b_{m-1} - b_m) \|u_h^{n-m}\| \|u_h^n\| + b_{n-1} \|u_h^0\| \|u_h^n\| \\ &\leq \frac{1}{2} \left( \sum_{m=1}^{n-1} (b_{m-1} - b_m) \|u_h^{n-m}\|^2 + b_{n-1} \|u_h^0\|^2 \right) + \frac{1}{2} \|u_h^n\|^2, \end{aligned}$$

that is

$$\begin{aligned} &\|u_h^n\|^2 + \beta \sum_{i=1}^{N_x} \int_c^d ([p_h^n]^2 + [s_h^n]^2)_{i-\frac{1}{2},y} dy + 2\beta\tau \sum_{j=1}^{N_y} \int_a^b [u_h^n]_{x,j-\frac{1}{2}}^2 dx \\ &\leq \sum_{m=1}^{n-1} (b_{m-1} - b_m) \|u_h^{n-m}\|^2 + b_{n-1} \|u_h^0\|^2. \end{aligned} \tag{3.7}$$

We will prove the Theorem 1 by mathematical induction. When  $n = 1$ , and from the inequality (3.7), we have

$$\|u_h^1\|^2 + \beta \sum_{i=1}^{N_x} \int_c^d ([p_h^1]^2 + [s_h^1]^2)_{i-\frac{1}{2},y} dy + 2\beta\tau \sum_{j=1}^{N_y} \int_a^b [u_h^1]_{x,j-\frac{1}{2}}^2 dx \leq \|u_h^0\|^2.$$

and  $\|u_h^1\| \leq \|u_h^0\|$ . Now suppose the following inequalities hold  $\|u_h^m\| \leq \|u_h^0\|$ ,  $m = 1, 2, \dots, K$ . Let take  $n = K + 1$  in the inequality (3.7), then we obtain

$$\begin{aligned} &\|u_h^{K+1}\|^2 + \beta \sum_{i=1}^{N_x} \int_c^d ([p_h^{K+1}]^2 + [s_h^{K+1}]^2)_{i-\frac{1}{2},y} dy + 2\beta\tau \sum_{j=1}^{N_y} \int_a^b [u_h^{K+1}]_{x,j-\frac{1}{2}}^2 dx \\ &\leq \sum_{m=1}^K (b_{m-1} - b_m) \|u_h^{K+1-k}\|^2 + b_K \|u_h^0\|^2 \\ &\leq \left( \sum_{m=1}^K (b_{m-1} - b_m) + b_K \right) \|u_h^0\|^2 \leq \|u_h^0\|^2. \end{aligned}$$

This finishes the proof of the stability result.  $\square$

Next we will state the error estimate of the scheme for the linear case  $g(u) = u$ , and use (3.4) as our flux choice. We have the following theorem.

**Theorem 2.** *Let  $u(x, y, t_n)$  be the exact solution of the problem (1.1),  $u_h^n$  be the numerical solution of the fully discrete LDG scheme (3.3), then there holds the following error estimates*

$$\|u(x, y, t_n) - u_h^n\| \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}).$$

*Proof.* For convenience we define the notations

$$\begin{aligned} \mathcal{R}(\omega, \widehat{\omega}; \phi) &= \int_a^b \int_c^d \omega \phi_x dx dy - \sum_{i=1}^{N_x} \int_c^d ((\widehat{\omega}\phi^-)_{i+\frac{1}{2},y} - (\widehat{\omega}\phi^+)_{i-\frac{1}{2},y}) dy, \\ \mathcal{Z}(\omega, \widetilde{\omega}; \phi) &= \int_a^b \int_c^d \omega \phi_y dx dy - \sum_{j=1}^{N_y} \int_a^b ((\widetilde{\omega}\phi^-)_{x,j+\frac{1}{2}} - (\widetilde{\omega}\phi^+)_{x,j-\frac{1}{2}}) dx. \end{aligned}$$



It is easy to verify that the exact solution of PDE (1.1) satisfies

$$\begin{aligned}
 & \int_a^b \int_c^d u(x, y, t_n) v \, dx \, dy + \beta \int_a^b \int_c^d \gamma^n(x, y) v \, dx \, dy - \beta \mathcal{R}(u(x, y, t_n), u(x, y, t_n); v) \\
 & \quad - \beta \mathcal{R}(q(x, y, t_n), q(x, y, t_n); v) - \beta \mathcal{Z}(r(x, y, t_n), r(x, y, t_n); v) \\
 & = \sum_{m=1}^{n-1} (b_{m-1} - b_m) \int_a^b \int_c^d u(x, y, t_{n-m}) v \, dx \, dy + b_{n-1} \int_a^b \int_c^d u(x, y, t_0) v \, dx \, dy, \\
 & \int_a^b \int_c^d q(x, y, t_n) \rho \, dx \, dy + \mathcal{R}(p(x, y, t_n), p(x, y, t_n); \rho) = 0, \\
 & \int_a^b \int_c^d p(x, y, t_n) \xi \, dx \, dy + \mathcal{R}(u(x, y, t_n), u(x, y, t_n); \xi) = 0, \\
 & \int_a^b \int_c^d r(x, y, t_n) \eta \, dx \, dy + \mathcal{R}(s(x, y, t_n), s(x, y, t_n); \eta) = 0, \\
 & \int_a^b \int_c^d s(x, y, t_n) \phi \, dx \, dy + \mathcal{Z}(u(x, y, t_n), u(x, y, t_n); \phi) = 0. \tag{3.8}
 \end{aligned}$$

Denote

$$\begin{aligned}
 e_u^n &= u(x, y, t_n) - u_h^n = \mathcal{P}^- e_u^n - (\mathcal{P}^- u(x, y, t_n) - u(x, y, t_n)), \\
 e_q^n &= q(x, y, t_n) - q_h^n = \mathcal{P} e_q^n - (\mathcal{P} q(x, y, t_n) - q(x, y, t_n)), \\
 e_p^n &= p(x, y, t_n) - p_h^n = \mathcal{P} e_p^n - (\mathcal{P} p(x, y, t_n) - p(x, y, t_n)), \\
 e_r^n &= r(x, y, t_n) - r_h^n = \mathcal{P} e_r^n - (\mathcal{P} r(x, y, t_n) - r(x, y, t_n)), \\
 e_s^n &= s(x, y, t_n) - s_h^n = \mathcal{P} e_s^n - (\mathcal{P} s(x, y, t_n) - s(x, y, t_n)). \tag{3.9}
 \end{aligned}$$

Subtracting (3.3) from (3.8), and with the fluxes (3.4) we can obtain the error equation

$$\begin{aligned}
 & \int_a^b \int_c^d e_u^n v \, dx \, dy - \beta \mathcal{R}(e_u^n, (e_u^n)^-; v) - \beta \mathcal{R}(e_q^n, (e_q^n)^+; v) - \beta \mathcal{Z}(e_r^n, (e_r^n)^+; v) \\
 & \quad + \int_{\Omega} e_q^n \rho \, dx \, dy + \mathcal{R}(e_p^n, (e_p^n)^+; \rho) + \int_{\Omega} e_p^n \xi \, dx \, dy + \mathcal{R}(e_u^n, (e_u^n)^-; \xi) \\
 & \quad + \int_{\Omega} e_r^n \eta \, dx \, dy + \mathcal{R}(e_s^n, (e_s^n)^+; \eta) + \int_{\Omega} e_s^n \phi \, dx \, dy + \mathcal{Z}(e_u^n, (e_u^n)^-; \phi) \\
 & \quad - \sum_{m=1}^{n-1} (b_{m-1} - b_m) \int_a^b \int_c^d e_u^{n-m} v \, dx \, dy - b_{n-1} \int_a^b \int_c^d e_u^0 v \, dx \, dy \\
 & \quad + \beta \int_a^b \int_c^d \gamma^n(x, y) v \, dx \, dy + \beta \tau \sum_{j=1}^{N_y} \int_a^b [e_u^n]_{x, j-\frac{1}{2}}^2 \, dx = 0.
 \end{aligned}$$

Taking  $v = \mathcal{P}^- e_u^n$ ,  $\rho = \beta \mathcal{P} e_p^n$ ,  $\xi = -\beta \mathcal{P} e_q^n$ ,  $\eta = \beta \mathcal{P} e_s^n$ ,  $\phi = -\beta \mathcal{P} e_r^n$ .

Using (3.9), the properties (2.2) and (2.3) gives

$$\begin{aligned}
 & \int_a^b \int_c^d (\mathcal{P}^- e_u^n)^2 dx + \beta\tau \sum_{j=1}^{N_y} \int_a^b [\mathcal{P}^- e_u^n]_{x,j-\frac{1}{2}}^2 dx \\
 & \quad + \frac{\beta}{2} \sum_{i=1}^{N_x} \int_c^d ([\mathcal{P}^- e_u^n]^2 + [\mathcal{P} e_p^n]^2 + [\mathcal{P} e_s^n]^2)_{i-\frac{1}{2},y} dy \\
 & = \int_a^b \int_c^d (\mathcal{P}^- u(x, y, t_n) - u(x, y, t_n)) \mathcal{P}^- e_u^n dx dy - \beta \int_a^b \int_c^d \gamma^n(x, y) \mathcal{P}^- e_u^n dx dy \\
 & \quad - \beta \sum_{i=1}^{N_x} \int_c^d \{(\mathcal{P} q(x, y, t_n) - q(x, y, t_n))^+ [\mathcal{P}^- e_u^n] \\
 & \quad - (\mathcal{P} s(x, y, t_n) - s(x, y, t_n))^+ [\mathcal{P} e_s^n] \\
 & \quad - (\mathcal{P} p(x, y, t_n) - p(x, y, t_n))^+ [\mathcal{P} e_p^n]\}_{i-\frac{1}{2},y} dy \\
 & \quad + \sum_{m=1}^{n-1} (b_{m-1} - b_m) \int_a^b \int_c^d \mathcal{P}^- e_u^{n-m} \mathcal{P}^- e_u^n dx dy + b_{n-1} \int_a^b \int_c^d \mathcal{P}^- e_u^0 \mathcal{P}^- e_u^n dx dy \\
 & \quad - \sum_{m=1}^{n-1} (b_{m-1} - b_m) \int_a^b \int_c^d (\mathcal{P}^- u(x, y, t_{n-m}) - u(x, y, t_{n-m})) \mathcal{P}^- e_u^n dx dy \\
 & \quad - \beta \sum_{j=1}^{N_y} \int_a^b (\mathcal{P} r(x, y, t_n) - r(x, y, t_n))^+ [\mathcal{P}^- e_u^n]_{x,j-\frac{1}{2}} dx \\
 & \quad - b_{n-1} \int_a^b \int_c^d (\mathcal{P}^- u(x, y, t_0) - u(x, y, t_0)) \mathcal{P}^- e_u^n dx dy \\
 & \leq \left( \|\mathcal{P}^- u(x, y, t_n) - u(x, y, t_n)\| + \beta \|\gamma^n(x, y)\| \right. \\
 & \quad + b_{n-1} \|\mathcal{P}^- u(x, y, t_0) - u(x, y, t_0)\| \\
 & \quad + \sum_{i=1}^{n-1} (b_{i-1} - b_i) \|\mathcal{P}^- u(x, y, t_{n-i}) - u(x, y, t_{n-i})\| \Big) \|\mathcal{P}^- e_u^n\| \\
 & \quad - \beta \sum_{i=1}^{N_x} \int_c^d \{(\mathcal{P} q(x, y, t_n) - q(x, y, t_n))^+ [\mathcal{P}^- e_u^n] \\
 & \quad - (\mathcal{P} s(x, y, t_n) - s(x, y, t_n))^+ [\mathcal{P} e_s^n] \\
 & \quad - (\mathcal{P} p(x, y, t_n) - p(x, y, t_n))^+ [\mathcal{P} e_p^n]\}_{i-\frac{1}{2},y} dy \\
 & \quad - \beta \sum_{j=1}^{N_y} \int_a^b (\mathcal{P} r(x, y, t_n) - r(x, y, t_n))^+ [\mathcal{P}^- e_u^n]_{x,j-\frac{1}{2}} dx \\
 & \quad + \left( \sum_{m=1}^{n-1} (b_{m-1} - b_m) \|\mathcal{P}^- e_u^{n-m}\| + b_{n-1} \|\mathcal{P}^- e_u^0\| \right) \|\mathcal{P}^- e_u^n\|.
 \end{aligned}$$

From the fact that  $\varphi\psi \leq \varepsilon\varphi^2 + \frac{1}{4\varepsilon}\psi^2$ , and the property (2.5), we know that

there exists a positive constant  $C$ , such that

$$\begin{aligned} & \int_a^b \int_c^d (\mathcal{P}^- e_u^n)^2 dx + \frac{\beta}{2} \sum_{i=1}^{N_x} \int_c^d ([\mathcal{P}^- e_u^n]^2 + [\mathcal{P} e_p^n]^2 + [\mathcal{P} e_s^n]^2)_{i-\frac{1}{2},y} dy \\ & + \beta\tau \sum_{j=1}^{N_y} \int_a^b [\mathcal{P}^- e_u^n]_{x,j-\frac{1}{2}}^2 dx \\ & \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}})^2 + \beta\varepsilon \sum_{j=1}^{N_y} \int_a^b [\mathcal{P}^- e_u^n]_{x,j-\frac{1}{2}}^2 dx \\ & + \frac{\beta}{4} \sum_{i=1}^{N_x} \int_c^d ([\mathcal{P}^- e_u^n]^2 + [\mathcal{P} e_p^n]^2 + [\mathcal{P} e_s^n]^2)_{i-\frac{1}{2},y} dy \\ & + \left( \sum_{m=1}^{n-1} (b_{m-1} - b_m) \|\mathcal{P}^- e_u^{n-m}\| + b_{n-1} \|\mathcal{P}^- e_u^0\| \right)^2 + \frac{1}{2} \int_a^b \int_c^d (\mathcal{P}^- e_u^n)^2 dx. \end{aligned}$$

Choosing a small enough  $\varepsilon \leq \tau$ , we can obtain

$$\begin{aligned} \|\mathcal{P}^- e_u^n\|^2 & \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}})^2 \\ & + 2 \left( \sum_{m=1}^{n-1} (b_{m-1} - b_m) \|\mathcal{P}^- e_u^{n-m}\| + b_{n-1} \|\mathcal{P}^- e_u^0\| \right)^2. \end{aligned} \tag{3.10}$$

We prove the error estimate by mathematical induction. When  $n = 1$ , the equation (3.10) becomes

$$\|\mathcal{P}^- e_u^1\|^2 \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}})^2 + 2\|\mathcal{P}^- e_u^0\|^2.$$

It is easy to see that  $\|\mathcal{P}^- e_u^0\| \leq Ch^{k+1}$ , then

$$\|\mathcal{P}^- e_u^1\| \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}).$$

Next we suppose the following inequality holds

$$\|\mathcal{P}^- e_u^m\| \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}), \quad m = 2, 3, \dots, K. \tag{3.11}$$

When  $n = K + 1$ , from the equation (3.10), we deduce

$$\begin{aligned} \|\mathcal{P}^- e_u^{K+1}\|^2 & \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}})^2 \\ & + 2 \left( \sum_{m=1}^K (b_{m-1} - b_m) \|\mathcal{P}^- e_u^{K+1-k}\| + b_K \|\mathcal{P}^- e_u^0\| \right)^2. \end{aligned}$$

By the expression (3.2), the assumption (3.11), we can get the following result immediately

$$\|\mathcal{P}^- e_u^{K+1}\| \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}).$$

Thus Theorem 2 follows by the triangle inequality and the interpolating property (2.5).  $\square$

## 4 Conclusion

In this paper an implicit fully discrete local discontinuous Galerkin (LDG) finite element method is presented for a class of two-dimensional time-fractional Zakharov–Kuznetsov equation. The method is based on a finite difference scheme in time and local discontinuous Galerkin methods in space. Stability is ensured by a careful choice of interface numerical fluxes. We prove that our scheme is unconditional stable and  $L^2$  error estimate for the linear case with the convergence rate  $O(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}})$ . To date we are not aware of any similar results in published papers. Although not addressed in this paper, the method and analytical technique can also be extended to other kinds of time-fractional equations and higher-dimensional problems easily.

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