

PLASMA DYNAMICS

VIII. PLASMA PHYSICS*

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A. ON THE NONLINEAR STABILIZATION OF THE CONVECTIVE LOSS CONE INSTABILITY NEAR THRESHOLD

1. Introduction

The purpose of this report is to examine the stabilization of the convective loss cone instability¹⁻⁴ near stability threshold in infinite media. A discussion of the quasi-linear stabilization has been given previously.⁵ Quasi-linear diffusion stabilizes instabilities in plasma physics, however, by removing the source of free energy which drives the instability.^{6,7} In the case of a loss cone instability, this means enough particles must diffuse into the loss cone to remove the anisotropy in the ion distribution function.

In a laboratory mirror machine, however, clearly the ions have a loss cone distribution function in any steady state that evolves. Therefore, it is of interest to examine stabilization mechanisms that allow a loss-cone distribution function in the steady state. We shall examine two stabilization mechanisms, resonant mode coupling and electron heating. The principal conclusion is that each mechanism does stabilize the instability, but that the final wave amplitude is so high that it is likely that some other nonlinear mechanism stabilizes the instability first. Also it can be concluded that electron heating generally stabilizes the plasma at lower wave amplitude than mode coupling.

2. Mode Coupling

We shall consider a plasma with a hot-ion and cold-electron component. The hot-ion component will be taken to have a loss-cone distribution and the electrons will be considered Maxwellian. We shall assume the density to be slightly above the threshold for convective instability, although sufficiently above that electron Landau damping may be neglected. This limit has been studied previously⁸ and it was found that two important simplifications result in the algebra: (a) The electrons, as long as they are sufficiently

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(VIII. PLASMA PHYSICS)

cold, are the active species and they determine the electric field and the dispersion relation $\omega = \omega_{pe} \frac{k_z}{k}$. (b) The instability is a resonant instability whose growth is given by

$$\gamma_L = \frac{\omega_{pi}^2}{2|k|^2} \int dv_{\perp} v_{\perp} J_1^2 \left(\frac{k_{\perp} v_{\perp}}{\omega_{ci}} \right) \left\{ k_z \frac{\delta f_i}{\partial v_z} + \frac{2\omega_{ci}}{v_{\perp}} \frac{\partial f_i}{\partial v_{\perp}} \right\}_{v_z = \frac{\omega - \omega_{ci}}{k_z}}. \quad (1)$$

In order to make the problem more specific, let us consider the ion distribution function

$$f_i = \frac{1}{\pi^{3/2} a^{3/2}} \left(\frac{v_{\perp}}{a} \right)^2 \exp \left[- \frac{v_{\perp}^2 + v_z^2}{a^2} \right], \quad (2)$$

where we also take $\left(\frac{H}{H} \right) = \frac{M_i a^2}{m_e a_e^2} \approx 10^3$. The linear analysis in this case has been worked out.⁴ The conditions for marginal stability are

$$\frac{\omega_{pi}^2}{\omega_{ci}^2} \approx 0.02 \quad \frac{k_z a}{\omega_{ci}} \approx 0.2 \quad \frac{k_{\perp} a}{\omega_{ci}} \approx 1.3. \quad (3)$$

If the density is just above threshold, the frequency of an unstable wave is approximately the ion cyclotron frequency ω_{ci} . As the density is increased, waves at the cyclotron harmonics become unstable. In general, for instability at the n^{th} harmonic, $\frac{k_{\perp} a}{\omega_{ci}} \gtrsim n$. We will be concerned here with densities such that only waves at the first cyclotron harmonic are unstable. In this limit, $k_{\perp} \approx k \gg k_z$.

For the above-mentioned parameters, it is not difficult to see that the ion contribution to the dispersion relation is smaller by an order of magnitude than the electron contribution.

Let us then consider a plasma unstable at the first cyclotron harmonic. We shall examine the coupling of two waves $(\omega_1, \underline{k}_1)$ and $(\omega_2, \underline{k}_2)$ at the ion cyclotron frequency to a wave $(\omega_3 = \omega_1 + \omega_2, \underline{k}_3 = \underline{k}_1 + \underline{k}_2)$ at twice the ion cyclotron frequency. In order for the wave $(\omega_3, \underline{k}_3)$ to satisfy the linear dispersion relation, we must have $k_{z1} \approx k_{z2}$ and $k_{3\perp} \approx k_{1\perp} \approx k_{2\perp}$. If $k_1 \approx k_2$, the angle between $\underline{k}_{1\perp}$ and $\underline{k}_{2\perp}$ is approximately 120° . If these conditions are satisfied, \underline{k}_1 and \underline{k}_2 couple to a wave \underline{k}_3 . Since $\frac{k_{3\perp} a}{\omega_{ci}} < 2$, this wave k_3 is a damped wave.

We shall then calculate the rate at which energy is produced at the second cyclotron

harmonic. Using the fact that for each plasmon produced at $2\omega_{ci}$, two are destroyed at ω_{ci} , we can calculate the rate at which energy is lost by the wave spectrum at ω_{ci} .⁹ By equating the nonlinear loss rate to the linear growth rate for waves in the spectrum at ω_{ci} , we may determine the equilibrium wave amplitude. Also, by equating the nonlinear production rate to the linear damping rate, we may determine the equilibrium amplitude for the spectrum at $2\omega_{ci}$. We shall assume, for the present, that the energy in the spectrum at ω_{ci} is much larger than the energy in the spectrum at $2\omega_{ci}$. This assumption will be clearly justified at the end of the calculation.

The rate at which energy is coupled to the second harmonic can be computed from the matrix element for the coupling of the waves at $\underline{k}_1 \underline{k}_2 \underline{k}_3$.¹⁰ The result is

$$\begin{aligned} \frac{d|\underline{E}(\underline{k}_3)|^2}{dt} = & \sum_{\underline{k}_2} \frac{\omega_{pe}^4 \omega^2}{4|\underline{k}_1 + \underline{k}_2|^2} \frac{e^2}{m^2} \frac{k_{1z}^2 k_{2z}^2}{k_1^2 k_2^2} |\underline{E}(\underline{k}_1)|^2 \delta[\omega_3 - \omega_1 - \omega_2] |\underline{E}(\underline{k}_2)|^2 \\ & \cdot 2\pi \left[\frac{(\omega_2^2 + 2\omega_1 \omega_2) k_{z1} (k_{z1} + k_{z2}) + (\omega_1^2 + 2\omega_1 \omega_2) k_{z2} (k_{z1} + k_{z2})}{(\omega_1 + \omega_2)^2 \omega_1^2 \omega_2^2} \right]^2 - 2\gamma_L^1 |\underline{E}(\underline{k}_3)|^2, \quad (4) \end{aligned}$$

where $\underline{k}_1 + \underline{k}_2 = \underline{k}_3$, and γ^1 is the linear damping rate for the wave at $2\omega_{ci}$. Making use of the fact that $\omega_1 \approx \omega_2$, $k_{1z} \approx k_{2z}$ and $|\underline{k}_1 + \underline{k}_2| \approx k_1$ near threshold, and

$$\frac{\omega_{pe}^2 k_z^2}{\omega^2 k^2} \approx 1,$$

Eq. 4 reduces to

$$\frac{d|\underline{E}(\underline{k}_3)|^2}{dt} \approx \sum_{\underline{k}_2} \frac{9e^2}{4k_1^2 m^2} |\underline{E}(\underline{k}_1)|^2 |\underline{E}(\underline{k}_2)|^2 \left(\frac{k_{z1}^2}{\omega_1} \right)^2 2\pi \delta[\omega_3 - \omega_1 - \omega_2] - 2\gamma_L^1 |\underline{E}(\underline{k}_3)|^2. \quad (5)$$

The kinetic equation for a wave at ω_{ci} immediately follows⁹:

$$\begin{aligned} \frac{d|\underline{E}(\underline{k}_1)|^2}{dt} = & 2\gamma_L |\underline{E}(\underline{k}_1)|^2 - \frac{9\pi}{2} \frac{e^2}{m^2} \frac{k_z^4}{k^2 \omega^2} |\underline{E}(\underline{k})|^2 \sum_{\underline{k}^1} |\underline{E}(\underline{k}^1)|^2 \\ & \cdot \delta[\omega(\underline{k} + \underline{k}^1) - \omega(\underline{k}) - \omega(\underline{k}^1)], \quad (6) \end{aligned}$$

where the summation over \underline{k}^1 is over all waves in the spectrum at the first cyclotron harmonic.¹¹ We would like to determine the order of magnitude of the last term on the

(VIII. PLASMA PHYSICS)

right-hand side of Eq. 6. To do so, let us say the spectrum at ω_{ci} has cylindrical symmetry and has characteristic width Δk_z about k_{zo} and Δk_\perp about $k_{\perp o}$ in the parallel and perpendicular directions, respectively. Within these limits $|\underline{E}(\underline{k})|^2 \approx E^2$. Then converting the summation over \underline{k}^1 into an integral in cylindrical coordinates, the second term on the right of Eq. 6 becomes

$$-\frac{9\pi}{2} \frac{e^2}{m^2} \frac{k_z^4}{k_\perp^2 \omega_{pe}^2} |\underline{E}(\underline{k})|^2 E^2 \frac{V}{(2\pi)^3} \int_{k_{oz}}^{k_{oz}+\Delta k_z} dk_z^1 \int_{k_o}^{k_o+\Delta k} k_\perp^1 dk_\perp^1 \int_0^{2\pi} d\theta$$

$$\cdot \delta \left[\frac{k_z + k_z^1}{\sqrt{k_\perp^2 + k_\perp^2 + 2k_\perp k_\perp^1 \cos \theta}} - \frac{k_z}{k_\perp} - \frac{k_z^1}{k_\perp^1} \right], \quad (7)$$

where V is the total volume of the system, $\omega = \omega_p \frac{k_z}{k}$, and $k_\perp \gg k_z$. We shall first perform the integrals over θ . Assuming that the spectrum is narrow so $k_\perp^1 \approx k_\perp$ and $k_z \approx k_z^1$, we have seen that the argument of the delta function is zero when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. Doing the integral over the delta function gives

$$-\frac{9\pi}{\sqrt{3}} \frac{e^2}{m^2} \frac{k_z^3}{k \omega_{pe}^2} |\underline{E}(\underline{k})|^2 E^2 \frac{V}{(2\pi)^3} \Delta k_z k_\perp \Delta k_\perp. \quad (8)$$

Then making use of the fact that $E^2 \frac{V}{(2\pi)^3} 2\pi k_\perp \Delta k_\perp \Delta k_z \approx E_{rms}^2$, Eq. 6 becomes

$$\frac{d|\underline{E}(\underline{k})|^2}{dt} = 2\gamma_L |\underline{E}(\underline{k})|^2 - \frac{9}{2\sqrt{3}} \frac{e^2}{m^2} \frac{k_z^3}{k \omega_{pe}^2} |\underline{E}(\underline{k})|^2 E_{rms}^2. \quad (9)$$

Equation 8 predicts a steady turbulent state when

$$\frac{E_{rms}^2}{2\pi} \approx \frac{2\sqrt{3}}{9\pi} \gamma_L \frac{m^2 k \omega_{pe}^2}{e^2 k_z^3} \approx \frac{1.7 \times 10^3}{2\pi} \frac{m_e}{M_i} \frac{\gamma_L}{\omega_{pe}} n M a^2, \quad (10)$$

where we have made use of Eq. 3. Assuming that the plasma is a hydrogen plasma with $\frac{m_e}{M_i} \approx \frac{1}{1836}$, then

$$E_{wave} \sim \frac{\gamma_L}{\pi \omega_{pe}} E_{ions}. \quad (11)$$

Thus we see that mode coupling does indeed stabilize the convective loss-cone instability near the stability threshold, and the total wave energy is considerably less than the ion energy. The wave energy is quite high, however, and it seems likely that some other mechanism may stabilize the loss cone instability at lower amplitudes than does resonant mode coupling.

Finally, let us use Eq. 5 to determine the amplitude of the spectrum at $2\omega_c$. Using the same techniques as previously yields

$$\frac{E_{\text{rms}}^2(2\omega_{ci})}{E_{\text{rms}}^2(\omega_{ci})} \approx \frac{\gamma_L}{2\pi\gamma_L^1}. \quad (12)$$

Assuming we are sufficiently near threshold, we find that $\gamma_L \ll \gamma_L^1$ so the rms spectrum at $2\omega_{ci}$ is much less than the rms spectrum at ω_{ci} .

3. Electron Heating

Another possible stabilization mechanism is electron heating. If the parallel electron temperature is initially T_e and the wave spectrum grows to final amplitude E_{rms} , the final electron temperature is roughly

$$T_e + \frac{e^2 E_{\text{rms}}^2}{2m\omega^2} \frac{k_z^2}{k^2}.$$

As the electron temperature increases, however, electron Landau damping becomes important. According to Callen,⁴ the instability will be stabilized when

$$\frac{\omega}{k_z \sqrt{\frac{2T_e}{m_e}}} \approx 2.5. \quad (13)$$

Assume initially that

$$\frac{\omega}{k_z \sqrt{\frac{2T_e}{m}}} \approx 3.$$

Then the loss-cone instability will stabilize by electron Landau damping when $E_{\text{wave}} \approx E_{\text{electron}}$. Since the electron energy density is less by a factor of 1000 than the ion energy, it appears likely that the convective loss instability near threshold, in infinite media will be stabilized by electron heating before it will be stabilized by mode coupling.

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(VIII. PLASMA PHYSICS)

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11. Actually, ref. 10 shows that there are other mode-coupling terms that should be included on the right-hand side of Eqs. 5 and 6. These terms have products like $E(\underline{k}_3)^2$, $E(\underline{k}_1 \text{ or } \underline{k}_2)^2$. Since the spectrum at \underline{k}_3 has a small amplitude compared with the spectrum at \underline{k}_1 , these terms will be small compared with those retained. Thus they are justifiably neglected.

B. STRIATIONS AND THE STEADY-STATE DISTRIBUTION FUNCTION IN A LOW-PRESSURE DISCHARGE

1. Introduction

The purpose of this report is twofold: to describe the problem on which I am working, and to present some theory of a low-pressure discharge.

First, I shall describe the experimental observations of striations in a low-pressure ($p \sim 5 \times 10^{-3}$ Torr). Second, I shall derive the equations for ion and metastable production and loss discharge, and give an overview of the theoretical problem. Third, I shall examine the electron equations, using the Boltzmann equation. Fourth, I shall make approximations germane to my experiment, and present calculations of the steady-state electron velocity distribution.

a. Problem

Barrett and Little,^{1, 2} among others, have observed waves in low-pressure glow discharges which have backward-wave characteristics (by "low-pressure," I mean the pressure regime where the electron and ion mean-free paths are larger than the discharge tube radius). The waves have several characteristics in common:

1. Their phase velocity is from anode to cathode.
2. Phase and group velocities are in opposite directions (hence the name "backward wave").
3. Phase velocities are usually slow, of the order of 10^3 - 10^5 cm/sec.
4. Frequencies are also low (very much less than plasma frequency, for example), with observed frequencies in the range 1-100 kHz.

Sometimes these waves have been self-excited, at other times they have been excited externally. They have generally been given the rather broad, hazily defined label of "striations." Recently, Pekarek^{3,4} and others have worked out theories for striations, but these theories are all made under the assumption that the discharge is diffusion-dominated (that is, mean-free paths much less than tube radius). Thus far, there has been no theory that has tried to explain these waves in the low-pressure region.

At present, I am observing these waves in a hot-cathode glow discharge under the following experimental conditions:

Gas: Argon

Pressure: 5×10^{-3} Torr

Tube radius: 1.8 cm

Discharge current: 20-150 mA

Cathode-Anode distance: 130 cm.

The waves are excited by applying a sinusoidal signal of from 1 Volt to 10 Volts peak-to-peak to an annular electrode located 30 cm from the cathode. The waves are observed on the anode side of the electrode, with a phase velocity directed toward the cathode. These waves are damped, their amplitude slowly decreasing as one moves away from the exciter. Frequencies of observation have been from 5 kHz to 45 kHz. Details of the experimental arrangement will not be given in this report. By measuring the wavelength λ as a function of the frequency ν of the waves, the highly dispersive nature of these waves is clearly shown. A plot of λ against ν is shown in Fig. VIII-1. The data can be roughly approximated by the equation

$$\nu = K\lambda^2$$

to give a phase velocity

$$v_p = \lambda\nu = K\lambda^3 = K^{-1/2} \nu^{3/2}$$

and a group velocity

$$v_g = -\lambda^2 \frac{d\nu}{d\lambda} = -2K\lambda^3 = -2v_p.$$

Now you have an idea of the experiment. The other part of the problem is to try to

(VIII. PLASMA PHYSICS)

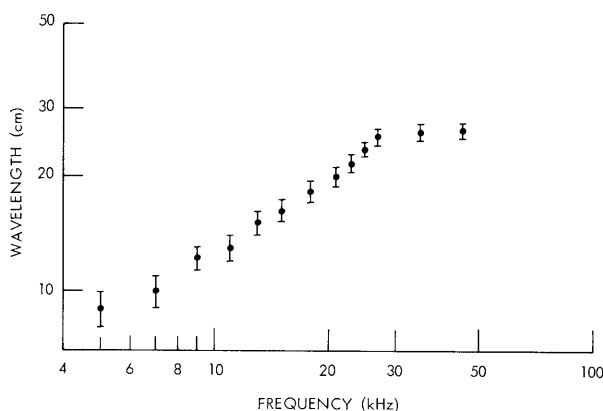


Fig. VIII-1. Frequency vs wavelength of the striations.

explain theoretically the properties of these waves. The rest of this report deals with the mathematical model used in the theory. At the present time, the theory is incomplete, but the basic equations have been formulated and the steady-state electron distribution function has been calculated.

b. Introduction to the Theory

The energy balance in the discharge will be discussed to give some insight into the physical processes present. The energy source is the axial electric field, E_0 . This field accelerates the electrons and ions, thereby giving them a velocity along the z axis. Since electron mobility is much greater than ion mobility, most of the energy is gained by the electrons.

Electrons lose their energy by collisions. They may suffer elastic collisions with neutrals, thereby losing a small fraction of their energy in each collision, or they may have inelastic collisions with neutrals or metastable atoms, in which they may lose a large fraction of their initial energy. Coulomb collisions between electrons occur, and tend to make the distribution function more nearly Maxwellian. There is, however, no net energy loss because of these collisions, since they are between particles of the same species. Electron-ion collisions will be neglected because of the low density ($n_e \approx 10^9 \text{ cm}^{-3}$) of the discharge under consideration. It will be shown that electron-electron collisions may be neglected also for this case. Finally, some electrons will strike the wall of the tube, and recombine with ions there. In the theory of higher pressure discharges, this effect is usually neglected, but in the low-pressure case it will be shown to be of major importance.

The inelastic collisions that the electrons undergo with the neutrals excite or ionize the neutrals. The ions and metastables created are lost by recombination or de-excitation, either in the discharge or at the wall. It is a good approximation to

consider the ions and metastables as having Maxwellian velocity distributions. The ion temperature will be somewhat higher than the neutral temperature, but will still be much less than the electron temperature. When this assumption of Maxwellian ions and metastables is made, it is then permissible to use the moment (or MHD) equations to describe the ion and metastable properties.

The electrons will not have a Maxwellian distribution, and it will be necessary to use the complete Boltzmann transport equation to obtain the distribution function.

2. Ion and Metastable Equations

The equations for ions and metastables will now be derived, with source terms (ionization and excitation rates) depending implicitly on the electron distribution function. The equations will also contain wall loss terms.

a. Moment Equations

Assume that ions and metastables have Maxwellian distributions, and the moment equations can be used. Then the conservation of particles and conservation of momentum equations for ions and metastables have the following forms:

$$\frac{\partial N_+}{\partial t} + \text{div}(N_+ \vec{V}_+) = S_i N_m \quad (1)$$

$$MN_+ \frac{d\vec{V}_+}{dt} = -\text{grad } p_+ + qN_+ \vec{E} - MN_+ \nu_+ \vec{V}_+ \quad (2)$$

$$\frac{\partial N_m}{\partial t} + \text{div}(N_m \vec{V}_m) = S_m N_o - \gamma N_- N_m \quad (3)$$

$$MN_m \frac{d\vec{V}_m}{dt} = -\text{grad } p_m - MN_m \nu_m \vec{V}_m, \quad (4)$$

where

m, M = electron, ion mass

N = density

\vec{V} = velocity

ν = momentum transfer collision frequency

S_i = ionization frequency of a metastable

$S_m N_o$ = creation rate of metastables

$\gamma N_- N_m$ = rate of metastable de-excitation by electrons

Subscripts +, -, m, o refer to ions, electrons, metastables, and neutrals.

(VIII. PLASMA PHYSICS)

Now we consider the problem in cylindrical geometry, and assume that all quantities are independent of the azimuthal angle θ .

Divide all quantities into two parts, DC (steady-state) and AC (time-variant) parts, of the form

$$X = X + x e^{i(kz - \omega t)}.$$

As usual, assume $x \ll X$ for all quantities, and then linearize the equations to obtain DC and linearized AC sets of equations.

i. D-C Equations

For the ions, the DC equations are

$$S_i N_m = -\frac{1}{r} \frac{\partial}{\partial r} (N_+ V_{r+}) \quad (5)$$

$$qE_z = Mv_+ V_{z+} \quad (6)$$

$$0 = -\frac{\partial}{\partial r} (N_+ T_+) + qN_+ E_r - MN_+ v_+ V_{r+}. \quad (7)$$

Now assume that $T_+ \approx T_o \approx \text{constant}$, where T_+ is the ion temperature in units of electron volts. Equations 5 and 7 yield

$$S_i N_m = -\frac{1}{Mv_+ r} \frac{\partial}{\partial r} \left(T_+ \frac{N_+}{r} - qN_+ E_r \right). \quad (8)$$

The left-hand side is the creation rate, and the right-hand side is the loss rate to the walls. The right-hand side is then the ion density divided by the characteristic time, τ_+ , of an ion lifetime. (The value of τ_+ will be calculated in section 2.b.) Thus the DC equation for the ions becomes

$$S_i N_m = \frac{N_+}{\tau_+}, \quad (9)$$

and similarly for the metastables the DC equation can be written

$$S_m N_o = \frac{N_m}{\tau_m}. \quad (10)$$

ii. A-C Equations

The linearized AC equations for the ions are

$$-i\omega n_+ + (N_+ v_{z+} + V_{z+} n_+) ik = S_i n_m + N_m s_i - \frac{n_+}{\tau_+} + \frac{N_+}{\tau_+} \cdot \frac{\delta\tau_+}{\tau_+} \quad (11)$$

$$-i\omega MN_+ v_{z+} = -ikT_+ n_+ + qN_+ e_z - \nu_+ MN_+ v_z, \quad (12)$$

where $\delta\tau_+$ = AC part of τ_+ . Substituting (6) and (12) in (11) and dropping z subscripts, since the radial dependence has been eliminated, yields

$$\left[-i\omega + \frac{k^2 D_+ + ik\mu_+ E}{1 - \frac{i\omega}{\nu_+}} + \frac{1}{\tau_+} \right] n + \frac{ik\mu_+ N}{1 - \frac{i\omega}{\nu_+}} e = S_i n_m + s_i N_m + N \frac{\delta\tau_+}{\tau_+}, \quad (13)$$

where

$$D_+ = \frac{T_+}{M\nu_+}, \quad \mu_+ = \frac{q}{M\nu_+}.$$

Here, the assumption of quasi neutrality has been made. That is, it has been assumed that

$$N_+ \approx N_- = N$$

$$n_+ \approx n_- = n.$$

This can be shown to be equivalent to the assumption that the Debye length, λ_D , is small compared with the radius of the tube and wavelength of the wave. This assumption is justified, for $\lambda_D \approx 0.65$ mm in this experiment. Now dividing (13) by (9) yields

$$\left[-i\omega\tau_+ + \frac{(k^2 D_+ + ik\mu_+ E)\tau_+}{1 - \frac{i\omega}{\nu_+}} + 1 \right] \frac{n}{N} + \frac{ik\mu_+ E\tau_+}{1 - \frac{i\omega}{\nu_+}} \frac{e}{E} = \frac{n_m}{N_m} + \frac{s_i}{S_i} + \frac{\delta\tau_+}{\tau_+}. \quad (14)$$

By plugging in appropriate values of $k \leq 2\pi/5$ cm, $E \approx 1$ V/cm, $\tau_+ \approx 6 \times 10^{-6}$ sec, $\mu_+ \approx 1.5 \times 10^5$ cm²/V-sec, one finds that $k^2 D_+ \tau_+ \approx 0.038 \ll 1$, but $k\mu_+ E \tau_+ \approx 1.08$, so the first term may be neglected, but the second may not. In practice, as the pressure increases, μ_+ decreases, so that at higher pressures the second term may be neglected also.

In the same manner as for the ions, the AC equation for the metastables may be derived. If the metastable diffusion is neglected, the metastable equations yield

$$(1-i\omega\tau_m) \frac{n_m}{N_m} = \frac{s_m}{S_m} - \gamma \tau_m n. \quad (15)$$

(VIII. PLASMA PHYSICS)

Finally eliminating n_m/N_m from (14) by the use of (15), we obtain

$$\left[1 - i\omega\tau_+ - \frac{k\mu_+ E\tau_+}{i + \frac{\omega}{v_+}} \right] \frac{n}{N} - \frac{k\mu_+ E\tau_+}{i + \frac{\omega}{v_+}} \frac{e}{E} = (1 - i\omega\tau_m)^{-1} \left[\frac{s_m}{S_m} - \gamma N\tau_m \frac{n}{N} \right] \frac{n}{N} + \frac{s_i}{S_i} + \frac{\delta\tau_+}{\tau_+}. \quad (16)$$

This is an equation in the AC quantities $\frac{n}{N}$, $\frac{s_i}{S_i}$, $\frac{s_m}{S_m}$, $\frac{\delta\tau_+}{\tau_+}$, and $\frac{e}{E}$. In section 3, the Boltzmann equation is solved for the DC and the AC components of the electron distribution function. When this is done, it will be seen that the first four quantities can be written as proportional to e/E . That is, for a given set of discharge conditions, $\frac{s_i}{S_i} = (\text{function of } k) \cdot \frac{e}{E}$, and the same for $\frac{n}{N}$, $\frac{s_m}{S_m}$, and $\frac{\delta\tau_+}{\tau_+}$. Therefore, (16) becomes a homogeneous equation in e/E , of the form $D(\omega, k) \frac{e}{E} = 0$. $D(\omega, k) = 0$ is then the dispersion relation for the waves, which must then be solved to obtain ω vs k .

iii. Relation of Macroscopic Quantities to the Electron Distribution Function

Assume that the cross section for ionization of a metastable atom under electron bombardment has the form

$$\begin{aligned} \sigma_i &= \sigma_{i0} (v^2 - v_i^2) & \text{for } v > v_i \\ &= 0 & \text{for } v < v_i, \end{aligned}$$

where $1/2 mv_i^2 = eU_i$, and U_i = ionization energy of the metastable in Volts. Then the total number of ionization/sec is

$$N_m S_i = N_m \int \sigma_i(v) v f(v) d^3v,$$

so

$$S_i = 4\pi \int_0^\infty \sigma_i(v) v^3 f(v) dv.$$

If we write $f(v) = F(v) + \not f(v) e^{i(kz - \omega t)}$, then

$$\frac{s_i}{S_i} = \frac{\int_{v_i}^\infty (v^2 - v_i^2) v^3 \not f(v) dv}{\int_{v_i}^\infty (v^2 - v_i^2) v^3 F(v) dv}, \quad (17)$$

and similarly,

$$\frac{s_m}{S_m} = \frac{\int_{v_m}^{\infty} (v^2 - v_m^2) v^3 f(v) dv}{\int_{v_m}^{\infty} (v^2 - v_m^2) v^3 F(v) dv} \quad (18)$$

$$\frac{n}{N} = \frac{\int_0^{\infty} f(v) v^2 dv}{\int_0^{\infty} F(v) v^2 dv}. \quad (19)$$

Also, since $\frac{3}{2} NT = \frac{1}{2} \int m v^2 f(v) d^3 v$, then to first order in the perturbations,

$$\left(\frac{\delta T}{T} + \frac{n}{N} \right) = \frac{\int_0^{\infty} v^4 f(v) dv}{\int_0^{\infty} v^4 F(v) dv}. \quad (20)$$

Note that in the equations above the "temperature" T is defined as being equal to $3/2$ the average energy of an electron. This is a generalized temperature, however, since $f(v)$ may not be Maxwellian. Although $\delta T/T$ does not appear in the ion equations, it will be shown below that

$$\frac{\delta \tau_+}{\tau_+} \approx - \frac{\delta T}{2T}, \quad (21)$$

so that $\delta \tau_+/\tau_+$ is also known.

Thus, Eqs. 17-20 give relations between the quantities appearing in Eq. 16 and f/F , and we shall find that $f/F \propto e/E$.

b. Evaluation of the Radial Loss Rate

The discharge under consideration is one in which all mean-free paths are longer than the tube radius, R , but shorter than the tube length. Under these circumstances, it is necessary to use a free-fall theory of ion loss. Self⁵ has a theory for this case in which he makes the following assumptions:

1. Electrons are almost Maxwellian.
2. Electron temperature does not depend on r .
3. Inertial effects of electrons may be neglected.
4. $S_i \gg \nu_{c+}$
5. $N_+ \approx N_-$,

where ν_{c+} = ion elastic collision frequency. He then derives a set of equations that are solved numerically, giving the electron density and electrostatic potential vs radius. In the process, by equating the creation and loss rates, an expression for the ionization frequency is obtained of the form

$$S_i = \nu_c \frac{s_b(K)}{K},$$

(VIII. PLASMA PHYSICS)

where s_b is Self's notation for a nondimensionalized radius, and $K = v_c R \sqrt{M/T}$. Now, for $K \leq 5 \times 10^3$, $S_b(K) \approx 1$, so that in this case

$$S_i \approx \sqrt{\frac{T}{M}} \frac{1}{R}, \quad T = \text{electron temperature.}$$

In an Argon discharge, with conditions which have been described, $T \approx 4 \text{ eV}$, $v_c \approx 2.5 \times 10^7 \text{ sec}^{-1}$, so

$$K \approx 1.4 \times 10^2 \ll 5 \times 10^3,$$

and

$$S_i \approx 1.75 \times 10^5 \text{ sec}^{-1}.$$

Equating creation and loss rates gives $\tau_+ = S_i^{-1} = R \sqrt{M/T}$, so

$$\frac{\delta \tau_+}{\tau_+} = - \frac{\delta T}{2T} \quad (\text{Eq. 21}).$$

Self's assumptions 1 and 4 are not really valid, since $f(v)$ is not Maxwellian and $v_{c+} \approx S_i$. The bulk of the distribution function does have, however, a form that may be approximated as Maxwellian. Since the ion losses are primarily influenced by ambipolar fields generated by the bulk of the electrons, this approximation is good. In another paper,⁶ Self gives a more approximate treatment for the case when S_i is not $\gg v_{c+}$, and finds that there is not an appreciable change between the cases $S_i \gg v_{c+}$ and $S_i \approx v_{c+}$.

c. Ion and Electron Current to the Wall

In the following analysis, a crude attempt will be made to solve for the ion and electron current to the walls, and the potential of the wall relative to the center of the discharge tube. First, the ion current to the wall is obtained for a Maxwellian distribution of electrons. Second, a generalization is made to distributions that are approximately Maxwellian in the body of the distribution function but may not be Maxwellian in the tail. The assumption of planar instead of cylindrical geometry may make quantitative results not very accurate, but a qualitative feel for what wall losses do will be obtained.

i. Solution for Maxwellian Electrons

Self⁷ obtains an exact numerical solution to the plasma-sheath equation in planar geometry. He makes three assumptions.

1. Electrons are Maxwellian.

2. Collisions are neglected for both ions and electrons.

3. Ion creation rate \propto to electron density.

He then solves for $N(r)$, $\phi(r)$, and $J_+(\phi)$, where $-\phi$ is electrostatic potential. The solutions for ion and electron current densities are

$$J_+(\phi) = q \sqrt{\frac{2T}{M}} N(r=0) I(q\phi/T), \quad (22)$$

and

$$J_-(\phi) = q \sqrt{\frac{T}{2\pi m}} N(r=0) e^{-\frac{q\phi}{T}} \quad (23)$$

$$I(\eta) = \int_0^\eta e^{-\eta} \frac{ds}{d\eta} d\eta, \quad (24)$$

where $\eta = q\phi/T$, and $s = x/L$, with L the width of the discharge tube. Because of the $e^{-\eta}$ in the integrand, the integrand becomes small for $\eta \gtrsim 2$ (also $ds/d\eta$ decreases with increasing η). Thus, $I(\eta) \approx \text{const.}$ for large values of η . The value of the constant is slightly dependent on the ratio λ_D/L . In the case under consideration, $\lambda_D/R \approx 10^{-3/2}$, which implies that $I(\eta) \approx 0.388$ for $\eta \gtrsim 2$.

For an insulating wall, the net current of charged particles to the wall must equal zero, so setting $J_-(\phi) = J_+(\phi)$ yields

$$\frac{q\phi_w}{T} \approx \ln \left[\frac{1}{2\pi^{1/2}} \left(\frac{M}{m}\right)^{1/2} \frac{1}{I\left(q\frac{\phi_w}{T}\right)} \right],$$

where $-\phi_w$ = wall potential, and $\phi(r=0) = 0$. For Argon, this expression gives

$$\phi_w \approx 5.29 \frac{T}{q}.$$

ii. Effects of a Non-Maxwellian Electron Distribution

For a discharge in which the electron distribution function departs from a Maxwellian, (22) is no longer valid. For the one-dimensional problem, the current in the x-direction is

$$J_-(\phi_w) = q \int_{v_x=v_w}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(\vec{v}) v_x d^3v, \quad (25)$$

where

$$v_w = \sqrt{\frac{2e\phi_w}{m}}.$$

(VIII. PLASMA PHYSICS)

If $f(\vec{v})$ is spherically symmetric (that is, $f(\vec{v}) = f(v)$), then Eq. 25 can be written

$$J_{-}(\phi_w) = \pi q \int_{v_w}^{\infty} (v^2 - v_w^2) f(v) v dv. \quad (26)$$

Therefore, if we consider only the spherically symmetrical part of $f(v)$, the requirement of equal electron and ion current to the walls yields

$$q \sqrt{\frac{2T}{M}} N I \left(\frac{q\phi_w}{T} \right) = \pi q \int_{v_w}^{\infty} (v^2 - v_w^2) f(v) v dv$$

or

$$\int_{v_w}^{\infty} (v^2 - v_w^2) f(v) v dv = \sqrt{\frac{2T}{M}} \frac{N}{\pi} I \left(\frac{q\phi_w}{T} \right). \quad (27)$$

Since I is approximately a constant, (27) is to be solved, given $f(v)$, for the value of v_w that satisfies the equation. This then gives the value of the wall potential, ϕ_w .

There is a question as to the validity of using (22) for the ion current, if the electrons have a non-Maxwellian distribution. If, however, one examines the form of $I(\eta)$ in Eq. 24, the main contribution is from those portions of the discharge where η is small. It is exactly in this region that the bulk of the electrons (which are at low energies) exist, so that the shape of the potential curve would not be changed much if the low-energy portion of the electrons has an approximately Maxwellian form. A more serious objection might be directed at the use of a planar geometry. The physical significance of Eq. 27 is, however, that the electron current represented by the integral on the left-hand side will equal the ion current, which will be approximately equal to the right-hand side. In practice, it will turn out that the value of v_w determined from Eq. 27 is not very sensitive to changes in the value of the right-hand side. The reason for this is that $f(v)$ is a rapidly decreasing function out in the tail, which is where v_w usually is. Thus, changing v_w by 10-20% may change the value of the integral by more than 100%, and hence make v_w relatively insensitive to the ion current magnitude.

3. Boltzmann Equation for Electrons

a. Expansion in Spherical Harmonics

The Boltzmann equation may be written

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \text{grad}_r f - \vec{a} \cdot \text{grad}_v f = B(\vec{v}), \quad (28)$$

where $\vec{a} = q \vec{E}/m$ is the acceleration of an electron, and $B(\vec{v})$ is the collision term, which will be described later.

Now, let f be expanded in Legendre polynomials,

$$f(\vec{r}, \vec{v}, t) = \sum_{n=0}^{\infty} f^n(z, v, t) P_n(\cos \theta), \quad (29)$$

where θ is the angle in the velocity space, between v_z and v . In the expansion above, the following assumptions have been made:

1. f does not depend on any spatial coordinate except z .
2. f has azimuthal symmetry in velocity space.

In making assumption 1, the mathematical model is departing somewhat from reality, because the electron velocity distribution function depends also on r . But the radial electric field is small, and the density relatively uniform until the sheath region is reached near the wall of the tube.⁵ The assumption is then fairly good for $r < r_{\text{sheath}}$. Most of the plasma is in the region $r < r_{\text{sheath}}$, and this region is where almost all of the ionizing collisions take place. Therefore, the first assumption is equivalent to the assumption that the characteristics of the striations are determined primarily by the electrons in the bulk of the plasma, and not by the electrons in the sheath. Of course, the presence of the wall does affect f , even for small values of r . The effect of the wall will be brought into the equation as a wall-loss collision term on the right-hand side of (28). This term will be derived (see Eq. 36).

If the collision term is also expanded in Legendre polynomials,

$$B(\vec{v}) = \sum_{n=0}^{\infty} B^n(v) P_n(\cos \theta),$$

then substituting (29) in (28) gives an equation in the f^n and the B^n . Now assume that $|f^1(v)| \ll |f(v)|$, and that all higher order terms may be neglected. Physically, this is equivalent to the assumption that the DC drift velocity attributable to the axial electric field, is much smaller than the average thermal velocity of the electrons.⁸ In the experimental case under consideration this assumption is valid. Neglect all higher orders than f^1 , then the equations are

$$\frac{\partial f^0}{\partial t} + \frac{v}{3} \frac{\partial f^1}{\partial z} - \frac{a}{3v^2} \frac{\partial}{\partial v} (v^2 f^1) = B^0 \quad (30)$$

and

$$\frac{\partial f^1}{\partial t} + v \frac{\partial f^0}{\partial z} - a \frac{\partial f^0}{\partial v} = B^1. \quad (31)$$

(VIII. PLASMA PHYSICS)

Before we can proceed further, it will be necessary to investigate the form and size of the collisional terms.

b. Collision Terms

i. Elastic Collisions

The evaluation of the collision terms for elastic collisions have been carried out in Dreicer.⁸ For $\frac{m}{M} \ll 1$, the results to lowest order in that parameter are

$$B_{\text{elas}}^1 = -\nu_c(v) f^1 \quad (32)$$

$$B_{\text{elas}}^0 = \frac{m}{M} \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^3 \nu_c(v) f^0 \right), \quad (33)$$

where $\nu_c(v)$ is the usual momentum transfer collision frequency, which is defined by

$$\nu_c(v) = 2\pi N_o v \int_0^\pi (1 - \cos \beta) \sin \beta \sigma_{\text{elas}}(\beta, v) d\beta,$$

where $\sigma_{\text{elas}}(\beta, v)$ is the differential cross section for an electron with velocity v being scattered by an angle β in a collision with a neutral atom.

ii. Inelastic Collisions

As with elastic collisions, inelastic collisions have a collision frequency associated with them of the forms

$$\nu_{\text{ex}}(v) = 2\pi N_o v \int_0^\pi \sin \beta \sigma_{\text{ex}}(\beta, v) d\beta = N_o v \sigma_{\text{ex}}(v)$$

$$\nu_i(v) = 2\pi N_m v \int_0^\pi \sin \beta \sigma_i(\beta, v) d\beta = N_m v \sigma_i(v).$$

Note that $\nu_i(v)$ is proportional to N_m , since the assumption has been made that ionization is predominately due to the ionization of metastables. Therefore, σ_i is the cross section for an electron to ionize a metastable. ν_{ex} is the rate of excitation from the ground state to the metastable level. Excitation to higher states is neglected.

An excitation collision by an electron with velocity v will then deplete $f(v)dv$, and will put the electron at $f(v') \frac{v'}{v} dv$, where $1/2 m v^2 = 1/2 m v'^2 + eU_{\text{ex}}$, with U_{ex} = excitation potential of the metastable level. In practice, most electrons suffering inelastic collisions will lose almost all of their energy, and will be returned to the distribution

with essentially zero velocity. To simplify computation, one can write

$$B_{\text{ex}}^0 = -\nu_{\text{ex}}(v) f^0(v) + \frac{\delta(v)}{v^2} \int_0^\infty \nu_{\text{ex}}(w) f^0(w) w^2 dw. \quad (34)$$

Note that this expression conserves particles exactly; that is,

$$\int_0^\infty B_{\text{ex}}^0 v^2 dv = 0.$$

An ionizing collision has the same form as Eq. 34, except that one ionizing collision creates two low-energy electrons, so

$$B_i^0 = -\nu_i(v) f^0(v) + 2 \frac{\delta(v)}{v^2} \int_0^\infty \nu_i(w) f^0(w) w^2 dw. \quad (35)$$

iii. Wall-Loss Collisions

It is difficult to treat wall losses exactly, so in the following analysis several approximations will be made to simplify the expressions. They should give a rough idea of the effects of loss to the walls. In order for an electron to reach the walls,

$$\frac{1}{2} m v_r^2 > q \phi_w.$$

For f^0 , which is isotropic, the number of electrons with $v_r > v_w = \sqrt{\frac{2q\phi_w}{m}}$ will be one third of the number having $v > v_w$.

For $v_r > v_w$, the time that it takes an electron near the center of the tube to reach the wall for a parabolic potential is

$$\tau_w \approx \frac{R}{v_w} \sin^{-1} \left(\frac{v_w}{v_r} \right)$$

and

$$\tau_w \approx \frac{R}{v_r}$$

for a potential that is zero except for the sheath region. These two expressions do not differ greatly from each other, and the second potential is closer to the real case, so set

$$B_{\text{wall}}^0 \approx -\frac{1}{3} \frac{f^0(v)}{\tau_w} \approx -\frac{1}{3} f^0(v) \frac{v}{R} S(v-v_w), \quad (36)$$

(VIII. PLASMA PHYSICS)

where

$$\begin{aligned} S(x) &= 1 & x > 0 \\ &= 0 & x < 0. \end{aligned}$$

For a given velocity distribution of electrons, v_w must be determined by equating ion and electron currents to the wall, as described previously.

iv. Coulomb Collisions

Allis⁹ and Dreicer⁸ give detailed calculations of the effects of electron-electron collisions on the distribution function. They use a Fokker-Planck collision model, and calculate the rate of change of f^n . The equations to be used in the numerical computation may be written in the form

$$B_{cl}^0 = A \left\{ (f^0)^2 + \left(\frac{I_2}{2} - \frac{I_4}{3v^4} + \frac{2I_{-1}}{3v} \right) \frac{\partial f^0}{\partial v} + \frac{1}{3} \left(\frac{I_4}{v^3} + I_{-1} \right) \frac{\partial^2 f^0}{\partial v^2} \right\} \quad (37)$$

$$A = \left(\frac{q^2}{\epsilon_0 m} \right)^2 \ln \Lambda = 0.835 \times 10^{19} \frac{\text{cm}^6}{\text{sec}^4} \ln \Lambda$$

$$\begin{aligned} I_k &= \int_0^v w^k f^0(w) dw; \quad \text{for } k \geq 0, \\ &= \int_v^\infty w^{-k} f^0(w) dw; \quad \text{for } k < 0. \end{aligned}$$

Here, Λ is the ratio of the Debye length to the average distance of closest approach. It is usually $\gg 1$, so that $\ln \Lambda$ is a slowly varying function of electron density and temperature.¹⁰ Λ will be treated as a constant. For the densities now under investigation, B_{cl}^0 will be negligible relative to B_{ex}^0 , for example.

v. Relative Sizes of Collision Terms

The dominant term in the equations is usually ν_c , since the average value of ν_c is greater than B_{cl} , B_{ex} , B_i , or B_{wall} when averaged over the distribution function. Note that B_{elas}^1 is proportional to ν_c , while B_{elas}^0 is proportional to $\frac{m}{M} \nu_c$. $\frac{m}{M} \nu_c$ is not the largest collision term, and, in fact, may be neglected, as we shall show.

c. Simplified Equations

Equation 31 may be written

$$\frac{\partial f^1}{\partial t} + v \frac{\partial f^0}{\partial z} - a \frac{\partial f^0}{\partial v} = -\nu_c f^1 + B_{\text{other}}^1.$$

$\frac{\partial f^1}{\partial t} = -i\omega f^1$, and may be neglected, since $\omega \ll \nu_c$. Neglecting B_{other}^1 also, as discussed above, yields

$$f^1(v) = \frac{1}{\nu_c(v)} \left[a \frac{\partial f^0}{\partial v} - v \frac{\partial f^0}{\partial z} \right]. \quad (38)$$

Also neglecting the time-dependent term in (30), the equation may be written

$$\frac{v}{3} \frac{\partial f^1}{\partial z} - \frac{a}{3v^2} \frac{\partial}{\partial v} (v^2 f^1) = B_{\text{elas}}^0 + B_{\text{ex}}^0 + B_i^0 + B_{\text{wall}}^0 + B_{\text{cl}}^0, \quad (39)$$

where the terms on the right-hand side are given in terms of f^0 by Eqs. 33-37. Thus (38) and (39) form a closed set for $f^0(v)$. These equations may be used to derive equations for the DC and the AC parts of f^0 , denoted by F^0 and f^0 , respectively, as before. At this time, however, the quantities will be made nondimensional, and then the equations will be derived.

d. Nondimensional Equations

Let

$$F^0 = \text{Nu}_0^{-3} Y^0$$

$$f^0 = \text{Nu}_0^{-3} y^0$$

$$\frac{1}{2} m u_0^2 = 1 \text{ eV}$$

$$x = v/u_0 \text{ (} x = \text{velocity in "square root Volts").}$$

Then, in terms of the nondimensional quantities Y and x , Eqs. 38 and 39, for the DC quantities, become

$$\begin{aligned} \frac{-A_0}{3u_0 x^2} \frac{d}{dx} (x^2 Y^1) &= \nu_{\text{cl}} C(Y^0, x) + \frac{m}{Mx^2} \frac{d}{dx} \left(\nu_c(x) x^3 Y^0 \right) - (\nu_{\text{ex}} + \nu_i) Y^0 + \frac{\delta(x)}{x^2} \\ &\quad - \int_0^\infty (\nu_{\text{ex}} + 2\nu_i) Y^0(w) w^2 dw - \frac{u_0 x Y^0}{3R} S(x - x_w) \end{aligned} \quad (40)$$

(VIII. PLASMA PHYSICS)

$$Y^1 = \frac{A_o}{\nu_c(x) u_o} \frac{dY^0}{dx}, \quad (41)$$

where

$$A_o = \frac{eE}{m}$$

$$\nu_{cl} = ANu_o^{-3}$$

$$C(Y^0, x) = \frac{B_{cl}^0}{A},$$

as given in Eq. 37, with f^0 and v replaced by Y^0 and x . Equations 40 and 41 are the equations that must be solved for the DC distribution function.

The equations for y , the AC part of the normalized distribution function, is more complicated, since to first order the AC equation will have products of the form $Y\alpha$ and $A_o y$, where $\alpha = \frac{qe}{m}$ is the AC part of electric field acceleration.

It will be shown in section 4 that the elastic and Coulomb collision terms in (40) will be negligible. Therefore, these terms will be omitted from the AC equation. Then Eqs. 38 and 39, after linearization in the AC quantities, become

$$\begin{aligned} & \frac{iku_o}{3} y^1 - \frac{A_o}{3u_o x^2} \frac{d}{dx} (x^2 y^1) - \frac{a}{3u_o x^2} \frac{d}{dx} (x^2 Y^1) \\ & = -(\nu_{ex} + \nu_i) y^0 + \frac{\delta(x)}{x^2} \int_0^\infty (\nu_{ex}(w) + \nu_i(w)) y^0(w) w^2 dw - \frac{u_o x y^0}{3R} S(x - x_w) \end{aligned} \quad (42)$$

$$y^1 = \frac{1}{\nu_c(x)} \left[\frac{A_o}{u_o} \frac{dy^0}{dx} + \frac{a}{u_o} \frac{dY^0}{dx} - iku_o x y^0 \right]. \quad (43)$$

In terms of normalization and evaluation of macroscopic quantities, the Y have been defined such that

$$4\pi \int_0^\infty Y^0(x) x^2 dx = 1 \quad (44)$$

$$4\pi \int_0^\infty y^0(x) x^2 dx = \frac{n}{N} \quad (45)$$

$$T \text{ (in eV)} = \frac{8\pi}{3} \int_0^\infty Y^0(x) x^4 dx \quad (46)$$

$$\left(\frac{\delta T}{T} + \frac{n}{N}\right) = \frac{8\pi}{3T} \int_0^\infty y^0 x^4 dx \quad (47)$$

$$\frac{s_i}{S_i} = \int_{x_i}^\infty (x^2 - x_i^2) x^3 y^0(x) dx \bigg/ \int_{x_i}^\infty (x^2 - x_i^2) x^3 y^0(x) dx \quad (48)$$

and similarly for s_m/S_m .

Substituting (43) in (42) yields a differential equation for y^0 . In symbolic notation, the equation can be written

$$L y^0 = (L' Y^0) \frac{a}{A_0} = (L' Y^0) \frac{e}{E}. \quad (49)$$

L and L' are differential operators and, since Y^0 is known, $L' Y^0$ is also a known function of x . As with all perturbation equations, only the inhomogeneous solution to Eq. (49) is desired. Therefore, the form of y^0 is independent of e/E , but the amplitude of y^0 is proportional to e/E . Thus, putting the solution for y^0 into Eqs. 45-48 gives relations between $\frac{n}{N}$, $\frac{\delta T}{T}$, $\frac{s_i}{S_i}$, $\frac{s_m}{S_m}$, and $\frac{e}{E}$. These may then be substituted in (16) to obtain the dispersion relation.

The AC equations will not be discussed further in this report, since work has not been completed on solving them. The solution to the DC equation will now be discussed.

4. D-C Equation—Numerical Results

a. Equations to be Solved Numerically

Multiply Eq. 40 by x^2 and integrate from ∞ to x . Substituting from (41) for Y^1 yields

$$\begin{aligned} \frac{3\nu_a^2 x^2}{\nu_c(x)} \frac{dY}{dx} = & \int_\infty^x \left[\nu_{ex}(w) + \nu_i(w) \right] Y(w) w^2 dw + \nu_L \int_\infty^x Y(w) w^3 S(w-x_w) dw \\ & - \nu_{cl} \int_\infty^x C(Y, w) w^2 dw - \frac{m}{M} \nu_c(x) x^3 Y, \end{aligned} \quad (50)$$

where

$$\nu_a = \frac{A_0}{3u_0} = 0.99 \times 10^7 \text{ E sec}^{-1} \quad (\text{E in V/cm})$$

(VIII. PLASMA PHYSICS)

$$\nu_L = \frac{u_o}{3R} = \frac{0.196 \times 10^8}{R} \text{ sec}^{-1} \quad (R \text{ in cm})$$

$$\nu_{cl} = 0.835 \times 10^{19} \frac{\text{cm}^6}{\text{sec}^4} \ln \Lambda N u_o^{-3}.$$

For $E = 1 \text{ V/cm}$, $R = 1.8 \text{ cm}$, $N \approx 10^9 \text{ cm}^{-3}$, $T = 5 \text{ eV}$, and $p = 5 \times 10^{-3} \text{ Torr}$,

$$\nu_a \approx 1 \times 10^7 \text{ sec}^{-1}$$

$$\nu_L \approx 1.1 \times 10^7 \text{ sec}^{-1}$$

$$\nu_{cl} \approx 1 \times 10^6 \text{ sec}^{-1}$$

$$\frac{m}{M} \nu_c \approx 1.2 \times 10^2 \text{ sec}^{-1}.$$

Here, the superscript o has been dropped. Bearing in mind that $C(Y, x)$ is usually $\ll 1$, we see that the last two terms in Eq. 50 may be neglected. Actual numerical calculations retaining these terms have been carried out, and the results differ negligibly from solutions without the terms. Therefore, they will be neglected.

The equation representing energy conservation may be derived by multiplying (40) by x^4 and integrating from 0 to ∞ . This gives

$$-6\nu_a^2 \int_0^\infty \frac{x^3}{\nu_c(x)} \frac{dY(x)}{dx} dx = \int_0^\infty (\nu_{ex} + \nu_i) x^4 Y dx + \nu_L \int_0^\infty x^5 Y S(x-x_w) dx. \quad (51)$$

This equation simply states that the power put in by the electric field equals the power lost in inelastic collisions and to the walls.

From Eq. 27, the equation for x_w may be obtained:

$$\int_{x_w}^\infty (x^2 - x_w^2) Y(x) x dx = \frac{I(\eta_w)}{\pi} \sqrt{\frac{m}{M}} T \approx 0.46 \times 10^{-3} \sqrt{T}. \quad (52)$$

Finally, the equation for the particle creation rate is

$$4\pi \int_0^\infty \nu_i(x) Y(x) x^2 dx = S_i \approx 2 \times 10^{-5} \text{ sec}^{-1}, \quad (53)$$

from section 2. b.

Equations 50, 52, and 53 completely specify the problem. The unknowns to be determined are $Y(x)$, x_w , and N_m which occurs in the definition of ν_i . These equations will be solved after the various collision frequencies are evaluated. $\nu = \nu_p P$ where p = pressure in Torr. This equation becomes

$$\nu(x) = 0.592 \times 10^8 p x P(x) \text{ sec}^{-1} \quad (54)$$

when expressed in terms of x , with P in collisions/(cm-Torr). For Argon, an analytic approximation to P_c is¹¹

$$\begin{aligned} P_c &= 3.4 & 0 < x < 1 \\ &= 7.87x^2 - 4.47 & 1 < x < 3.42 \\ &= 172 e^{-.06x^2} & x > 3.42 \end{aligned} \quad (55)$$

which will be used in the numerical calculations.

ν_{ex} can be determined from P_{ex} as given by Brown.¹² A rough analytic fit to $P_{ex}(x)$ for energies between 10.75 and 16 eV is given by

$$\begin{aligned} P_{ex}(x) &\approx 0.185(x^2 - 10.75)^{1.35} \frac{\text{collision}}{\text{cm-Torr}}. \\ P_{ex} &= 0 \quad \text{for } x^2 < 10.75 \text{ eV} \end{aligned} \quad (56)$$

b. Evaluation of ν_i

There do not seem to be good data on the ionization cross section of an Argon metastable by electron bombardment. Therefore the approximation will be made that the cross section increases linearly with energy, starting at zero at the threshold energy of 4.2 eV.

$$\begin{aligned} \sigma_i &= \sigma_{io}(x^2 - 4.2); & x^2 > 4.2 \\ &= 0; & x^2 < 4.2 \end{aligned}$$

If the metastable cross section were expressed in similar form, $\sigma_{ex} = \sigma_{exo}(x^2 - 10.75)$, the value for σ_{exo} would be approximately $\sigma_{exo} \approx 0.23 \times 10^{-16} \text{ cm}^2$. This value will also be taken for σ_{io} . Thus

(VIII. PLASMA PHYSICS)

$$\begin{aligned}\sigma_i &\approx 0.23 \times 10^{-16} \text{ cm}^2 (x^2 - 4.2); \quad \text{for } x^2 > 4.2 \\ \nu_i &= N_m \sigma_i v = 1.36 \times 10^{-9} N_m (x^2 - 4.2)x; \quad \text{for } x^2 > 4.2 \\ &= 0; \quad \text{for } x^2 < 4.2\end{aligned}\tag{57}$$

c. Method of Numerical Solution

Equation 50 is now

$$\frac{dY}{dx} = \frac{\nu_c(x)}{3\nu_a x^2} \int_0^x [\nu_{ex}(w) + \nu_i(w) + \nu_L w S(w - x_w)] Y(w) w^2 dw.\tag{58}$$

Starting at $x = 10$, where $Y(x)$ should be very small, we set

$$Y(10) = 0$$

$$\frac{dY(10)}{dx} = -0.0001$$

$$x_w = x_w^{(1)}$$

$$N_m = N_m^{(1)}$$

Then the interval $(0, 10)$ is broken up into N intervals, each of length Δx , and (58) is integrated by Euler's method into $x = 0$. This gives an unnormalized Y . The integral $4\pi \int_0^\infty Y(x) x^2 dx$ is then calculated, and $Y(x)$ is divided by this number, so that it is now normalized.

By using Y , Eqs. 52 and 53 are checked to see if they balance. If they do not, x_w and N_m are changed in the appropriate directions, and Y is calculated again. This cycle is iterated until values of Y , x_w , and N_m are determined which simultaneously satisfy the three equations. As a check on the computations, energy balance is checked by using (51).

d. Results

Figure VIII-2 is a plot of $Y(x)$ against x , with the values of N_m and x_w determined for equilibrium. Also shown is $Y(x)$ for $x_w = 10$. It is evident that the value of x_w has a strong effect on the tail of $Y(x)$. This shows that neglecting losses to the wall would be a very poor approximation for this discharge. The value of N_m does not have a

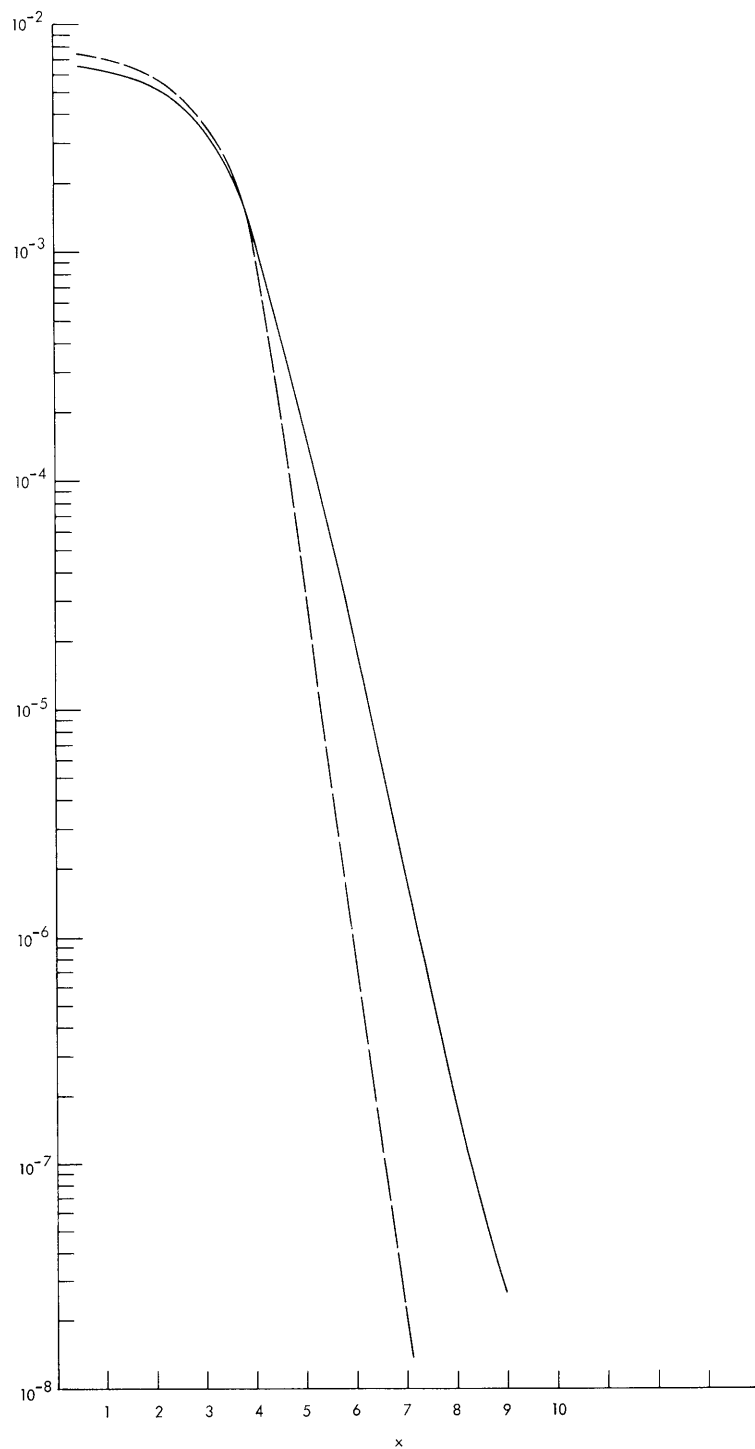


Fig. VIII-2. $Y(x)$ as a function of x . Solid line: $x_w = 10.0$.
Dashed line: $x_w = 4.46$, which is the value that
balances wall current.

(VIII. PLASMA PHYSICS)

significant effect on Y , until metastable densities get to be approximately a factor of 10 larger than they are here. It should be pointed out that N_m is not a well-determined number, since the value of σ_{io} in Eq. 57 is only a rough guess; however, $N_m \sigma_{io}$ is fairly well determined, since it is this product that enters into v_i , and the calculation of S_i .

Work is now in progress on the solution to the AC equations. The solution and the evaluation of the dispersion relation will be the subject of future reports.

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