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PERIODIC FOCUSING OF INTENSE BEAMS†

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The theory of periodic focusing with space charge is reviewed and general relations between beam current, emittance, and acceptance of magnetic channels are derived by the "smooth approximation" method. The current limits of quadrupole (FODO) and solenoid channels are calculated and the pertinent scaling laws are discussed. Numerical examples are given to illustrate the application of the theory to high-power beam transport for heavy-ion fusion.

1 INTRODUCTION

Recent proposals[1–3] to use high-power heavy-ion beams in inertial confinement schemes for controlled fusion ("pellet fusion") have created a new interest in the focusing capabilities and space charge limits of magnetic channels for intense beams. The design, size, and cost of an accelerator for pellet fusion depend strongly on the current limits and scaling laws of the magnet systems that are employed to focus and transport the heavy-ion beams. All of the accelerator schemes proposed so far start with long pulses of the relatively low currents available from conventional ion sources. During acceleration, these bunches must then be compressed into short pulses of a few nanoseconds with instantaneous power of order 10^{15} W/cm^2 when they hit the target at full energy. (As an example, with uranium beams one needs final energies of 20 to 60 GeV and peak particle currents of 4 to 10 kA for igniting the pellet.) The focusing problem is most severe at full energy and at low energy near the ion source. In view of the strong space charge effects, superconducting magnets (quadrupoles or solenoids) with large apertures are required to transport the beam.

An approximate formula for the maximum beam power P that can be transported in such a magnetic channel was first derived by Maschke and Courant.[4] For a quadrupole channel, they find the relationship

$$P = \text{const}(A/Z)^{4/3}\varepsilon^{2/3}(\beta\gamma)^{7/3}(\gamma - 1)B_0^{2/3},$$

where A, Z are the atomic mass number and charge state, $\varepsilon\pi$ the emittance, B_0 the pole-tip field, and β, γ the relativistic velocity and mass factors, while the constant depends on the channel parameters. More recently, the problem was treated in three papers at the 1977 Particle Accelerator Conference at Chicago. Khoe and Martin[5] and Lambertson, Laslett, and Smith[6] obtained results which are essentially presented in the form of the Maschke-Courant formula except that the constant is calculated in more detail and shown to be dependent upon the phase shift. This author,[7] on the other hand, derived a scaling law with the thin-lens approximation, which is of the form

$$P = \text{const}\ \beta\gamma(\gamma - 1)(B_0 a)^2[1 - (\varepsilon/\alpha)^2],$$

where $\varepsilon\pi$ is the beam emittance, $\alpha\pi$ the channel acceptance without space charge, and $2a$ the aperture of the channel. The most important feature here is that the power through a channel (with given acceptance) decreases with increasing emittance, and, furthermore, it is independent of the charge state, i.e., one could operate with high-charge state ions and proportionally shorter accelerator columns. By contrast, in the Maschke-Courant formula, the power increases with emittance and decreases as $Z^{-4/3}$, which would

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necessitate the use of singly charged ions and long linacs or induction accelerator systems.

In an effort to clarify this problem of high-power beam transport, we present in this paper a general theory on beam focusing in periodic channels in the presence of strong space charge forces, which is more rigorous than the thin-lens approximation in the previous paper.[7] First, in Section 2, we will solve the equations for the beam envelopes by the "smooth approximation" method which will yield a general formula for the current that can be transported in a periodic magnetic channel. Next, in Section 3, we will apply the theory to a quadrupole channel in the FODO configuration and to a solenoid channel. Finally, we will discuss the validity of the theory and the scaling laws, present results for a few examples of high-power beam transport for heavy-ion fusion, and explain the differences between the two scaling laws.

2 GENERAL THEORY OF PERIODIC FOCUSING WITH SPACE CHARGE

2.1 *Brief Review of Existing Theory*

The theory of strong focusing in periodic magnetic channels when space charge effects are negligible is treated in the classic paper by Courant and Snyder.[8] The inclusion of the space charge forces usually follows the self-consistent method first developed by Kapchinsky and Vladimirsky.[9] In the K–V model, the beam has an elliptic cross section with uniform particle density, and the envelopes X and Y in the two perpendicular transverse directions are determined by the two differential equations

$$\frac{d^2X}{ds^2} + \kappa_x X - \frac{2K}{X+Y} - \frac{\varepsilon_x^2}{X^3} = 0, \qquad (1)$$

$$\frac{d^2Y}{ds^2} + \kappa_y Y - \frac{2K}{X+Y} - \frac{\varepsilon_y^2}{Y^3} = 0. \qquad (2)$$

The independent variable s is the path length along the direction of beam propagation; $\varepsilon_{x(y)}\pi$ is the emittance in $x(y)$ direction. The function $\kappa_{x(y)}$ represents the applied forces and is periodic in the variable s with period S, i.e.,

$$\kappa(s+S) = \kappa(s). \qquad (3)$$

For a quadrupole channel with pole-tip field B_0 and aperture $2a$, we have

$$\kappa_{x(y)} = \pm\frac{qB'}{m_0 c\beta\gamma} = \frac{qB_0/a}{m_0 c\beta\gamma}, \qquad (4)$$

where, in the ideal case, the gradient B' is constant (B_0/a) inside the magnets and zero between magnets.

The factor K in the envelope equations is the "generalized perveance"[10] and represents the space-charge forces. It is defined in terms of the total beam current as (mks units):

$$K = \frac{qI}{2\pi\varepsilon_0 m_0 c^3\beta^3\gamma^3} = \frac{I}{I_0}\frac{2}{\beta^3\gamma^3}, \qquad (5)$$

where

$$I_0 = \frac{4\pi\varepsilon_0 m_0 c^3}{q}$$

is the limiting current.

The beam envelopes are related to the amplitude function $\beta(s) = w^2(s)$ introduced by Courant and Snyder,[8] namely,

$$X(s) = \sqrt{\varepsilon_x \beta_x(s)} = \sqrt{\varepsilon_x}\, w_x(s), \qquad (7)$$

and likewise for $Y(s)$.

An important parameter in the theory of periodic focusing is the phase shift per period, μ, which is defined by

$$\mu = \int_s^{s+S}\frac{ds}{\beta(s)} = \int_s^{s+S}\frac{ds}{w^2(s)}, \qquad (8)$$

where we dropped the subscripts x, y. In view of (7), we may also write

$$\mu_x = \int_s^{s+S}\frac{\varepsilon_x\, ds}{X^2}, \qquad \mu_y = \int_s^{s+S}\frac{\varepsilon_y\, ds}{Y^2}. \qquad (9)$$

The normal procedure is to solve the envelope equations for the case $K = 0$ (zero space charge) and then integrate numerically by gradually increasing K to the desired value. The disadvantage of these numerical techniques is that they do not readily reveal the functional relationships between the various parameters and the scaling laws that apply to a given channel configuration. We will therefore, in the following, try to obtain an analytical solution of the envelope equations that, at least to good approximation, reveals the parametric dependence for the beam current that can be transported in a periodic channel. However, for easy comparison and in order to gain some physical insight into the nature of the problem, we will first present and discuss the results for a uniformly focused beam in a long solenoid from our previous paper[7] (with a few changes in notation and presentation).

2.2 *Uniformly Focused Beam in a Long Solenoid*

For a long solenoid with uniform magnetic field B_s, one obtains in the paraxial approximation a single equation for the beam envelope R which is similar to Eqs. (1) and (2) and of the form

$$\frac{\mathrm{d}^2R}{\mathrm{d}s^2} + \kappa_s R - \frac{K}{R} - \frac{\varepsilon^2}{R^3} = 0. \qquad (10)$$

$\varepsilon\pi$ is the emittance in the rotating Larmor frame and κ_s is defined by

$$\kappa_s = \left(\frac{qB_s}{2m_0 c\beta\gamma}\right)^2. \qquad (11)$$

For a matched beam, i.e., $R = a = \text{const}$, $\mathrm{d}^2R/\mathrm{d}s^2 = 0$, one gets from (10) the solution[7]

$$K = \kappa_s a^2\left[1 - \frac{\varepsilon^2}{\kappa_s a^4}\right]. \qquad (12)$$

In the absence of space charge (i.e., $K = 0$), the envelope is given by

$$R = R_0 = \left(\frac{\varepsilon}{\sqrt{\kappa_s}}\right)^{1/2} \qquad (13)$$

and we can define the acceptance α of the channel by

$$a = \left(\frac{\alpha}{\sqrt{\kappa_s}}\right)^{1/2} \quad \text{or} \quad \alpha = a^2\sqrt{\kappa_s}. \qquad (14)$$

Using relations (13) and (14), we obtain for K the alternate expression

$$K = \sqrt{\kappa_s}\,\alpha\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right] = \left(\frac{\alpha^2}{a^2}\right)\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]. \qquad (15)$$

In view of (5), the beam current is then given by

$$I = \frac{I_0}{2}\beta^3\gamma^3\sqrt{\kappa_s}\,\alpha\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]$$

$$= \frac{I_0}{2}\beta^3\gamma^3\frac{\alpha^2}{a^2}\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]. \qquad (16)$$

When numerical values for the various parameters are substituted, one finds for ions with charge state Z and mass number A the expression[7]

$$I_{[A]} = 4.0 \times 10^5\,\frac{Z}{A}\,\beta\gamma(B_s a)^2\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]. \qquad (17)$$

Note that $(\varepsilon/\alpha)^2 = (R_0/a)^4$ in view of (13) and (14).

The physical interpretation of the above results for beam transport in long solenoids is straightforward: in the limit of zero intensity ($K = 0$), a beam with given emittance will have an envelope radius R_0 which is less than the semi-aperture a. As the current is raised, the envelope radius increases due to the defocusing action of the space-charge forces. The current maximum is reached when the beam fills the entire aperture (i.e., when $R = a$) and its magnitude is determined by Eqs. (16), (17). From these two equations, we see that the current increases with channel acceptance α and decreases with emittance ε as one would expect on physical grounds. High-current transport requires that $\varepsilon \ll \alpha$. Furthermore, we see from (17) that the particle current, I/Z, is independent of charge state Z (as long as $\varepsilon \ll \alpha$ when Z is increased). This last result may be attributed to the fact that both the focusing force as well as the space-charge force on a given particle vary as Z^2 (i.e., $\kappa_s \propto Z^2$, $K \propto Z^2$), and hence there is complete cancellation between the two forces.

2.3 *Current Limit in a Periodic Channel*

After the preceding analysis of the long solenoid case, we shall now attempt to derive analogous scaling laws for the beam current in periodic channels. At the outset, we recognize that important differences exist between the two focusing systems. For one thing, the beam envelope in the periodic channel is rippled; and, secondly, periodic channels have a focusing limit (defined by $\mu = 180°$) which does not exist in long solenoids. Nevertheless, when the beam ripple is not too large (i.e., sufficiently below the focusing limit $\mu = 180°$), one would expect on physical grounds that a scaling law for the beam current exists which is similar to the one for long solenoids (except for a geometry factor). More specifically, this is to say that a differential equation of the form (10) should be obtained for the mean radius of the beam in a periodic system. In the following, we shall derive such an equation using the smooth-approximation method. As an example for a general periodic focusing system, Figure 1 illustrates a typical ideal quadrupole channel of the FODO type. The function κ represents alternating-gradient (AG) focusing and defocusing in this case. The X-envelope has a local maximum where κ_x is focusing and a local minimum where it is defocusing. The Y-envelope follows the same pattern except that it has its minimum where X is a maximum and vice versa. When the beam is "matched," the maxima and minima will have the same magnitude along the channel, i.e., $X_{\max}(s) = X_{\max}(s + S)$, etc. For an unmatched beam, there will be a slow variation of the envelope function with a wavelength that is generally large compared to the period S. Superimposed on this slow variation is a "ripple" of short wavelength that exhibits the periodic structure of the channel. Thus, we can represent the solutions to the envelope equations in the form

$$X(s) = \bar{X}(s)[1 + \delta_x(s)], \quad Y(s) = \bar{Y}(s)[1 + \delta_y(s)], \tag{18}$$

where the mean values $\bar{X}$ and $\bar{Y}$ vary slowly with s and the "fast" amplitude functions $\delta(s)$ are periodic with the period S of the lattice, i.e.,

$$\delta_x(s + S) = \delta_x(s), \qquad \delta_y(s + S) = \delta_y(s). \tag{19}$$

In the smooth approximation, it is assumed that $\bar{X}$, $\bar{Y}$ and their derivatives vary slowly enough that they may be considered as constant within one period S and that the amplitude of $\delta(s)$ is much smaller than unity, i.e.,

$$|\delta_{x(y)}| \ll 1. \tag{20}$$

This condition implies that the phase shift per period is small, or

$$\mu_{x(y)} \ll 2\pi. \tag{21}$$

The variation of the functions δ_x, δ_y with s is determined by the focusing functions κ_x, κ_y. For a periodic channel of the type depicted in Figure 1, the mean values of $\kappa(s)$ and $\delta(s)$ and of their derivatives are zero, i.e.,

$$\bar{\kappa}(s) = \frac{1}{S}\int_s^{s+S} \kappa(s)\mathrm{d}s = 0, \tag{22}$$

$$\bar{\delta}(s) = \frac{1}{S}\int_s^{s+S} \delta(s)\mathrm{d}s = 0,$$
$$\frac{1}{S}\int_s^{s+S} \frac{\mathrm{d}\delta(s)}{\mathrm{d}s}\mathrm{d}s = 0, \text{ etc.} \tag{23}$$

From the comparison of (7) and (18), we see that the functions $\delta(s)$ are related to the amplitude functions $w(s)$, i.e., we may write (for $\bar{X} \approx$ const within one period)

$$w(s) = C[1 + \delta(s)]. \tag{24}$$

The constant may be determined from relation (8)

$$\mu = \int_s^{s+S} \frac{\mathrm{d}s}{w^2} = \frac{1}{C^2}\int_s^{s+S} \frac{\mathrm{d}s}{[1 + \delta(s)]^2} \approx \frac{1}{C^2}\int_s^{s+S} [1 - 2\delta(s)]\mathrm{d}s. \tag{25}$$

Considering condition (23), one finds $C = \sqrt{S/\mu}$ and thus

$$w(s) = \sqrt{\frac{S}{\mu}}[1 + \delta(s)]. \tag{26}$$

When (7) is substituted into Eq. (1), one obtains for the function $w(s)$ the equation

$$\frac{\mathrm{d}^2 w_x}{\mathrm{d}s^2} + \kappa_x w_x - \frac{2K}{\varepsilon_x[w_x + \sqrt{\varepsilon_y/\varepsilon_x}\, w_y]} - \frac{1}{w_x^3} = 0. \tag{27}$$

In the following, we will assume that the emittance and the phase shift per period in each transverse direction are the same, i.e.,

$$\varepsilon_x = \varepsilon_y = \varepsilon; \qquad \mu_x = \mu_y = \mu. \tag{28}$$

By substitution of $w = \sqrt{S/\mu}(1 + \delta)$ into Eq. (27), Taylor-expanding and keeping only first-order terms in δ, one obtains

$$\frac{\mathrm{d}^2\delta_x}{\mathrm{d}s^2} + \kappa_x(1 + \delta_x) - \frac{\mu K}{S\varepsilon}\left(1 - \frac{\delta_x}{2} - \frac{\delta_y}{2}\right) - \left(\frac{\mu}{S}\right)^2(1 - 3\delta_x) = 0. \tag{29}$$

If this equation is integrated over one period and condition (23) is used, one finds

$$\bar{\kappa}_x + \frac{1}{S}\int_s^{s+S} \kappa_x(s)\delta_x(s)\mathrm{d}s = \frac{\mu^2}{S^2} + \frac{\mu K}{S\varepsilon}. \tag{30}$$

From (22), the average of the gradient function, $\bar{\kappa}_x(s)$, is zero for a symmetric quadrupole channel. The integral in Eq. (30) represents the average focusing force along the envelope of the beam within one period of the lattice. This average force is thus related to the phase shift μ. For negligible space charge ($K \approx 0$, $\mu = \mu_0$) and with $\bar{\kappa}_x = 0$, we can write

$$\frac{1}{S}\int_s^{s+S} \kappa_x(s)\delta_x(s)\mathrm{d}s = \frac{\mu_0^2}{S^2}. \tag{31}$$

Our main objective is to find a relationship between the mean values $\bar{X}$, $\bar{Y}$ of the envelopes and the focusing functions $\kappa(s)$, the emittance ε, and the generalized perveance K. To accomplish this task, we substitute the expressions (18) into the envelope Eqs. (1) or (2) and take the average over one period S. After Taylor-expansion to first order in δ, this substitution yields

$$\frac{d^2\bar{X}}{ds^2}(1+\delta_x) + 2\frac{d\bar{X}}{ds}\frac{d\delta_x}{ds} + \bar{X}\frac{d^2\delta_x}{ds^2} + \kappa_x\bar{X}(1+\delta_x) - \frac{2K}{\bar{X}+\bar{Y}} + \frac{2K\bar{X}}{(\bar{X}+\bar{Y})^2}\delta_x + \frac{2K\bar{X}}{(\bar{X}+\bar{Y})^2}\delta_y + \frac{\varepsilon_x}{\bar{X}^3}(1-3\delta_x) = 0. \quad (32)$$

The only surviving terms after integration of this equation are

$$\frac{d^2\bar{X}}{ds^2} + \bar{\kappa}_x\bar{X} + \frac{\bar{X}}{S}\int_s^{s+S}\kappa_x(s)\delta_x(s)ds - \frac{2K}{\bar{X}+\bar{Y}} - \frac{\varepsilon_x^2}{\bar{X}^3} = 0, \quad (33)$$

where $\bar{\kappa}_x = 0$ for a symmetric quadrupole channel. Substitution of the relation (30) for the average of the focusing force in Eq. (33) then yields the following differential equation for the mean value of the X-envelope

$$\frac{d^2\bar{X}}{ds^2} + \left(\frac{\mu^2}{S^2} + \frac{\mu K}{S\varepsilon}\right)\bar{X} - \frac{2K}{\bar{X}+\bar{Y}} - \frac{\varepsilon^2}{\bar{X}^3} = 0. \quad (34)$$

An analogous equation is obtained for the mean of the Y-envelope.

In principle, Eq. (34) can be solved by successive approximation starting with the phase shift for zero space charge ($\mu = \mu_0$, $K = 0$) in the second term and then progressively changing μ as K varies. As it will turn out in the following analysis, μ decreases as K increases and, to good approximation, the factor in the second term remains constant, namely,

$$\frac{\mu^2}{S^2} + \frac{\mu K}{S\varepsilon} \approx \frac{\mu_0^2}{S^2}. \quad (35)$$

The phase shift μ_0 is readily found from the solutions of the envelope equations for the case $K = 0$.

In an unmatched beam, $\bar{X}$ and $\bar{Y}$ are slowly varying functions of s with a wavelength that is generally large compared to the lattice period S. We are, however, interested primarily in the *matched* beam which permits the maximum current to flow through a channel of given strength and aperture. In this case, $\bar{X} = \text{const}$ and $d^2\bar{X}/ds^2 = 0$. We have already assumed that the emittance and the phase shift are the same in both x and y direction. For a matched beam, the average magnitude of the two envelopes are therefore the same, i.e.,

$$\bar{X} = \bar{Y} = \bar{R}. \quad (36)$$

Eq. (34) then yields, for the mean radius $\bar{R}$ of the matched beam, the algebraic equation

$$\frac{\mu_0^2}{S^2}\bar{R} - \frac{K}{\bar{R}} - \frac{\varepsilon^2}{\bar{R}^3} = 0. \quad (37)$$

This relation has the same form as Eq. (10) for the long solenoid (with $d^2R/ds^2 = 0$). The only difference is that κ_s is replaced by μ_0^2/S^2 which represents the average focusing force in one channel period S. Solving (37) for K yields

$$K = \frac{\mu_0^2}{S^2}\bar{R}^2\left[1 - \frac{\varepsilon^2 S^2}{\mu_0^2\bar{R}^4}\right]. \quad (38)$$

This may be directly compared with the expression (12) for a long solenoid where we had κ_s in place of μ_0^2/S^2 and the semi-aperture a in place of the mean radius $\bar{R}$.

Alternatively, we can solve Eq. (37) for the mean radius $\bar{R}$ which yields

$$\bar{R} = \sqrt{\frac{\varepsilon S}{\mu_0}}[u + \sqrt{1+u^2}]^{1/2} = \bar{R}_0[u + \sqrt{1+u^2}]^{1/2} \quad (39)$$

Here we introduced the dimensionless parameter u, which is defined by

$$u = \frac{KS}{2\mu_0\varepsilon} = \frac{I}{I_0}\frac{S}{\mu_0}\frac{1}{\beta^3\gamma^3\varepsilon} \quad (40)$$

and which measures the ratio of the space charge force to the product of the average focusing force and the emittance of the beam. $\bar{R}_0$ is the mean radius of the beam in the limit of zero intensity ($K = 0$, $u = 0$) and is given by

$$\bar{R}_0 = \sqrt{\frac{\varepsilon S}{\mu_0}}. \quad (41)$$

In the high-density regime where emittance may be neglected compared with the space-charge effect, we find, with $\varepsilon = 0$, from (37):

$$\bar{R} = \bar{R}_1 = \frac{\sqrt{K}S}{\mu_0}. \quad (42)$$

Returning now to relation (38), we will find it more useful to introduce the semi-aperture of the channel in place of the mean radius $\bar{R}$. From (18), we may write

$$X_{\max} = \bar{R}(1 + \delta_{\max}). \tag{43}$$

The function $\delta(s)$ is predominantly determined by the periodic structure of the focusing function $\kappa(s)$. This is to say that $\delta(s)$ has only a very weak dependence on the space-charge parameter K which can be neglected for our purpose. With (26) we may therefore write

$$1 + \delta_{\max} = \sqrt{\frac{\mu_0}{S}}\, w_{0,\max} \tag{44}$$

where $w_{0,\max}$ is the maximum of the amplitude function $w(s)$ for zero space charge ($K = 0$). If $2a$ is the aperture of the channel[13], the maximum beam current that can be transmitted is obtained when $X_{\max} = a$; from (43), (44) then follows

$$\bar{R} = \frac{a}{w_{0,\max}}\sqrt{\frac{\mu_0}{S}}. \tag{45}$$

Substitution of (45) into (38) then yields for the generalized perveance K the expression

$$K = \frac{\mu_0}{S}\frac{a^2}{w_{0,\max}^2}\left[1 - \frac{\varepsilon^2 w_{0,\max}^4}{a^4}\right]. \tag{46}$$

The acceptance α of the channel for zero space charge is defined by

$$a = \sqrt{\alpha}\, w_{0,\max}, \quad \text{or} \quad \alpha = \frac{a^2}{w_{0,\max}^2}. \tag{47}$$

Consequently, we can write (46) in the form

$$K = \frac{\mu_0}{S}\alpha\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right], \tag{48}$$

which compares directly with expression (15) for a long solenoid except that μ_0/S has replaced $\sqrt{\kappa_s}$ as one would expect. Since the maximum of the beam envelope for zero intensity ($K = 0$) is defined by

$$X_{0,\max} = \sqrt{\varepsilon}\, w_{0,\max}, \tag{49}$$

we have

$$\left(\frac{\varepsilon}{\alpha}\right)^2 = \frac{X_{0,\max}^4}{a^4}. \tag{50}$$

With (5), one obtains from (48) for the beam current that can be transported through a periodic channel the result

$$I = \frac{I_0}{2}\beta^3\gamma^3\frac{\mu_0}{S}\alpha\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]. \tag{51}$$

For electrons, the limiting current is $I_0 \approx 1.7 \times 10^4$ amperes; for ions with mass number A (in terms of the atomic rest mass $Mc^2 = 931.48$ MeV) and charge state Z we have

$$I_0 \approx 3.1 \times 10^7 \frac{A}{Z}\ \text{[amperes]}. \tag{52}$$

Equation (51) is the desired general relation for high-current beam transport in an arbitray periodic channel. In view of our assumptions (20), (21), it is accurate when the phase shift μ_0 is small. (We will discuss the validity of our formula in more detail in Section 4.1 after we applied the theory to FODO and solenoid channels.) From (51), the beam current is seen to be proportional to the phase shift per period, μ_0, and to depend on the channel acceptance α. Both μ_0 and α are the channel functions for zero space charge and can therefore be calculated in a straightforward way.

Last, we note that the above results allow us now to determine the amplitude function and the phase shift for the beam with nonzero space charge ($K \neq 0$). Since, by definition, $w = X/\sqrt{\varepsilon} = \bar{R}(1 + \delta)/\sqrt{\varepsilon}$, we find with (39) the result

$$\begin{aligned} w(s) &= \sqrt{\frac{S}{\mu_0}}[1 + \delta(s)][u + \sqrt{1 + u^2}]^{1/2} \\ &= w_0(s)[u + \sqrt{1 + u^2}]^{1/2}, \end{aligned} \tag{53}$$

and

$$\mu = \int_s^{s+S} \frac{1}{w^2(s)}\, ds = \mu_0[\sqrt{1 + u^2} - u]. \tag{54}$$

The last relation shows that the phase shift decreases as the space-charge parameter u increases. In view of condition (21), the accuracy of the smooth approximation method should therefore improve when the beam current gets stronger.

Finally, with the aid of (54) and definition (40), we can prove the validity of relation (35). We find

$$\frac{\mu^2}{S^2}+\frac{\mu K}{S\varepsilon}=\frac{\mu_0^2}{S^2}\left[\left(\frac{\mu}{\mu_0}\right)^2+2\frac{KS}{2\mu_0^2\varepsilon}\frac{\mu}{\mu_0}\right]$$
$$=\frac{\mu_0^2}{S^2}[(\sqrt{1+u^2}-u)^2$$
$$+2u(\sqrt{1+u^2}-u)]$$
$$=\frac{\mu_0^2}{S^2}[1+u^2-2u\sqrt{1+u^2}+u^2$$
$$+2u\sqrt{1+u^2}-2u^2]=\frac{\mu_0^2}{S^2},\quad \text{q.e.d.}$$

3 HIGH-CURRENT BEAM TRANSPORT IN QUADRUPOLE AND SOLENOID CHANNELS

3.1 *FODO Channels*

We now apply the general theory to an ideal quadrupole channel of the FODO type as shown in Figure 1. To determine the beam current that can be transported in the channel, we need $w_{0,\max}$ and μ_0 for this configuration. The maximum of the amplitude function, $w_{0,\max}$, is obtained from the transfer matrix for a half period $S/2 = L + l$ of the channel (from the center of the focusing magnet to the center of the defocusing magnet) which is

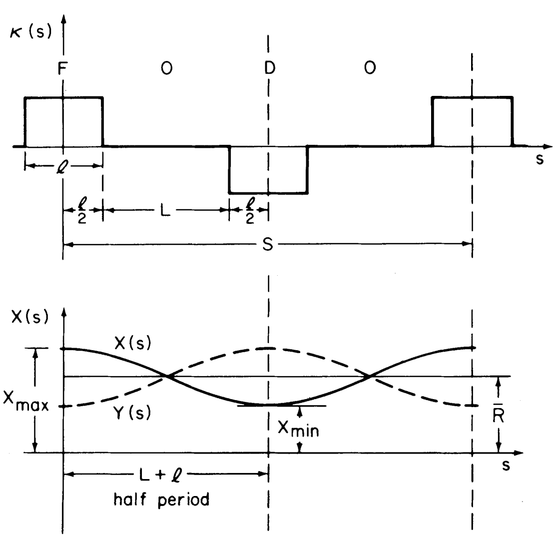

FIGURE 1 Gradient and envelope functions in a periodic quadrupole channel of the FODO type.

$$\tilde{M}=\begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}=\begin{pmatrix}\cosh\phi & \dfrac{l}{2\phi}\sinh\phi\\ \dfrac{2\phi}{l}\sinh\phi & \cosh\phi\end{pmatrix}$$
$$\times\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos\phi & \dfrac{l}{2\phi}\sin\phi\\ -\dfrac{2\phi}{l}\sin\phi & \cos\phi\end{pmatrix}\quad (55)$$

Multiplication yields

$$\left.\begin{aligned}
m_{11}&=\cos\phi\cosh\phi-\sin\phi\sinh\phi\\
&\quad-\frac{2L}{l}\phi\sin\phi\cosh\phi\\
m_{12}&=\frac{l}{2\phi}\Big[\cos\phi\sinh\phi+\sin\phi\cosh\phi\\
&\quad+\frac{2L}{l}\phi\cos\phi\cosh\phi\Big]\\
m_{21}&=\frac{2\phi}{l}\Big[\cos\phi\sinh\phi-\sin\phi\cosh\phi\\
&\quad-\frac{2L}{l}\phi\sin\phi\sinh\phi\Big]\\
m_{22}&=\cos\phi\cosh\phi+\sin\phi\sinh\phi\\
&\quad+\frac{2L}{l}\phi\cos\phi\sinh\phi\Big]
\end{aligned}\right\}\quad (56)$$

The parameter ϕ is defined by

$$2\phi=\Theta=\sqrt{\kappa}\,l=\left(\frac{qB_0}{m_0c\beta\gamma a}\right)^{1/2}l=\left(\frac{qB_0a}{m_0c\beta\gamma}\right)^{1/2}\frac{l}{a},\quad (57)$$

where l is the width of a magnet section and L the separation between the magnets (see Figure 1).

To obtain $w_{0,\max}$, we make use of the fact that the transfer matrix between two points may be represented in terms of the amplitude and phase functions $w(s)$, $\psi(s)$ as follows:[8]

$$\tilde{M}\left(\frac{s_2}{s_1}\right) = \begin{pmatrix} \dfrac{w_2}{w_1}\cos\psi - w_2 w_1' \sin\psi & w_2 w_1 \sin\psi \\ \left(\dfrac{w_2'}{w_1} - \dfrac{w_1'}{w_2}\right)\cos\psi - \dfrac{1 + w_2 w_2' w_1 w_1'}{w_2 w_1}\sin\psi & \dfrac{w_1}{w_2}\cos\psi - w_2' w_1 \sin\psi \end{pmatrix}, \tag{58}$$

where $\psi = \psi(s_2) - \psi(s_1)$ and $w' = dw/ds$. If this is applied to the half period in the FODO channel of Figure 1, we see that

$$w_1' = w_2' = 0. \tag{59}$$

Consequently,

$$\tilde{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{w_2}{w_1}\cos\psi & w_1 w_2 \sin\psi \\ -\dfrac{1}{w_1 w_2}\sin\psi & \dfrac{w_1}{w_2}\cos\psi \end{pmatrix}. \tag{60}$$

Moreover, w_1 and w_2 represent the maximum and minimum of the amplitude function in the channel, i.e.,

$$w_1 = w_{0,\max}, \qquad w_2 = w_{0,\min}. \tag{61}$$

We thus obtain, with $\Theta = 2\phi$, from (56) to (60) the following equations for the maximum and minimum of the amplitude function:

$$\frac{w_1^2}{w_2^2} = \frac{m_{22}}{m_{11}} = \frac{1 + \tanh(\Theta/2)[\tan(\Theta/2) + (L/l)\Theta]}{1 - \tan(\Theta/2)[\tanh(\Theta/2) + (L/l)\Theta]} = n_1^2, \tag{62}$$

$$w_1^2 w_2^2 = -\frac{m_{12}}{m_{21}} = \frac{l^2}{\Theta^2}\,\frac{1 + \coth(\Theta/2)[\tan(\Theta/2) + (L/l)\Theta]}{\tan(\Theta/2)[\coth(\Theta/2) + (L/l)\Theta] - 1} = \frac{l^2}{\Theta^2} n_2^2. \tag{63}$$

Solving for w_1^2, w_2^2 yields

$$w_1^2 = w_{0,\max}^2 = \frac{l}{\Theta} n_1 n_2 = \frac{1}{\sqrt{\kappa}} n_1 n_2, \tag{64}$$

$$w_2^2 = w_{0,\min}^2 = \frac{l}{\Theta}\frac{n_2}{n_1} = \frac{1}{\sqrt{\kappa}}\frac{n_2}{n_1}. \tag{65}$$

For $\Theta \ll 1$, we get the approximations

$$n_1 \approx \left[\frac{1 + (1 + 2(L/l))\Theta^2/4}{1 - (1 + 2(L/l))\Theta^2/4}\right]^{1/2}, \tag{66}$$

$$n_2 \approx \frac{2\sqrt{3}(1 + (L/l))^{1/2}}{\Theta(1 + 3(L/l))^{1/2}}. \tag{67}$$

The phase shift μ_0 is obtained from the transfer matrix (M_{ij}) for a full period. With $\Theta = 2\phi$, one obtains

$$\cos\mu_0 = \tfrac{1}{2}(M_{11} + M_{22}) = \cos\Theta\cosh\Theta - \frac{L}{l}\Theta(\sin\Theta\cosh\Theta - \cos\Theta\sinh\Theta) - \frac{1}{2}\left(\frac{L}{l}\right)^2\Theta^2\sin\Theta\sinh\Theta. \tag{68}$$

For $\mu_0 \ll 1$, we can expand both sides and find

$$\mu_0 \approx \frac{1}{\sqrt{3}}\Theta^2\left[1 + 4\frac{L}{l} + 3\left(\frac{L}{l}\right)^2\right]^{1/2}. \tag{69}$$

It is therefore convenient to write

$$\mu_0 = \frac{1}{\sqrt{3}}\Theta^2\left[1 + 4\frac{L}{l} + 3\left(\frac{L}{l}\right)^2\right]^{1/2} h_0\left(\Theta, \frac{L}{l}\right), \tag{70}$$

where

$$h_0\left(\Theta, \frac{L}{l}\right) = \frac{\sqrt{3}\cos^{-1}[\cos\Theta\cosh\Theta - (L/l)\Theta(\sin\Theta\cosh\Theta - \cos\Theta\sinh\Theta) - \tfrac{1}{2}(L/l)^2\Theta^2\sin\theta\sinh\Theta]}{\Theta^2[1 + 4(L/l) + 3(L/l)^2]^{1/2}} \tag{71}$$

is close to unity when $\mu_0 < 1$.

Particle motion in the FODO channel is stable when

$$|\cos\mu_0| < 1 \tag{72}$$

Substitution of the expression (70) for the phase shift μ_0 and (64) for $w_{0,\max}^2$ into Eq. (51) yields for the beam current that can be transported in a FODO channel of period $S = 2l(1 + L/l)$ the result

$$I = \frac{I_0}{2}\beta^2\gamma^3\left(\frac{a}{l}\right)^2 H\left(\Theta, \frac{L}{l}\right)\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]. \quad (73)$$

The function $H(\Theta, L/l)$ is defined as

$$H(\Theta, L/l) = \Theta^3 \frac{[1 + 4(L/l) + 3(L/l)^2]^{1/2}}{2\sqrt{3}(1 + L/l)} \frac{h_0(\Theta, L/l)}{n_1 n_2} \quad (74)$$

and is plotted in Figure 2 versus Θ for several values of the ratio L/l. It rises initially as Θ^4, reaches a maximum at $\Theta = \Theta_{max}$, and then drops rapidly as it approaches the stability limit $|\cos \mu_0| = 1$ at $\Theta = \Theta_s$. For $\Theta < 0.6\Theta_{max}$ and $L/l < 3$, one may use the approximation

$$H\left(\Theta, \frac{L}{l}\right) \approx \Theta^4 \frac{[1 + 4(L/l) + 3(L/l)^2]^{1/2}}{12} \times \frac{(1 + 3(L/l))^{1/2}}{(1 + L/l)^{3/2}} \left[\frac{1 - (1 + 2(L/l))\Theta^2/4}{1 + (1 + 2(L/l))\Theta^2/4}\right]^{1/2}. \quad (75)$$

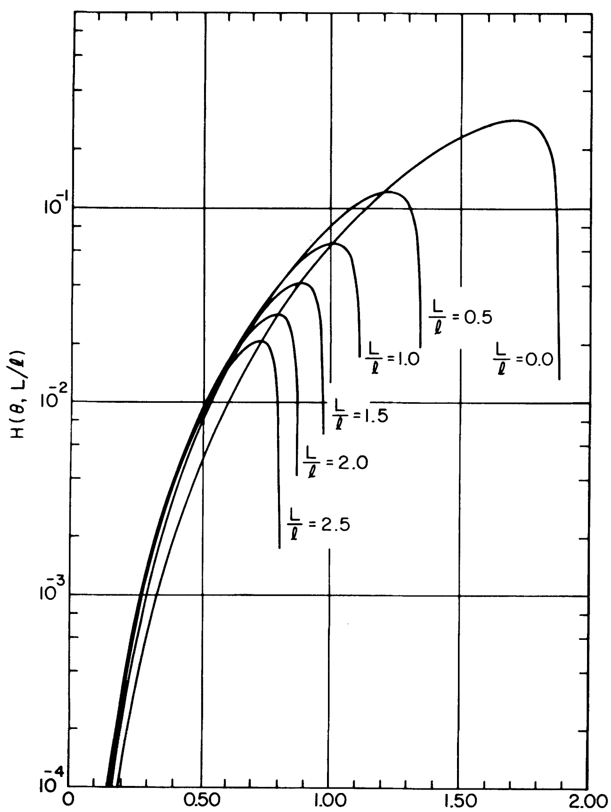

FIGURE 2 The function $H(\theta, L/l)$ of Eq. (74) for a FODO channel.

In the case $L/l = 1$, for instance, one gets the simple expression

$$H(\Theta, 1) \approx \frac{\Theta^4}{6}\left[\frac{1 - 0.76\Theta^2}{1 + 0.75\Theta^2}\right]^{1/2}, \quad (76)$$

and as Figure 2 indicates, there is very little difference between the various $H(\Theta, L/l)$ curves in this region of small Θ values.

The maximum of $H(\Theta, L/l)$ occurs at $\Theta = \Theta_{max}$ and one finds approximately

$$\Theta_{max} \approx 4\left[5\left(1 + 2\frac{L}{l}\right)\right]^{1/2}, \quad (77)$$

which is less than 5 per cent larger than the exact value in Figure 2. At small values of Θ, say, $\Theta \le 0.3\Theta_{max}$, the quadratic term $(1 + 2(L/l))\Theta^2/4$ in the expressions (75), (76) may be neglected (the error is less than 10 per cent). $H(\Theta, L/l)$ is then practically proportional to Θ^4, and in this region Eq. (73) for the current takes the form

$$I = \frac{I_0}{2}\beta^3\gamma^3\left(\frac{a}{l}\right)^2 \Theta^4 g(L/l)[1 - (\varepsilon/\alpha)^2], \quad (78)$$

where the geometry factor $g(L/l)$ is given by

$$g\left(\frac{L}{l}\right) = \frac{[1 + 4(L/l) + 3(L/l)^2]^{1/2}(1 + 3(L/l))^{1/2}}{12(1 + L/l)^{3/2}}. \quad (79)$$

For ions of mass number A and charge state Z, the focusing parameter Θ, which is defined in (57) and measures the strength of the focusing lenses, is given by

$$\Theta = 0.5675\left(\frac{Z}{A}\right)^{1/2}\left(\frac{B_0 a}{\beta\gamma}\right)^{1/2}\frac{l}{a}. \quad (80)$$

In the nonrelativistic region, we may use for $\beta\gamma$ the relation

$$\beta\gamma = 0.04634\sqrt{\frac{W}{A}}, \quad (81)$$

where W/A is the kinetic energy per nucleon in MeV. By substituting (52) for I_0 and (80) for Θ, the expression (78) for the beam current becomes

$$I_{[A]} = 16.1 \times 10^5 \frac{Z}{A}\beta\gamma(B_0 l)^2 g\left(\frac{L}{l}\right)\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right], \quad (82)$$

and the corresponding beam power

$$P = (\gamma - 1)(m_0 c^2/q)I = 9.3148 \times 10^8(\gamma - 1)(A/Z)I$$

is

$$P_{[W]} = 1.5 \times 10^{15}\beta\gamma(\gamma - 1)(B_0 l)^2 g\left(\frac{L}{l}\right)\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]. \tag{83}$$

For $L/l = 1$, the geometry factor is $g(1) = 0.167$.

Thus, in the region of small Θ values where $H(\Theta, L/l)$ varies as Θ^4 and where the phase shift μ_0 is small, we obtain expressions for the ion current and power that, apart from the numerical factors, exhibit the same functional structure as our previous results.[7] Specifically, in this parameter range, the particle current I/Z and the power P are independent of the charge state Z for a given channel structure and emittance (provided $\varepsilon \ll \alpha$). As the focusing parameter Θ, and hence the phase shift μ_0, increases, the amount of current that can be transported in the FODO channel is determined by the behavior of the function $H(\Theta, L/l)(a/l)^2[1 - (\varepsilon/\alpha)^2]$. For particles with given charge, mass, and energy, the beam current reaches an upper limit, $I_{\max}$, at the maximum of this function and then drops rapidly as the focusing limit $\mu_0 = 180°$ is approached. We should note, however, that near and beyond this current maximum, i.e., at large values of μ_0, our formula (73) is not too accurate (see discussion of validity in Section 4.1).

As in the case of long solenoids, our results show that the emittance ε must be significantly smaller than the channel acceptance α for transport of high currents. When this is not the case, i.e., when $X_{0,\max}$ is only slightly less than the semi-aperture a, we find that the current may vary considerably with small changes in the focusing parameter Θ (or phase shift μ_0) in view of the fourth-power variation of the factor $X_{0,\max}/a$ according to Eq. (50).

We conclude that the acceptable current in a FODO channel for particles of given energy depends on the focusing parameter Θ, the lattice structure, L/l, the emittance ε of the beam, and the channel acceptance α in a way that, in general, does not yield a simple scaling law except in the region where the phase shift μ_0 is small. We will discuss this question further in Section 4 where examples will be given to illustrate the application of these results to various parameter regimes for high-current beam transport.

3.2 *Solenoid Channels*

The general theory presented in Section 2 is applicable also to channels with short solenoid magnets. For this case, Eq. (10) can be used except that κ_s is now a periodic function of the variable s. For an ideal channel, we assume that κ_s be a rectangular function which, in contrast to the quadrupole case, is always positive. As in Figure 1, the length of each magnet region, where $\kappa_s = \text{const}$, is l and the separation between magnets, where $\kappa_s = 0$, is L. The period of this lattice is then $S = (l + L)$ and the transfer matrix for one half period (from center of magnet to center of drift space) is given by

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & \frac{l}{2\phi}\sin\phi \\ -\frac{2\phi}{l}\sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi - \frac{L}{l}\phi\sin\phi & \frac{l}{2\phi}\sin\phi + \frac{L}{2}\cos\phi \\ -\frac{2\phi}{l}\sin\phi & \cos\phi \end{pmatrix},$$

where the lens parameter ϕ is defined as

$$2\phi = \Theta = \sqrt{\kappa_s}\, l = \frac{qB_s l}{2m_0 c\beta\gamma}. \tag{85}$$

The amplitude functions w_1, w_2 are determined by the relations

$$\frac{w_1^2}{w_2^2} = \frac{w_{0,\max}^2}{w_{0,\min}^2} = \frac{m_{22}}{m_{11}} = n_1^2 = \frac{1}{1 - (L/l)(\Theta/2)\tan(\Theta/2)}, \tag{86}$$

$$w_1^2 w_2^2 = -\frac{m_{12}}{m_{21}} = \frac{l^2}{\theta^2} n_2^2 = \frac{l^2}{\Theta^2}\left[1 + \frac{L}{l}\frac{\Theta}{2}\cot\left(\frac{\Theta}{2}\right)\right], \tag{87}$$

from which follows

$$w_{0,\max}^2 = \frac{l}{\Theta} n_1 n_2 = \frac{l}{\Theta}\left[\frac{1 + (L/l)(\Theta/2)\cot(\Theta/2)}{1 - (L/l)(\Theta/2)\tan(\Theta/2)}\right]^{1/2}. \quad (88)$$

The transfer matrix for one full period is

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} \cos\Theta - \frac{L}{l}\Theta\sin\Theta & \frac{l}{\Theta}\sin\Theta + L\cos\Theta \\ -\frac{\Theta}{l}\sin\Theta & \cos\Theta \end{pmatrix}. \quad (89)$$

The phase shift per period is then given by

$$\cos\mu_0 = \tfrac{1}{2}(M_{11} + M_{22}) = \cos\Theta - \frac{1}{2}\frac{L}{l}\Theta\sin\Theta, \quad (90)$$

and the motion is stable for $|\cos\mu_0| \leq 1$ as in the FODO case.

At small values of Θ and μ_0, we have the approximation $\mu_0 \approx \Theta(1 + L/l)^{1/2}$, and it is therefore convenient to write

$$\mu_0 = \Theta\left(1 + \frac{L}{l}\right)^{1/2} f_0\left(\Theta, \frac{L}{l}\right). \quad (91)$$

The function $f_0(\Theta, L/l)$ is defined as

$$f_0\left(\Theta, \frac{L}{l}\right) = \frac{\cos^{-1}(\cos\Theta - \frac{1}{2}(L/l)\Theta\sin\Theta)}{\Theta(1 + L/l)^{1/2}} \quad (92)$$

and is approximately unity for $\mu_0 < 1$. The substitution of (91) and (88), with $S = l(1 + L/l)$, into Eq. (51) yields for the current in the solenoid channel the expression

$$I = \frac{I_0}{2}\beta^3\gamma^3\frac{a^2}{l^2}F\left(\Theta, \frac{L}{l}\right)\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right], \quad (93)$$

where

$$F\left(\Theta, \frac{L}{l}\right) = \Theta^2\frac{f_0(\Theta, L/l)}{(1 + L/l)^{1/2}} \times \left[\frac{1 - (L/l)(\Theta/2)\tan(\Theta/2)}{1 + (L/l)(\Theta/2)\cot(\Theta/2)}\right]^{1/2}. \quad (94)$$

When the phase shift μ_0 is small, one may use the approximation

$$F\left(\Theta, \frac{L}{l}\right) \approx \Theta^2\left(1 + \frac{L}{l}\right)^{-1}. \quad (95)$$

For ions with charge state Z and mass number A, the focusing parameter Θ is given by

$$\Theta = 0.16\frac{Z}{A}\frac{B_s a}{\beta\gamma}\frac{l}{a}. \quad (96)$$

By substitution of (95), (96), and (52) into (93), one obtains for the current that can be transported in a solenoid channel with small phase shift μ_0 the result

$$I_{[A]} = 4.0 \times 10^5\frac{Z}{A}\beta\gamma(B_s a)^2\frac{l}{l + L}\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]. \quad (97)$$

For a long solenoid channel without drift space, we have $L = 0$, and (97) reduces to Eq. (17) as it should.

For larger values of the parameter Θ, the current is determined by the function $F(\Theta, L/l)$. As in the quadrupole case, the current reaches a maximum at some value $\Theta_{\max}$ and then rapidly drops as the stability limit $\mu_0 = 180°$ is approached. When $L = l$, for instance, we find

$$\Theta_{\max} \approx 1.47; \qquad F(\Theta, 1)_{\max} \approx 0.71. \quad (98)$$

This gives for the ion current the result

$$I = 1.1 \times 10^7\frac{A}{Z}\beta^3\gamma^3\left(\frac{a}{l}\right)^2\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right], \quad (99)$$

which is valid only when the condition

$$0.16\frac{Z}{A}\frac{B_s a}{\beta\gamma}\frac{l}{a} = 1.47 \quad (100)$$

is satisfied simultaneously. However, as noted before, our formula is not very accurate at large μ_0 values; furthermore, as will be discussed in the next section, the region near the theoretical current maximum is not useful for high-power beam transport due to envelope instabilities.

4 DISCUSSION

4.1 *Validity of the Theory*

The derivation of the general formula (51) for the beam current that can be transported in a periodic channel was based on the assumptions that the ripple in the beam envelope and the phase shift are small, i.e., $\delta_{max} \ll 1$ and $\mu \ll 2\pi$ [Eqs. (20) and (21)]. Both parameters are determined by the zero-intensity phase shift, μ_0, and the beam current. Specifically, δ_{max} increases with μ_0, and the phase shift μ decreases when the space-charge parameter $u = KS/2\mu_0\varepsilon$, and hence the current, is increased [according to Eq. (54)]. Thus the larger the phase shift μ_0, the larger is the error in formula (51); at the same time, for a given phase shift μ_0, the accuracy of the theory improves with increasing beam intensity. We can therefore obtain an upper limit for the relative error by examining the validity of the assumptions for the case of zero space charge. As an example, we choose a FODO channel with $L/l = 1$ and for the phase shift the value $\mu_0 = 90°$. The maximum and minimum of the envelope function $w(s)$ can be determined from the exact equations (62) to (65) (with $\Theta = 0.93$). One finds $w_{0,max}/w_{0,min} = n_1 = 2.2667$, $w_{0,max} = 2.531\sqrt{l}$, $\delta_{max} = (n_1 - 1)/(n_1 + 1) = 0.388$. Obviously the assumption $\delta_{max} \ll 1$ is not well-satisfied in this case.

We now calculate $w_{0,max}$ by our approximate relation (26), i.e.,

$$w_{0,max} = \sqrt{\frac{S}{\mu_0}}(1 + \delta_{max})$$

$$= \sqrt{\frac{2l(1 + L/l)}{\mu_0}}(1 + \delta_{max}).$$

Using the above values for L/l, μ_0, and δ_{max}, we find

$$w_{0,max} = 2.220\sqrt{l}.$$

This result indicates that at a phase shift μ_0 of about 90°, the maximum error in the calculation of the beam envelope by the smooth approximation is roughly 12 per cent. The accuracy of the theory improves with increasing current.

From the calculations of the function $H(\Theta, L/l)$ in Figure 2, one finds that the maxima occur at values of the phase shift μ_0 near 110°. The maximum error in the determination of w_{max} is about 20 per cent in this case. However, it is known from the literature on high-current electron beams that the envelopes are unstable against small variations in the initial conditions near the current maximum.[11] Recently, Laslett did some numerical work on the stability of the matched-beam envelopes which yielded similar, though quantitatively more accurate, results for intense ion beams in FODO channels; he found[12] that the envelopes are unstable against perturbations in the starting conditions for values of the phase shift μ_0 above about 90°. Thus, one must design a focusing channel for currents that are safely below the maximum, and a phase shift μ_0 of 90° would appear to be a reasonable upper limit. We conclude therefore, that the accuracy of our theory is quite adequate in the parameter regime that is of practical interest for transport of intense beams.

4.2 *The Scaling Laws*

The current that can be transported through a periodic magnetic channel is determined by the charge, mass and energy of the particles, the magnetic field, the lattice structure, the average focusing force, the space-charge force, and the emittance. In the smooth approximation method, the average focusing force is represented by the square of the phase shift per period, μ_0^2, and the mean radius of the beam can be calculated by integration of Eq. (37). For zero space charge ($K = 0$), the maximum of the beam envelope, $X_{0,max}$, for given lattice and beam parameters is determined by the emittance ε and the maximum of the amplitude function, $w_{0,max}$. The space charge forces increase the maximum of the envelope to a value $X_{max} > X_{0,max}$, which depends on the parameter $u = KS/2\mu_0\varepsilon_0$. For a channel with given aperture $2a$, the limit is reached when $X_{max} = a$. This condition then leads to the expression (51) for the current which is seen to be proportional to $\mu_0\alpha[1 - (\varepsilon/\alpha)^2]$, where ε/α represents the ratio of beam emittance to channel acceptance for zero space charge. Physically, the last term implies that high-current beam transport in a periodic channel requires that the emittance of the beam must be considerably smaller than the acceptance of the channel.[13]

The application of the general theory to a FODO channel results in Eq. (73). This equation shows that the beam current is largely determined by the ratio $(a/l)^2$ and the function $H(\Theta, L/l)$ which depends on the focusing parameter Θ and the spacing ratio L/l of the magnets in the periodic lattice. (Note that we use the parameter Θ rather than the phase shift μ_0 since Θ is a simple function of the channel and particle parameters while μ_0 is not.) Thus, there is no simple, generally valid scaling law that shows explicitly the dependence of the current in FODO channels (or other periodic channels) on the experimental parameters. The main exception is the fact that an increase in particle energy always increases the beam current. Apart from that, the acceptable current depends on the way the pertinent parameters are changed and on the constraints that are imposed by design requirements, stability limits, etc. In the following, we shall discuss three cases of beam transport design in FODO channels which illustrate this point.

Case 1: Consider first the case in which the zero-intensity phase shift is small, i.e., $\mu_0 \ll \pi$, but otherwise not constrained by the condition that it have a fixed value. In this case, the function $H(\Theta, L/l)$ varies as Θ^4 and we obtain the scaling law (82), i.e., $I \propto (Z/A)\beta\gamma(B_0 l)2[1 - (\varepsilon/\alpha)^2]$. The particle current is then independent of the charge state and the beam power is independent of both charge state Z and mass number A as long as the emittance is significantly smaller than the acceptance of the channel ($\varepsilon \ll \alpha$).

Case 2: Let us impose now the constraints $\mu_0 = \text{const}$, $L/l = \text{const}$, i.e., the particle parameters A, Z, $\beta\gamma$, and the channel parameters B_0, a, l are changed in such a way that the phase shift μ_0 and the ratio L/l remain fixed. In this case, the focusing parameter Θ is a constant and we have from (80)

$$\frac{Z}{A}\frac{B_0}{\beta\gamma}\frac{l^2}{a} = C_1, \tag{101}$$

where C_1 is a constant which is determined by the fixed value of Θ or μ_0. From (73) we get, with $H(\Theta, L/l) = \text{const}$, for the particle current in this case

$$\frac{I}{Z} \propto \frac{A}{Z^2}\beta^3\gamma^3\frac{a^2}{l^2}\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]. \tag{102}$$

By substitution of (101), we find

$$\frac{I}{Z} \propto \frac{\beta^2\gamma^2}{Z}B_0 a\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]. \tag{103}$$

We can cast this equation into a form similar to the Maschke–Courant formula by introducing the channel acceptance α in place of the semi-aperture a. From Eq. (47) we have $a^2 = w_{0,\max}^2\alpha$ which may be written as

$$a^2 \propto \alpha l \tag{104}$$

since, from (64), $w_{0,\max}^2/l$ is a constant when Θ and L/l are fixed. Elimination of l in (104) with the aid of (101), yields

$$a \propto \alpha^{2/3}\left(\frac{A\beta\gamma}{ZB_0}\right)^{1/3}. \tag{105}$$

Hence, we have

$$\frac{I}{Z} = C_2\frac{A^{1/3}}{Z^{4/3}}(\beta\gamma)^{7/3}B_0^{2/3}\alpha^{2/3}\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right], \tag{106}$$

where C_2 is a constant which depends on the value of the phase shift μ_0. For $\mu_0 = 90°$ and $L/l = 1$, we find $\Theta = 0.9318$, $w_{0,\max} = 2.5312\sqrt{l}$, $H(\Theta, L/l) = 0.0612$, With these numbers and using mks units, we obtain for the two constants the numerical values $C_1 = 2.696$, $C_2 = 1.689 \times 10^6$, while relation (104) becomes

$$a^2 = 6.407\alpha l. \tag{107}$$

Thus, with B_0 given in teslas, α in m-rad, we find for the particle current

$$\left(\frac{I}{Z}\right)_{[A]} = 1.69 \times 10^6\frac{A^{1/3}}{Z^{4/3}}(\beta\gamma)^{7/3}B_0^{2/3}\alpha^{2/3}\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right], \tag{108}$$

and for the power

$$P_{[W]} = 1.57 \times 10^{15}\left(\frac{A}{Z}\right)^{4/3}(\beta\gamma)^{7/3}(\gamma - 1)B_0^{2/3}\alpha^{2/3}\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]. \tag{109}$$

These expressions, which are valid when $\mu_0 = 90°$, $L/l = 1$, are practically identical to Maschke's formula except that we have $\alpha^{2/3}[1 - (\varepsilon/\alpha)^2]$ where he has $\varepsilon^{2/3}$ and our numerical factor is slightly different. The major difference between this result and Maschke's expression is that the beam power (or particle current) in this case decreases with the emittance as $1 - (\varepsilon/\alpha)^2$ and increases with the channel acceptance α. Note that while the emittance ε remains unchanged in an ideal beam transport system, the acceptance α varies when the radius a or the magnet length l is changed. As is evident from Eq. (103), one can make the current in this case arbitrarily large by increasing the radius a and thus the acceptance α. A practical constraint is imposed by the fact that the length l of the magnet should not be less than the magnet aperture, say, $l \gtrsim 2a$.

Case 3: When the current I, and thus the space-charge parameter u defined in Eq. (40), increases, the phase shift μ decreases according to Eq. (54). There may be a minimum permissible value for the phase-shift ratio μ/μ_0 where the beam propagation becomes unstable. The existence of such a limit would have to be determined by a stability analysis, which is beyond the scope of this paper. However, we can examine how the scaling law for the beam current is affected by such a limit. Suppose then that, in addition to the constraints of case 2 (μ_0 = const, L/l = const), we impose a restriction on the space-charge parameter u which from (54) is given by

$$u \leq u_{\max} = \frac{1}{2(\mu/\mu_0)_{\min}}\left[1 - \left(\frac{\mu}{\mu_0}\right)^2\right], \quad (110)$$

where $(\mu/\mu_0)_{\min}$ defines the lower limit of the phase-shift ratio. Now, from (40) and (48) we have the relation

$$u = \frac{1}{2}\frac{\alpha}{\varepsilon}\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right], \quad (111)$$

from which we get

$$\frac{\varepsilon}{\alpha} = \frac{\mu}{\mu_0} = \sqrt{1 + u^2} - u. \quad (112)$$

The constraint (110) implies that the current cannot exceed a limit given by

$$I_{\max[A]} = 1.55 \times 10^7 \frac{A}{Z}(\beta\gamma)^3 \frac{\mu_0 \varepsilon}{l(1 + L/l)} u_{\max}, \quad (113)$$

or, equivalently, that the acceptance α cannot exceed the limit

$$\alpha_{\max} = \frac{\varepsilon}{(\mu/\mu_0)_{\min}} = f_1 \varepsilon, \quad (114)$$

where the constant f_1 is given by the ratio $(\mu_0/\mu)_{\min}$. The existence of a lower limit of the phase-shift ratio thus fixes the value of the acceptance α with respect to the emittance ε, and we obtain

$$\alpha^{2/3}\left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right] = f_1^{2/3}\varepsilon^{2/3}\left[1 - \left(\frac{1}{f_1^2}\right)\right] = f_2 \varepsilon^{2/3}. \quad (115)$$

Substitution of (115) into (106) yields for the particle current the expression

$$\frac{I}{Z} = C_2 f_2 \frac{A^{1/3}}{Z^{4/3}}(\beta\gamma)^{7/3} B_0^{2/3} \varepsilon^{2/3}. \quad (116)$$

Thus, by using the conditions μ_0 = const, L/l = const, and $\mu/\mu_0 = (\mu/\mu_0)_{\min} = 1/f_1$, we obtain from our general formula the Maschke-Courant scaling law, i.e., $I \propto \varepsilon^{2/3}$. The constant C_2 depends on the value of the zero-intensity phase shift, μ_0, while f_2 is determined by the lower limit of the phase shift with space charge, i.e., $(\mu/\mu_0)_{\min}$. For $\mu = 90°$, $f_1 = \alpha/\varepsilon = 2$, for instance, one finds $f_2 = 1.191$ and $C_2 f_2 = 2.01 \times 10^6$; when $\mu_0 = 90°$ and $f_1 = 4$, the constants are $f_2 = 2.362$ and $C_2 f_2 = 3.99 \times 10^6$. Note that in this case there is almost no flexibility in the channel design. If the magnetic field B_0, and the particle constants Z, A, $\beta\gamma$ are given, the magnetic parameters a and l are uniquely determined by the condition a^2/l = const, from Eq. (104) with α fixed by (114), and l^2/a = const from Eq. (101).

4.3 *Numerical Examples of High-Current Beam Transport*

In the following, we present a few examples that serve to illustrate the application of the theory to the design of high-power beam-transport systems for pellet fusion with heavy ions. We consider a uranium beam ($A = 238$) and determine the beam current that can be transported in a FODO channel at a final particle energy of 40 GeV (example 1), at an intermediate energy of 238 MeV (example 2), and at a low energy of 1 MeV (example 3) with the constraint that the magnetic field B_0 and/or the phase shift μ_0 are fixed. It will be assumed that the spacing between magnets and the length of the magnets are the same, i.e., $L/l = 1$, and that the beams have a fixed normalized emittance of $\varepsilon_N = 1.8 \times 10^{-5}$ m-rad. For the magnetic field strength B_0, we take a value of 1 Tesla when iron quadrupoles are used and 3.5 Tesla in the case of a superconducting channel. We first assume a charge state of $Z = 1$ and then examine what happens when Z is increased.

Example 1: $W = 40$ GeV, $\beta\gamma = 0.627$.

We want to transport a particle current in the range of 2 kA, and we take first an iron channel ($B_0 = 1$ T) with $Z = 1$ and $\mu_0 = 90°$ (case *a*), then a superconducting channel ($B_0 = 3.5$ T) with $Z = 1$, $Z = 10$, and $\mu_0 = 90°$ (case *b*), and finally, a superconducting channel ($B_0 = 3.5$ T) operating in the region of small phase shift where the particle current is not strongly dependent on the charge state.

a) With $B_0 = 1$ T, $Z = 1$, $a = 0.030$ m, $\mu_0 = 90°$, we get $\Theta = 0.9318$, $l = 3.474$ m, $H(\Theta, 1) = 0.0612$, $w_{0,\max} = 2.531\sqrt{l} = 4.718$. The maximum of the zero-intensity beam envelope is then

$$X_{0,\max} = \sqrt{\frac{\varepsilon_N}{\beta\gamma}}\, w_{0,\max} = 0.0253;$$

hence,

$$1 - \left(\frac{\varepsilon}{\alpha}\right)^2 = 1 - \left(\frac{X_{0,\max}}{a}\right)^4 = 0.4959.$$

By substitution of these values into Eq. (73), one then finds for the particle current $I = 2.058 \times 10^3$ A. The space charge parameter u, defined in Eq. (40), has in this case the value $u = 0.349$, and from (54) one obtains for the phase shift with space charge the result $\mu = 63.93°$. An increase of the charge state Z without any change of the channel parameters would make the phase shift μ_0 larger than the 90° upper limit and is therefore ruled out.

b) Let us now consider a superconducting channel ($B_0 = 3.5$ T) with a somewhat larger aperture ($a = 0.043$ m), a charge state $Z = 10$, and a phase shift $\mu_0 = 90°$ for this charge state. We then obtain the following results: $l = 0.703$ m, $w_{0,\max} = 2.122$, $X_{0,\max} = 0.011$ m, $1 - (\varepsilon/\alpha)^2 = 0.995$; and for the particle current $I/Z = 2.059 \times 10^3$ A. The space-charge parameter is $u = 14.355$, and the phase shift with space charge is $\mu = 3.13°$. The advantages of operating with high charge state are quite obvious. One can build magnets with considerably shorter lengths, and the energy gain per gap is increased by a factor Z. In our specific example, for instance, the length of a linear (or induction) accelerator could be reduced by a factor of $10 \times (3.474/0.703) \simeq 50$ for a given energy gain compared with the channel in case (a). At the same time, the phase shift with space charge is relatively low, and one would have to examine if there are any stability problems. Such a channel can also transport beams with particles at lower charge state. For $Z = 1$, for instance, one obtains a current of 3.458×10^3 A, which is considerably larger than the 2 kA obtained in the $Z = 10$ case.

c) In the region where μ_0 is small and the particle current is independent of the charge state, one obtains the desired 2 kA with the following parameters: $B_0 = 3.5$ T, $a = 0.118$ m, $l = 0.5$ m. For particles with charge state $Z = 10$, one then finds $\Theta = 0.40$, $\mu_0 = 15°$, $w_{0,\max} = 0.707$, $X_{0,\max} = 0.0158$ m, and $I/Z = 1.919 \times 10^3$ A. When the charge state is lower, one gets essentially the same particle current independent of Z. For $Z = 1$, for instance, one finds $I = 2.097 \times 10^3$ A. The main question is whether or not stable transport can be achieved at very low values of the phase shift. In view of the obvious advantages of operating in such a regime, this question merits detailed examination by designers of heavy ion accelerators for pellet fusion.

Example 2: $W = 238$ MeV, $\beta\gamma = 0.046$.

With $B_0 = 3.5$ T, $a = 0.10$ m, $Z = 1$, $\mu_0 = 90°$, one finds $l = 0.932$ m, $X_{0,\max} = 0.048$ m, and $I = 246.8$ A. The particle current at this lower energy is thus down by an order of magnitude. Transport of particles with higher charge state is more restricted in this case. For $Z = 3$, $\mu_0 = 90°$, for instance, we find $l = 0.538$ m and $I/Z = 82.6$ A, i.e., the particle current is reduced by a factor 3.

Example 3: $W = 1$ MeV, $\beta\gamma = 3 \times 10^{-3}$.

With $B_0 = 3.5$ T, $a = 0.10$ m, $Z = 1$, $\mu_0 = 90°$, one obtains $l = 0.235$ m, $X_{0,\max} = 0.095$ m, and $I = 0.208$ A. These numbers illustrate the difficulties of transporting high-current beams at low energy. In our specific example, the current is only a fraction of an ampere, and, furthermore, the length l of the magnets approaches the size of the aperture $2a$. At these low energies, solenoids become comparable in effectiveness to quadrupoles, as was pointed out in Ref. 6.

4.4 *Conclusions*

By solving the KV-envelope equations with the smooth approximation method, we obtained the general formula (51) for a matched beam. It allows one to calculate the current that can be transported in a periodic channel as a function of the particle parameters (Z, A, $\beta\gamma$), the beam emittance ε, the channel acceptance α, and the zero-intensity phase shift μ_0. The application to quadrupole channels of the FODO type yielded Eq. (73), and for solenoid channels we obtained the analogous expression (93). We found that, with the exception of the long solenoid, there is no simple, generally valid scaling law which relates the beam current in an explicit form to all experimental parameters. The form of the current law for a given channel depends instead on the type of constraints that are used by the designer and the value of the phase shift μ_0. When μ_0 (or the focusing parameter Θ) is small ($\mu_0 \ll \pi$) but otherwise not constrained by the condition that it have a fixed value, we obtain a scaling law of the form $I \propto (Z/A)\beta\gamma(B_0 l)^2[1 - (\varepsilon/\alpha)^2]$, which was first derived in our previous paper.[7] On the other hand, if the phase shift μ_0 is fixed when other parameters are changed, we obtain the expression (111) for the beam current, i.e.,

$$I \propto \left(\frac{A}{Z}\right)^{1/3} (\beta\gamma)^{7/3} B_0^{2/3} \alpha^{2/3} \left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right].$$

This result is identical with the Maschke–Courant formula except for the factor $\alpha^{2/3}[1 - (\varepsilon/\alpha)^2]$. Physically, this factor implies that for high-current beam transport the acceptance α of the channel must be significantly larger than the emittance ε of the beam. If we make the additonal restriction that the phase shift ratio μ/μ_0 has a lower limit, we find that the ratio of the channel acceptance α to the emittance ε is fixed by this limit. With $\alpha \propto \varepsilon$, we obtain a scaling law for the beam power in the form that was first derived by Maschke, namely,

$$P = \text{const}\left(\frac{A}{Z}\right)^{4/3} (\beta\gamma)^{7/3} (\gamma - 1) B_0^{2/3} \varepsilon^{2/3}.$$

Thus, the different scaling laws for FODO channels are seen to be special solutions of our general formula (73), which was derived by the smooth approximation technique from the K-V envelope equations. The differences and seeming contradictions are the result of different constraints used in the design of a focusing channel.

With regard to heavy-ion beams for pellet fusion, our numerical examples show that there could be a great advantage in the use of superconducting magnets and particles with high charge state in the high-energy range. The focusing and accelerating structures could be considerably shorter compared with iron channels and singly charged particles. The main question that needs to be examined in this case is how low one can go with the phase shift μ before instabilities arise.

Our analysis also confirms the well-known difficulties to focus high-current beams at low energies. As discussed in the previous paper, electrical breakdown severely limits the energies obtainable from conventional ion sources, and it will be important for heavy-ion fusion that new ion source/injector systems, such as collective accelerators, be developed with energies above 1 MeV per nucleon and beam currents in the 100 A range.[7]

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13. Note that, in practice, the maximum beam radius is somewhat smaller than the semi-aperture of the magnets. Thus, $2a = 2X_{max}$ is to be taken as the maximum width of the beam and B_0 as the magnetic field at the edge of the beam ($x = a$).