VIII. APPLIED PLASMA RESEARCH[*]

A. Active Plasma Systems

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1. STABILITY OF ELECTRON BEAMS WITH VELOCITY SHEAR

In our last report[1] we showed that unneutralized zero-temperature electron beams focused by an infinite magnetic field were stable if the transverse velocity profile, $v_z(x)$ was linear. In this report we discuss the step-ramp velocity profile shown in Fig. VIII-1. We find that this configuration is unstable as long as the slope of the ramp remains finite.

As explained previously,[1] our approach to these problems is to find a neutrally stable (ω and k_z real) solution with $dv_z(x)/dx > 2\omega_p$. If no such solution exists, the beam is stable. If such a solution does exist, we must perturb it to see if there is a neighboring unstable solution.

The differential equation for the linearized small-signal potential ϕ is given by[2]

$$\frac{d^2\phi}{dx^2} - k^2\phi + \frac{\omega_p^2(x)}{[v_z(x)-u]^2}\phi = 0, \tag{1}$$

where $u \triangleq \omega/k$. Using the previous results,[1] we can write the potential in the three regions so that $\phi = 0$ at the walls ($x = \pm a$):

$$\phi_I = A_I \sin\left[(a+x)\sqrt{\left(\frac{\omega_p}{u - v_{min}}\right)^2 - k_z^2}\right] \tag{2}$$

$$\phi_{II} = A_{II}\, s^{1/2} J_{\pm\nu}(jk_z s); \qquad s \triangleq x + \frac{v_o - u}{\alpha}; \qquad \nu^2 = \frac{1}{4} - \frac{\omega_p^2}{\alpha^2} \tag{3}$$

$$\phi_{III} = A_{III} \sin \left[(a-x) \sqrt{\left(\frac{\omega_p}{u-v_{max}} \right)^2 - k_z^2} \right].$$

(4)

Since there can only be one singular point, either the plus or minus sign in Eq. 3 can be used but not both.[1] Since

$$v_{min} < u < v_{max}$$

(5)

for instability,[2] the singular point does not lie at $x = \pm\delta$, and ϕ'/ϕ will have to be continuous at these boundaries.
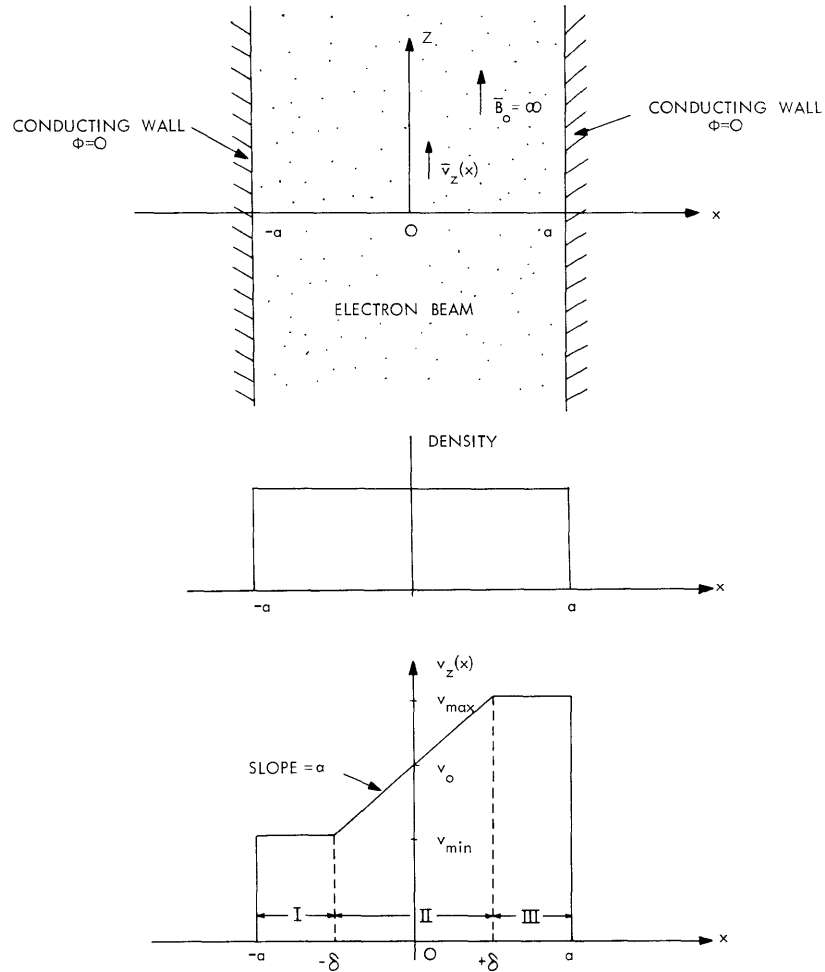

Fig. VIII-1. Geometry of the problem.

To solve the resulting determinantal equations we make two simplifying assumptions:

$$k_z \delta \ll 1 \tag{6}$$

$$v_o = u. \tag{7}$$

Thus, in analogy with the two-stream case, we assume that the phase velocity of the waves lies halfway between v_{min} and v_{max}.

These assumptions yield the following equation for k_z^2 which holds at both $x = \pm\delta$:

$$\sqrt{\left(\frac{\omega_p}{\Delta v}\right)^2 - k_z^2} \, \cot\left[(a-\delta)\sqrt{\left(\frac{\omega_p}{\Delta v}\right)^2 - k_z^2}\right] = -\frac{1}{2\delta}. \tag{8}$$

We have defined $\Delta v \triangleq a\delta = (v_o - v_{min})$. Equation 8 can be solved graphically for k_z^2. The first thing to note is that the left-hand side of Eq. 8 is positive until the argument of the cotangent is greater than $\pi/2$. Thus neutrally stable solutions with $\omega/k_z = v_o$ only exist when

$$k_z^2 < \left(\frac{\omega_p}{\Delta v}\right)^2 - \left[\frac{\pi}{2(a-\delta)}\right]^2. \tag{9}$$

This implies that for instability

$$\Delta v < \frac{2a\omega_p}{\pi}. \tag{10}$$

Additional neutral modes appear as ω_p is increased above this initial value.

From previous work,[2] we know that a necessary condition for instability is $v_z' > 2\omega_p$, so we expect instability for Δv in the range

$$2\omega_p \delta < \Delta v < \frac{2a\omega_p}{\pi}. \tag{11}$$

Now that we have shown that at least some neutral solutions exist, we must show that these modes are the limit of adjacent unstable modes. We do this by means of a perturbation method formulated by L. N. Howard.[3] By variational techniques, it can be shown that in the limit as the neutrally stable solution is approached,

$$\frac{du}{dk_z} = 2k_z \frac{\int_{x_1}^{x_2} \phi^2 \, dx}{\int_{x_1}^{x_2} \frac{2\omega_p^2}{(v_z-u)^3} \phi^2 \, dx}. \tag{12}$$

If du/dk_z has a negative imaginary part there is an adjacent unstable mode.

The integrals can be carried out if the solution is properly analytically continued around the singular point. The end result is messy unless δ is small. In this case, the imaginary part is given by

$$\frac{du}{dk_z} \to jk_z \frac{(\Delta v)^3}{\omega_p^2} \cot(\pi \nu). \tag{13}$$

(This approaches zero as $\delta \to 0$ since $\nu \to \frac{1}{2}$.)

Thus, the step-ramp profile is in general unstable for Δv's satisfying Eq. 11. This result is in contrast with the results of Harrison and Stringer[4] for the $\delta = 0$ case; they reported that the step profile is unstable for any stream width.

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References

1. J. A. Rome and R. J. Briggs, "Stability of Electron Beams with Velocity Shear," Quarterly Progress Report No. 93, Research Laboratory of Electronics, M. I. T., April 15, 1968, pp. 131-138.

2. R. J. Briggs and J. A. Rome, "Stability of Electron Beams with Velocity Shear," Quarterly Progress Report No. 90, Research Laboratory of Electronics, M. I. T., July 15, 1968, pp. 106-107.

3. L. N. Howard, "Neutral Curves and Stability Boundaries in Stratified Flow," J. Fluid Mech. 16, 333-342 (1963).

4. E. R. Harrison and T. E. Stringer, "Longitudinal Oscillations in Velocity-Gradient Plasmas, II. The Adjacent-Stream Mode," Proc. Phys. Soc. (London) 82, 700 (1963).