III. ATOMIC RESONANCE AND SCATTERING*

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A. WORK COMPLETED

1. SPIN-EXCHANGE COLLISIONS BETWEEN ALKALI ATOMS AND TARGETS OF ARBITRARY SPIN IN A UNIFORM MAGNETIC FIELD

This research has been completed and a thesis with the same title was submitted by Gary M. Carter to the Department of Physics, M.I.T., in April 1969, in partial fulfillment of the requirements for the degree of Master of Arts.

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B. RELATIONSHIP OF THE ACTION AND THE SEMI-CLASSICAL SCATTERING AMPLITUDE

1. Introduction

We shall show how to interpret Ford and Wheeler's semi-classical expression[1] for the scattering amplitude as an integral of the action over all impact parameters. This formulation clarifies the relationship between the action and the phase of the semi-classical expression for the scattering amplitude and lends physical insight into the angular dependence of interference processes in scattering.

In quantum-mechanical treatments of scattering the impact parameter cannot be known because of the uncertainty principle (the transverse momentum before the collision is fixed by the incident angle and momentum). If the de Broglie wavelength

$$\lambda\!\!\!^{-} = \frac{\hbar}{p} = \frac{\hbar}{\sqrt{2mE}} \tag{1}$$

is much smaller than the scale of variation of the potential, however, the impact parameter, b, becomes physically significant. One can imagine that each portion of the incident wave can be followed through the collision; its deflection being determined only

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by b (and the reduced potential). In this report we show how to express the scattering amplitude as an integral of the action over the impact parameter, and we show how this treatment simplifies the understanding of interference phenomena observed in thermal-energy atom-atom scattering experiments. We also present a simple derivation of the impact approximation for the phase difference in resonant exchange collisions (for example, spin exchange or resonant charge exchange).

Ford and Wheeler[1] have shown that a considerable mathematical simplification in the usual partial-wave treatment of scattering results when the de Broglie wavelength is smaller than the scale of variation of the potential. By making a related set of approximations, collectively called the semi-classical approximation, they show that the scattering amplitude may be written

$$f(\theta) = -\lambda\!\!\!^{-}(2\pi \sin\theta)^{-1/2} \int_0^\infty \left(\ell + \frac{1}{2}\right)^{1/2} \left[e^{i\Phi_+} - e^{i\Phi_-}\right] d\ell \tag{2}$$

where

$$\Phi_\pm = 2\eta(\ell) \pm \left(\ell + \frac{1}{2}\right)\theta \pm \frac{\pi}{4} \tag{3}$$

and the phase shift, $\eta(\ell)$, is to be considered a continuous function of ℓ (and may be found consistently from the JWKB approximation).

We introduce the impact parameter by means of the usual correspondence relationship

$$\left(\ell + \frac{1}{2}\right) \longleftrightarrow |L|/\hbar \longleftrightarrow b/\lambda\!\!\!^{-}. \tag{4}$$

The angular momentum, L, can have either sign, but ℓ (and also b) is restricted to positive values. We rewrite Eqs. 2 and 3, using the impact parameter

$$f(\theta) = -(2\lambda\!\!\!^{-}\pi \sin\theta)^{-1/2} \int_0^\infty b^{1/2} \left[e^{i\Phi_+} - e^{i\Phi_-}\right] db \tag{5}$$

where

$$\Phi_\pm = 2\eta(b) \pm \frac{b}{\lambda\!\!\!^{-}}\theta \pm \frac{\pi}{4}. \tag{6}$$

The function $\eta(b)$ equals $\eta(\ell)$ when b equals $\ell\lambda\!\!\!^{-}$. This follows naturally from the JWKB expression for $\eta(\ell)$; $\eta(b)$ is also closely related to the classical phase.[2]

The expression in Eq. 5 is physically misleading because only regions with $b > 0$ contribute to the scattering amplitude, while it is clear (see Fig. III-1) that processes with either a net repulsive interaction and positive impact parameter or a net attractive

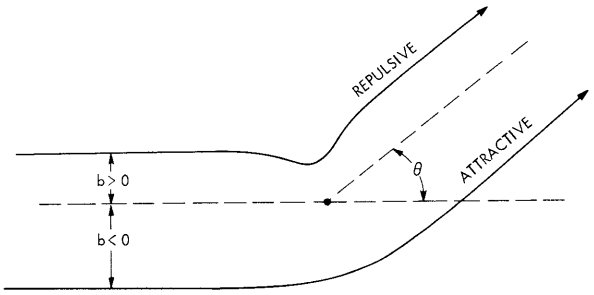

Fig. III-1. Trajectories for a collision with attractive interaction (and negative impact parameter) and a collision with dominantly repulsive interaction (and positive impact parameter), both of which result in the same deflection.

interaction and negative impact parameter can contribute to the scattering at positive angles. This fact is accounted for by the two terms in Eq. 5; the term Φ_- contributes for repulsive scattering, the Φ_+ for attractive scattering. A more meaningful expression may be obtained by changing the variable of integration for the Φ_+ term from b to -b, since then the attractive scattering will occur with negative impact parameter, as it should. We accomplish this by making the substitution $b' = e^{-i\pi}b$, so that the integral with Φ_+ in Eq. 5 becomes

$$\int_0^\infty db\, b^{1/2} \exp\left\{ i\left[2\eta(b) + \frac{b}{\chi}\theta + \frac{\pi}{4}\right]\right\} = -\int_0^{-\infty} db'(e^{-i\pi}b')^{1/2} \exp\left\{ i\left[2\eta(-b') - \frac{bb'}{\chi\theta} + \frac{\pi}{4}\right]\right\}$$

$$= \int_{-\infty}^0 db'\, b'^{1/2} \exp\left\{ i\left[2\eta(-b') - \frac{\chi\theta}{b'} - \frac{\pi}{4}\right]\right\}. \qquad (7)$$

Note that $\eta(-b) = \eta(b)$, so that the exponent in this expression is the same as the exponent in the Φ_- integral in Eq. 5. We can now write the scattering amplitude as an integral containing Φ_- over the whole range of b.

$$f(\theta) = \chi(2\pi \sin\theta)^{-1/2} \int_{-\infty}^{\infty} db\, b^{1/2} \exp\left\{ i\Phi\left[2\eta(b) - \frac{\theta b}{\chi} - \frac{\pi}{4}\right]\right\}. \qquad (8)$$

This expression is simpler and more natural than Eq. 5, since all impact parameters now can contribute to the scattering amplitude. An additional simplification results because the exponent is closely related to the action. F. T. Smith[2] has defined the collision action A (hereafter called simply the action) to be the difference between the total action with the potential V "on" and with it "off" (in which case the path is straight):

$$A(\theta, b) = \oint p \cdot dq - \oint_{V=0} p_o \cdot dq_o, \tag{9}$$

where the total energy is fixed. He shows that each term may be broken into radial and tangential components, the radial component being twice the JWKB phase shift, and the tangential component being proportional to the angular momentum and to the scattering angle.

$$A(\theta, b) = 2\hbar\eta(b) - \frac{\hbar b}{\chi}\theta. \tag{10}$$

The radial component is independent of the sign of b (that is, $\eta(-b) = \eta(b)$), while the angular term shows more action for collisions with $b < 0$, since this corresponds to the "outside track."

Our expression for the scattering amplitude (Eq. 8) is simply expressed in terms of the action

$$f(\theta) = e^{-i\pi/4}(2\chi\pi\sin\theta)^{-1/2}\int_{-\infty}^{\infty} db\, b^{1/2}\, e^{iA(\theta, b)/\hbar}. \tag{11}$$

This expression is similar to the formulation of quantum mechanics that is due to Feynman,[3] which was extended by Motz,[4] in which the amplitude for a particle to move from one place to another is expressed as an integral over all possible trajectories of $\exp(iS/\hbar)$, where S is the action. (In our analogy the $b^{1/2}$ weights the trajectories with larger b more heavily because there are more of them.)

Our expression for the scattering amplitude may be integrated by the stationary phase technique, since the action is a rapidly changing function of b except at a few impact parameters, b_i, where

$$\frac{\partial}{\partial b} A(\theta, b)\Big|_{b_i} = 0, \tag{12}$$

which occurs where

$$2\eta'(b_i) = \frac{\theta}{\chi}. \tag{13}$$

In the neighborhood of these stationary points the action may be expressed

$$A(\theta, b) \cong A(\theta, b_i) + \frac{1}{2}\frac{\partial^2 A}{\partial b^2}\Big|_{b=b_i}(b-b_i)^2$$

$$\cong A(\theta, b_i) + \hbar\eta''(b_i)(b-b_i)^2. \tag{14}$$

The stationary phase integration must be performed at each impact parameter where the action is stationary, so the scattering amplitude becomes a sum over all impact parameters b_i that satisfy Eq. 13.

$$f(\theta) = \sum_{b_i(\theta)} |b_i| (2b_i \pi \sin\theta \eta''(b_i))^{-1/2} e^{iA(\theta, b_i)/\hbar}. \tag{15}$$

(This expression fails if $\theta = 0$, or π [glory scattering], or $\eta''(b_i) = 0$ [rainbow scattering].) All quantities must be treated as complex numbers, so that a factor $e^{-i\pi/2}$ will result if either b_i or $\eta''(b_i)$ is negative. (If $b < 0$ in Eq. 11, $b^{1/2}$ must be interpreted as $e^{-i\pi/2} |b|^{1/2}$, because of the transformation used in Eq. 7.)

Equation 15 shows that contributions to the scattering amplitude come from (the neighborhood of) discrete values of the impact parameter. The magnitude of each contribution is determined by the angle, the impact parameter, and the second derivative of the phase. The phase of each contribution is determined solely by the action (apart from a possible constant). This makes our analysis especially enlightening when applied to scattering processes that are sensitive to the phase of the scattering amplitude (or its components).

2. Simple Elastic Scattering

Let us now consider the application of these results to elastic scattering by a single potential. Since the proceeding results are most helpful in simplifying the discussion of scattering when more than one impact parameter contributes to the scattering, we consider a potential that, like most interatomic potentials, is attractive at long distances and repulsive at short distances. The phase function $\eta(b)$ for this potential is shown in Fig. III-2. We also show the classical deflection function

$$\chi(b) = 2\pi\eta'(b), \tag{16}$$

which is the locus of points where the action is stationary (that is, where Eqs. 12 and 13 hold). The lower half of the deflection function is shown dashed to emphasize that only impact parameters for which $\chi(b) = \theta$ contribute to $f(\theta)$ in Eq. 15 (θ is the angle of observation, and is always positive). Thus at angle θ_o (in Fig. III-2), three impact parameters contribute to the scattering amplitude, b_1, b_2, and b_3; b_1 and b_2 are both negative and correspond to predominantly attractive scattering, while b is positive and corresponds to predominantly repulsive scattering.

When two (or more) impact parameters contribute to the scattering amplitude in Eq. 15 the differential cross section will contain interference terms where phase difference varies as $[A(\theta, b_1(\theta)) - A(\theta, b_2(\theta))]/\hbar$. The angular spacing of the resulting

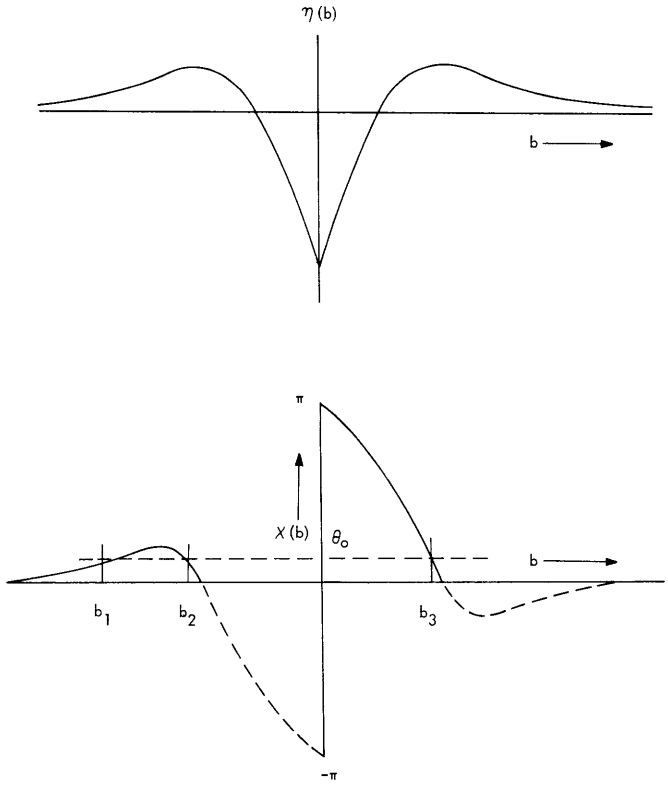

Fig. III-2. Phase function, $\eta(b)$, and deflection function, $\chi(b)$; each shown for both positive and negative values of impact parameter, b.

maxima and minima in the cross section depends on the total rate of change of the action with θ. This is

$$\frac{d}{d\theta} A(\theta, b_i(\theta)) = \left.\frac{\partial A}{\partial \theta}\right|_{b_i} + \left.\frac{\partial A}{\partial b}\right|_{b_i} \frac{db_i}{d\theta}$$

$$= -\hbar b_i/\chi \qquad (17)$$

from Eq. 10 (the second term is zero from Eq. 13). If b_1 and b_2 both contribute to scattering at some angle, then the angular spacing of successive maxima or minima in that neighborhood is given by

$$\Delta\theta \frac{d}{d\theta} (A(\theta, b_1) - A(\theta, b_2)) = 2\pi, \qquad (18)$$

so that

$$\Delta\theta = \frac{2\pi\lambda\!\!\!^{-}}{b_1(\theta) - b_2(\theta)}. \tag{19}$$

Thus the local spacing of maxima and minima is determined solely by distance between b_1 and b_2; the potential plays no role, once b_1 and b_2 have been determined.

Equation 19 also describes the spacing of maxima and minima in the two-slit diffraction of monochromatic light (Young's experiment), stressing once again the wave nature of matter. The b_i change with angle so that the spacing of successive maxima and minima is not constant in the scattering cross section, as it is in Young's experiment.

Equation 10 may be used to determine the distance between the impact parameters that contribute to the scattering amplitude. As an example we consider Hundhausen and Parly's[6] data for Na-Hg scattering at $V_{rel} = 1.475 \times 10^5$ cm/sec (λ is 0.1303 Å at this velocity). They observe interference between b_1 and b_2 (supernumerary rainbows) with a period of 0.09 rad, which implies that $b_1 - b_2 = 1.5$ Å; and they observe interference between b_1 and b_3 (rapid oscillations) with a period of 0.0127 rad, which implies that $b_1 - b_3 = 10.2$ Å. These distances are quite compatible with the size of the potentials which they determined by partial-wave analysis.

3. Resonant Exchange

Let us now consider a typical exchange cross section

$$\sigma_{ex}(\theta) = \left| f_+(\theta) - f_-(\theta) \right|^2, \tag{20}$$

where f_+ and f_- are two independent scattering amplitudes. Such expressions arise in spin exchange and in resonant charge exchange; in both processes the expected oscillatory structure has been observed. As in single-channel scattering, the oscillatory behavior is due to interference between the two scattering amplitudes whose phase difference varies as the difference between the two actions times $1/\hbar$. The local change of action with θ depends only upon b (and not on the potential, see Eq. 17), so we find the angular spacing of successive maxima and minima in the exchange cross section to be

$$\Delta\theta = \frac{2\pi\lambda\!\!\!^{-}}{b_+(\theta) - b_-(\theta)} \tag{21}$$

in the neighborhood of θ.

Frequently, it is possible to infer the phase difference between the two scattering amplitudes exactly, in which case we can measure the relative phase

$$\delta(\theta) = \left[A_+(\theta, b_+) - A_-(\theta, b_-)\right]/\hbar. \tag{22}$$

We assume that the potentials have similar shapes, so the additional $\pi/4$ terms in Eq. 15 cancel out. Although the impact parameters in the plus and minus states are different, the action is stationary for small variations of b, so if b_+ and b_- are approximately equal (as they will be if the potentials for the two states are nearly equal), we can approximate

$$\delta(\theta) \cong [A_+(\theta, \bar{b}) - A_-(\theta, \bar{b})]/\hbar$$

$$= 2\eta_+(\bar{b}) - 2\eta_-(\bar{b}), \tag{23}$$

where $\bar{b} = (b_+ + b_-)/2$, and the second line follows from Eq. 10. Use of the first term of the impact approximation[2] for $\eta \pm (\bar{b})$ yields the familiar result

$$\delta(\theta) \approx \frac{1}{\hbar v} \oint_{\bar{b}} V_+ \, d\ell - \frac{1}{\hbar v} \oint_{\bar{b}} V_- \, d\ell$$

$$\equiv \frac{1}{\hbar v} \oint_{\bar{b}} \delta V \, d\ell, \tag{24}$$

where δV is the difference potential $V_+ - V_-$, and v is the relative velocity of the collision.

The argument frequently used[7] to derive Eq. 24, under the assumption that the impact parameter is the same for collisions in both states, is incorrect; the "particle" in the state with weaker potential must travel closer to the target in order to sustain the deflection θ. In so doing, it travels in a region of deeper potential and picks up some extra phase (η). The phase of the scattering amplitude is governed, however, by the action, which is also affected by the decrease in path length caused by passing closer to the target. These two effects cancel exactly, since the action is stationary, so Eq. 26 is correct. A similar argument also applies for dominantly repulsive potentials.

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B. NEW DETERMINATION OF HYDROGEN-DEUTERIUM g-FACTOR RATIO

1. Introduction

The one-electron atom has a rich history as a proving ground for comparing theory and experiment to extraordinarily high precisions. In this report we propose a new comparison of the electronic g-factor in hydrogen and deuterium as a check on high-order terms to the electronic shielding corrections.

The nonrelativistic theory of hydrogen predicts that the g-factor in the ground state $\left(2\,^2S_{1/2}\right)$ is $g_j(H) = g_s$, where g_s is the g factor of the free electron. As shown by Breit,[1] in an early paper, relativistic effects alter this to $g_j(H) = g_s(1-\alpha^{2/3})$. This is correct to order α^2. There are further corrections of order α^3 and $\alpha^2 m/M$, which are due to the effect of binding on the anomalous moment, and to reduced mass effects. Hegstrom[2] has derived the following result.

$$g_j(H) = g_s\left\{1 - \frac{1}{3}\alpha^2 Z^2\left(\frac{M}{m+M}\right)^2 + \frac{1}{12\pi}\alpha^3 Z^2\left(\frac{M}{m+M}\right)^2 + \delta_e\right\},$$

where δ_e represents the correction to the anomalous moment caused by binding of the electron,[3] and is of order $\alpha^3 Z^3$.

This result can be checked by comparing electronic transitions in hydrogen and deuterium in a hydrogen maser. Details of our apparatus have been given in previous reports, and we describe here only the more important modifications that are needed.

The energy levels for hydrogen and deuterium are shown in Fig. III-3. At the field of our magnet, 3500 G, the transition frequencies are tabulated as follows:

Hydrogen		Deuterium	
Transition	Frequency (GHz)	Transition	Frequency (GHz)
1-2	0.6444	3-4	9.6465
3-4	0.7760	3-5	9.7603
2-3	9.2029	2-5	9.8671
2-4	9.9979	3-6	9.8717
1-4	10.6233	2-6	9.9785
		1-6	10.0829

Several of the transitions lie relatively close to each other. In particular, the 2-6 transition in deuterium lies only 0.39 MHz below the hydrogen 2-4 transition. This presents the possibility of measuring the two transitions simultaneously, which has important practical advantages.

To put the proposed experiment into perspective, it should be pointed out that

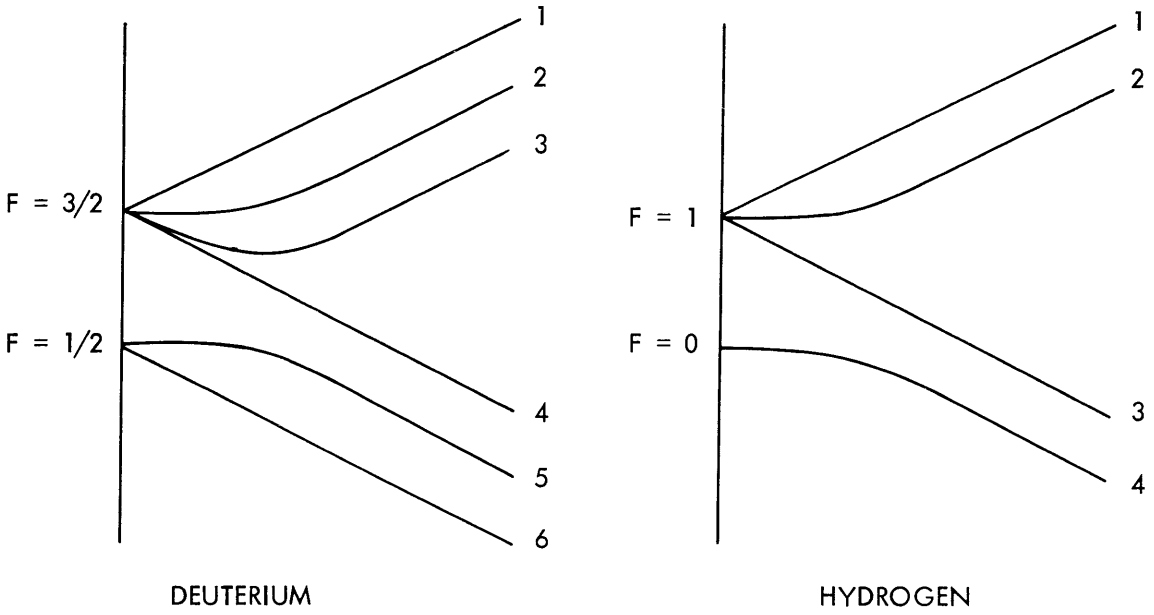

Fig. III-3. Energy levels as a function of external magnetic field.

g-factor comparisons traditionally consist in successive determination of transition frequencies of two species. Since the frequencies depend linearly on the applied magnetic field, the experimental precision is usually limited by the field stability. Thus, if the two species can be examined simultaneously, problems of field stability, and other problems having to do with sample interchange, are much reduced. This technique underlies our current determination of the electron-proton magnetic moment ratio.[4]

At the present time, our maser produces an electronics transition in hydrogen with a lifetime of 5 msec, at a frequency of 9 GHz. This corresponds to a linewidth of 70 Hz, or a resonance Q of $\nu/\Delta\nu = 1.3\times 10^8$. The signal-to-noise ratio is high, so that the center of the line can be determined to a small fraction of its width. Unfortunately, short-term fluctuations in the field broaden the line significantly. The fluctuations $\Delta H/H$ are typically a few parts in 10^8, so that the observed linewidths are often 200 Hz wide. The hydrogen and deuterium transitions, however, have a very similar field dependence, and their ratio is practically field-independent. In the proposed experiment, the hydrogen and deuterium are put simultaneously into a radiating state, and the ratio of the frequencies is determined in the course of a single radiation lifetime. The ratio is averaged many times over, in contrast to the more familiar technique of averaging each transition frequency many times over before taking the ratio.

An error analysis of the proposed experiment has been carried out. It indicates that $g_j(H)/g_j(D)$ can be measured to adequate precision to observe terms of order α^3, $\alpha^2 m/M$, and possibly α^4.

A fundamental technical problem in the proposed experiment is that it requires a coherent microwave structure with response at two well-separated frequencies. The procedure for achieving this will now be presented.

2. Doubly Resonant Cavity

We require a detection system that is capable of simultaneously measuring the frequencies of two microwave signals in X-band which are separated approximately 100 MHz. Because the signals are so weak, the microwave detection system must have a response Q of $\sim 10^4$; this rules out the possibility of detecting both signals in a single cavity. The required system, therefore, must respond at two frequencies, each at approximately 10 GHz, separated ~100 MHz, with each response having a bandwidth of 1 MHz.

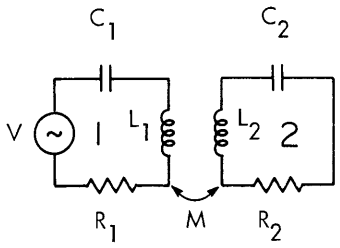

Fig. III-4. Equivalent circuit for doubly resonant cavity.

The system that we have developed comprises two microwave cavities with adjustable, weak coupling between them. The response characteristics can be approximated by using an electrical RCL circuit analog (see Fig. III-4). Here circuits 1 and 2 are simple RCL circuits, M is a mutual inductance between them, which represents the coupling, and circuit 1 contains a voltage-driving element, which represents the signal source. In terms of the circulating charge in each circuit, the differential equations of the system may be written

$$L_1\dot{q}_1 + R_1\dot{q}_1 + \frac{q_1}{C_1} + M\ddot{q}_2 = V \tag{1a}$$

$$L_2\ddot{q}_2 + R_2\dot{q}_2 + \frac{q_2}{C_2} + M\ddot{q}_1 = 0. \tag{1b}$$

In terms of microwave cavity parameters, these equations may be rewritten

$$\ddot{q}_1 + \frac{\omega_1\dot{q}_1}{Q_1} + \omega_1^2 q_1 + \kappa_1\ddot{q}_2 = \frac{V}{L_1} \tag{2a}$$

$$\ddot{q}_2 + \frac{\omega_2\dot{q}_2}{Q_2} + \omega_2^2 q_2 + \kappa_2\ddot{q}_1 = 0, \tag{2b}$$

where

$$\omega_i^2 = \frac{1}{L_i C_i}, \qquad \kappa_i = \frac{M}{L_i}, \qquad Q_i = \frac{\omega_i L_i}{R_i} \qquad (i=1,2).$$

Since the system is driven, we assume $V = V_o e^{i\omega t}$ and look for solutions $q_1 \approx a_1 e^{i\omega t}$; $q_2 = a_2 e^{i\omega t}$, where a_1 and a_2 are complex constants representing the responses of the two cavities. Equations 2 become

$$a_2 = \frac{a_1}{\kappa_1}\left(\left(\frac{\omega_1}{\omega}\right)^2 - 1 + \left(\frac{\omega_1}{\omega}\right)\frac{i}{Q_1}\right) - \frac{V_o}{L_1\kappa_1\omega^2}$$

$$= a_1 \frac{F_1(\omega)}{\kappa_1} - \frac{V_o}{L_1\omega^2\kappa_1} \qquad (3a)$$

$$a_1 = \frac{a_2}{\kappa_2}\left(\left(\frac{\omega_2}{\omega}\right)^2 - 1 + \left(\frac{\omega_2}{\omega}\right)\frac{i}{Q_2}\right)$$

$$= a_2 \frac{F_2(\omega)}{\kappa_2}. \qquad (3b)$$

Solving Eqs. 3 for a_1, we find

$$a_1(\omega_1\kappa) = \frac{\left(\frac{V_o}{L_1\omega^2}\right)}{F_1(\omega) - \left(\frac{\kappa^2}{F_2(\omega)}\right)},$$

where $\kappa^2 \equiv \kappa_1\kappa_2$. The interesting parameter is R, the relative response of circuit 1, normalized to its response when $\kappa = 0$ and $\omega = \omega_1$.

$$R_1(\omega_1\kappa) \equiv \frac{a_1(\omega_1\kappa)}{a_1(\omega_1, 0)} = \left(\frac{\omega_1}{\omega}\right)^2 \left[\frac{F_1(\omega_1)}{F_1(\omega) - \kappa^2 F_2^{-1}(\omega)}\right]. \qquad (4)$$

We now wish to find the frequencies for which R_1 is a maximum to first order. These correspond to the poles of the denominator of Eq. 4, which are given by

$$\mathrm{Re}\left\{F_1(\omega)F_2(\omega)\right\} - \kappa^2 = 0. \qquad (5)$$

If we write $\omega_2 = \omega_1 + \Delta$, and note that $\kappa^2 \ll 1$ and $(\Delta/\omega_1)^2 \ll 1$, the solutions of Eq. 5 to order (Δ/ω_1) are

$$\omega^2 = \omega_1^2 \left(1 + \frac{\Delta}{\omega_1} \pm \sqrt{\kappa^2 + \left(\frac{\Delta}{\omega_1}\right)^2}\right). \tag{6}$$

If we assume $\Delta = 0$, that is, $\omega_1 = \omega_2$, Eq. 5 has the solutions

$$\omega = \omega_1\left(1 \pm \frac{\kappa}{2}\right). \tag{7}$$

This result indicates that two frequency responses will occur, centered at ω_1 and split by $\omega_1 \kappa$.

The effective Q of the response lines can be estimated from Im $\{F_1(\omega)F_2(\omega)\}$, and is found to be

$$Q_{eff} \approx \frac{Q_1 Q_2}{Q_1 + Q_2},$$

independent of κ in this approximation. Then assuming $Q_1 = Q_2$, we have

$$Q_{eff} = \frac{1}{2} Q_1.$$

To summarize, the equivalent circuit theory predicts that one of two identical microwave cavities, weakly coupled, will respond at two frequencies, given by Eq. 7, with an effective Q approximately one half of the natural Q.

A simple prototype double cavity was constructed to test the predicted results. The system consists of two right circular cylindrical cavities, designed for optimum response at ~9200 MHz in the TE_{011} mode. The cavities are coupled through a hole in the middle of the side walls, which are cut to .010 inch on the adjoining surface. The coupling holes may be displaced from one another to reduce the coupling. The cavities are each separately tunable by means of adjustable end caps. The cavity responses are observed by means of standard X-band waveguide coupled to the cavities through small coupling holes in the side walls.

The response function was observed and compared with a computer-generated solution of Eq. 4, thereby substantially verifying the results predicted in the approximate treatment reported here. Using coupling holes of 1/2 inch diameter, we were able to obtain responses split up to 170 MHz with Q's of one-half the uncoupled Q.

F. G. Walther, D. Kleppner

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