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ROBINSON-TYPE STABILITY CRITERIA FOR BEAM AND RF CAVITY WITH DELAYED, VOLTAGE-PROPORTIONAL FEEDBACK

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The objective is to generalize the Robinson stability criteria, for a charged particle beam interacting with the radio-frequency (RF) cavity resonator that is responsible for accelerating it, to the case that the resonator is equipped with a delayed voltage-proportional feedback. The delay introduces exponential terms into the system characteristic leading to a transcendental equation for the free oscillation frequencies. We start by formulating an algebraic version of the Nyquist criterion that can be used to determine whether a polynomial containing exponentials has poles/zeros in the right half complex plane. The method is first applied to determining the limiting feedback gain of a resonator alone, and then to the problem of finding analogues of the Robinson criteria. Unfortunately, the criteria alone do not provide much insight as to the nature of the instability. To remedy this shortcoming, we shall apply the Sacherer formalism for computing longitudinal bunched beam instability, and as a consequence of considering the resistive and reactive parts of the impedance for various frequencies we shall explain the origin of the stability criteria.

Keywords: Instabilities; Robinson; Feedback; Collective effects; Impedances; RF devices

1 INTRODUCTION

Robinson gave criteria[3] for the stability of a charged particle beam interacting with the RF cavity resonator that is responsible for accelerating that beam. A widely adopted procedure[5] for high current beams, to avoid the power-limited instability, is to reduce the apparent cavity impedance by voltage-proportional feedback. Inevitably, the feedback is delayed; and this introduces exponential terms into the system characteristic equation. We give a general, exact, analytic procedure for determining whether there are poles/zeros in the right-half complex plane, and apply the method to find analogues of the Robinson stability criteria for the case that the resonator is equipped with a delayed voltage-proportional feedback. (Note, though there are many numerical packages for time-domain simulation and for finding poles or zeros in frequency-domain, a numerical result cannot prove stability in the absolute sense; and this is why analytic methods are still important.) Of course, it is essential that one should have a detailed understanding of the stability of the delayed-feedback-resonator alone, before embarking on a beam loading analysis. Whereas the analysis of the resonator alone is for arbitrarily high frequencies, the analysis of the Robinson problem is for low modulation frequencies about the carrier. It is also desirable that the criteria be explained in physical terms, and we shall later use the Sacherer[4] theory of beam instability to provide this understanding. This article is a precis of a rather more pedagogic exposition given in Refs. [9,10].

2 NYQUIST STABILITY CRITERION

Let $s=\sigma+\mathrm{j}\omega$ be the Laplace frequency and $\mathrm{j}=\sqrt{-1}$. Let us suppose that the system transfer function $F(s)$, which relates input to output in the s-domain, is the quotient of two polynomials. Values of s for which $F=0$ are called 'zeros' and are the roots of the numerator. Values of s for which F is infinite (or undefined) are called 'poles' and are the roots of the denominator. If any poles fall in the right half complex plane, then the system is unstable. Suppose that $F(s)$ has P poles and Z zeros in the right half plane. We adopt the usual convention of the complex plane that positive rotations are counter-clockwise. The rotation angle is often called the phase or 'argument' and denoted Arg. Nyquist[2] realized that the difference $(P-Z)$ is equal to the number of counter-clockwise encirclements of the origin by the locus of the function $F(s)$ as $s=\sigma+\mathrm{j}\omega$ varies along the contour of Figure 1. Hence, Nyquist reduced the stability analysis to merely counting up loops about the origin.

Given that we are searching for poles, it is often simplest to decompose a transfer function into a numerator and denominator. Then, for the stability analysis, we investigate under what conditions the

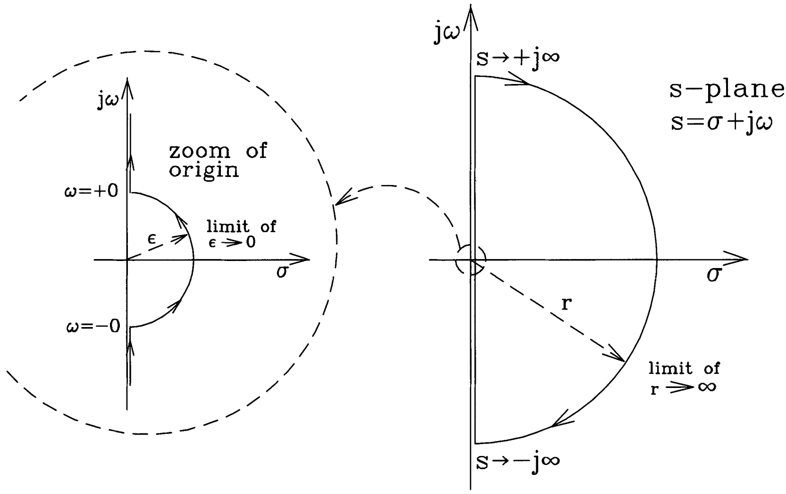

FIGURE 1 Clockwise semi-circular contour in the right half complex plane.

denominator has poles/zeros in the right half plane and from now on F shall stand for the denominator of the transfer function.

2.1 Analytic Stability Criteria

It is clear that we should like to have a criterion that is easy to apply, and preferably algebraic rather than geometric – so that curve sketching is avoided. Let us recall that if the origin is encircled in a clockwise manner (as ω sweeps along the contour of Figure 1) then there is a zero, whereas if the origin is encircled in a counter-clockwise fashion then there is a pole.

We divide $F(s)$ into a real part $\mathcal{A} = \Re[F]$ and an imaginary part $\mathcal{B} = \Im[F]$. Let us take cartesian coordinates with abscissa and ordinate equal to $\mathcal{A}$ and $j\mathcal{B}$, respectively. For stability, we want the denominator F to have no zeros in the right half plane. Thinking back to the Nyquist plots, it is clear that we want the locus of F to rotate counter-clockwise, and so we could consider studying the angular rotation rate

$$\frac{d}{d\omega}\text{Arg}(F) = \frac{[\mathcal{A}\mathcal{B}' - \mathcal{A}'\mathcal{B}]}{[\mathcal{A}^2 + \mathcal{B}^2]}, \tag{1}$$

which is positive for positive rotations (and has units of seconds). Here, the superfix prime denotes derivative w.r.t. frequency. Fortunately, we do not need to consider this quantity (1) for all values of ω.

Let us think how our function F could encircle the origin. In order to encircle the origin, the curve traced by $F(\omega)$ has to move through the four quadrants of the complex plane; and to do this there must be places where either $\mathcal{A}$ or $\mathcal{B}$ changes sign. Hence we are interested in the roots ω_A of $\mathcal{A}=0$ and the roots ω_B of $\mathcal{B}=0$.

2.1.1 *Criterion for Zeros and for Poles*

Now consider F at a root ω_A; here the location is $(0, \mathrm{j}\mathcal{B}(\omega_A))$ and let us assume $\mathcal{B}>0$. To get a clockwise motion of the point $(\mathcal{A}, \mathrm{j}\mathcal{B})$ as ω increases, $\mathcal{A}$ must be increasing. Consequently, the condition for a zero is $\mathcal{B}(\omega_A) \times \mathcal{A}'(\omega_A) > 0$.

Now consider F at a root ω_B; here the location is $(\mathcal{A}(\omega_B), \mathrm{j}0)$ and let us assume $\mathcal{A}>0$. To get a clockwise motion of the point $(\mathcal{A}, \mathrm{j}\mathcal{B})$ as ω increases, $\mathcal{B}$ must be decreasing. Consequently, the condition for a zero is $\mathcal{A}(\omega_B) \times \mathcal{B}'(\omega_B) < 0$. The above two conditions, which guarantee zeros in the right half plane, are sketched in Figure 2.

One may restate these conditions so that F has only poles but no zeros, that is in terms of criteria that ensure the counter-clockwise encirclement of the origin. There are only poles of F in the right half

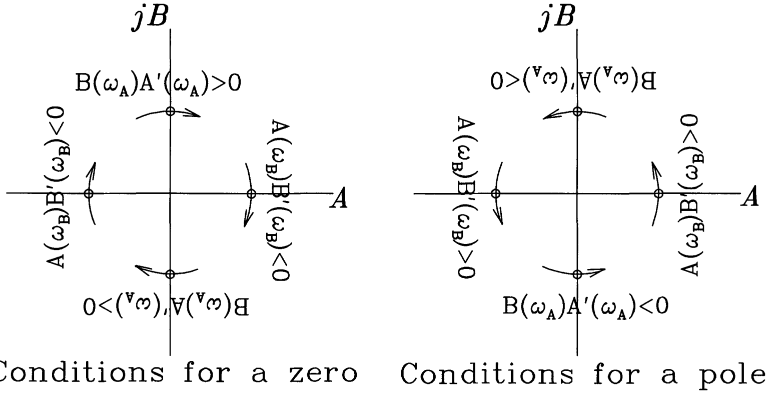

FIGURE 2 Geometric interpretation of stability criteria.

plane and the system is stable if:

$$\text{1st condition:} \qquad \mathcal{A}(\omega_B) \times \mathcal{B}'(\omega_B) > 0 \tag{2}$$

and/or

$$\text{2nd condition:} \qquad \mathcal{B}(\omega_A) \times \mathcal{A}'(\omega_A) < 0. \tag{3}$$

These conditions are sketched in Figure 2. Notice that what has been done is to formulate an algebraic version of the Nyquist criterion, whereas the formulation of Nyquist is essentially geometric. The conditions (2) and (3) are used in Ref. [9] to reproduce the Routh–Hurwitz[1] criteria for zeros of polynomials.

2.1.2 Comments

Notice that a true pole of F requires $\mathcal{A}$ and $\mathcal{B}$ to be simultaneously zero, which in general requires complex s. Our approach is to replace the problem by a potentially easier one: study the zeros of $\mathcal{A}$ and $\mathcal{B}$ independently as function of pure imaginary $s = \mathrm{j}\omega$.

Notice that, to get a single encirclement of the origin, we must have four, cyclic axis crossings.† The condition for this is that we find a consecutive pair of roots $\omega_B(n+1) > \omega_B(n) \geq 0$ that *alternate* with a root of $\mathcal{A}$ such that $\omega_B(n+1) > \omega_A > \omega_B(n)$. The fourth crossing occurs because the roots ω_A come in pairs with opposite signs, and $\mathcal{B}(-\omega_A) = -\mathcal{B}(+\omega_A)$. Continuity of a smooth function then guarantees that the sense of rotation $\mathrm{dArg}(F)/\mathrm{d}\omega$ will be the same at all four, cyclic crossings; and this is the reason why we can work with either the 1st or 2nd stability condition, but we do not have to explicitly verify them both. For simple, smooth functions (such as polynomials) the roots most often have this *alternate* property. If there is no root ω_A between the nearest neighbour roots $\omega_B(n)$ and $\omega_B(n+1)$, then there is no possibility for encirclement, and we do not need to bother evaluating the stability conditions $\mathcal{A}\mathcal{B}' > 0$ or $\mathcal{A}'\mathcal{B} < 0$. We shall return to this subject again, when we consider time delays.

If F is an impedance, then its locus is mirror symmetric about the real axis; because the real part $\mathcal{A}$ is even and the imaginary part $\mathcal{B}$ is odd

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†Of course, the properties of F at $s = 0$ and $s = +\infty$ usually guarantee at least two crossings of the $\mathcal{A}$-axis.

with respect to ω. This means it is sufficient to consider only $\omega > 0$, provided one remembers the implicit symmetry.

3 STABILITY OF CAVITY RESONATOR

In the neighbourhood of resonance, a cavity behaves like a simple LCR parallel resonator; and the impedance, $Z(\omega)$, which models this has two poles in the left half plane and a zero at d.c. (i.e. at the origin of the s-plane). Let the cavity resonance angular frequency be Ω, the shunt resistance R_s and the time constant be $\tau_c = 1/\alpha = 2Q/\Omega$ where Q is the quality factor. Let dots placed above a variable denote derivatives with respect to time, t. The voltage, V, and driving current, I, obey the equation

$$\ddot{V} + 2\alpha\dot{V} + \Omega^2 V = 2\alpha R_s \dot{I}. \tag{4}$$

The cavity complex impedance is the response when driven by a sinusoidal excitation exp(+jωt), and can be obtained from the equation

$$[(\Omega^2 - \omega^2) + 2\mathrm{j}\alpha\omega]\mathbf{V}^0 = 2\mathrm{j}\alpha\omega R_s \mathbf{I}^0, \tag{5}$$

where the steady state current drive and voltage response are the complex quantities $\mathbf{I}^0$ and $\mathbf{V}^0$ respectively. The impedance is $Z(\omega) = \mathbf{V}^0/\mathbf{I}^0$.

Let us apply the algebraic Nyquist criterion to our model problem of the cavity resonator,

$$\frac{Z(\omega)}{R} = \frac{\mathrm{j}\mathcal{B}}{\mathcal{A} + \mathrm{j}\mathcal{B}} \quad \text{with } \mathcal{A} = \Omega^2 - \omega^2 \text{ and } \mathcal{B} = 2\alpha\omega. \tag{6}$$

We want the denominator to have no zeros in the right half plane; or equivalently we want the denominator to have only poles in the right half plane. The conditions that the function $\mathcal{A} + \mathrm{j}\mathcal{B}$ has only poles are $\mathcal{A}'(\omega_A)\mathcal{B}(\omega_A) < 0$ and/or $\mathcal{B}'(\omega_B)\mathcal{A}(\omega_B) > 0$.

The roots of $\mathcal{A}$ are $\omega_A = \pm\Omega$. $\mathcal{A}'\mathcal{B}$ evaluated at these roots is equal to $-4\alpha\Omega^2$ which is less than zero provided that $\alpha > 0$. The root of $\mathcal{B}$ is $\omega_B = 0$. $\mathcal{B}'\mathcal{A}$ evaluated at this root is equal to $+2\alpha\Omega^2$ which is greater than zero provided that $\alpha > 0$. Hence, the system described by Eq. (6) is stable provided that $\alpha > 0$.

3.1 Detuning Angle

For convenience in later working, let us introduce the detuning angle by the definition:

$$\tan \Psi = (\Omega^2 - \omega^2)/(2\alpha\omega) = Q(\Omega - \omega)(\Omega + \omega)/(\Omega\omega). \tag{7}$$

Then the impedance is

$$Z(\omega) = \frac{\mathbf{V}^0}{\mathbf{I}^0} = \frac{R_s}{1 - \mathrm{j}\tan \Psi} = R_s \cos \Psi \mathrm{e}^{+\mathrm{j}\Psi}. \tag{8}$$

Note that when driven at the resonance frequency ($\omega = \Omega$), the impedance is purely resistive and has its maximum modulus.

3.2 Delayless Feedback

Now let us consider the case of delayless voltage-proportional negative-feedback but with some phasing denoted by the angle θ, as sketched in Figure 3. Naturally, $|\theta| \le \pi$. Given that the real part of an impedance is symmetric about $\omega = 0$, while the imaginary part is antisymmetric about $\omega = 0$, so it follows that θ is a shorthand for $\theta_0 \times \mathrm{sign}(\omega)$. Assuming an

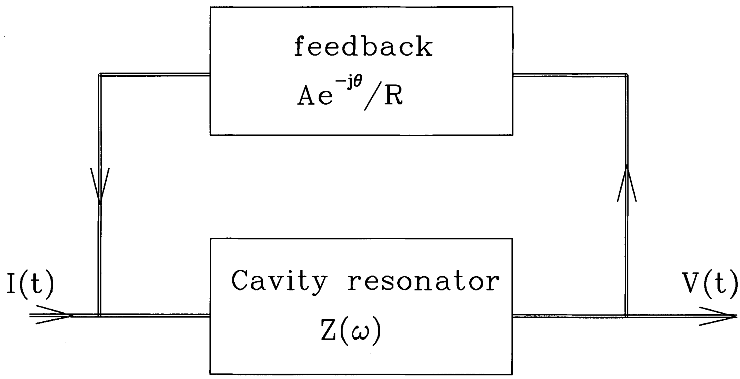

FIGURE 3 Cavity resonator with voltage-proportional feedback.

$\exp(+j\omega t)$ excitation, the system is governed by the equation

$$[(\Omega^2 - \omega^2) + 2j\alpha\,\omega]\mathbf{V}^0 = 2j\alpha\,\omega R_s[\mathbf{I}^0 - (A/R)e^{-j\theta}\mathbf{V}^0]. \tag{9}$$

The closed loop transfer function is

$$T(\omega) = \frac{Z(\omega)}{1 + (A/R)e^{-j\theta}Z(\omega)} = \frac{R_s}{1 - j\tan\Psi + A^{-j\theta}}. \tag{10}$$

Note, if $\theta \neq 0$ (assuming negative feedback) or if $\theta \neq \pi$ (assuming positive feedback) then the cavity does not appear as a pure real load when driven at the resonance frequency. The new resonance frequency is the solution of the equation

$$\tan\Psi + A\sin\theta = 0 \quad \text{or} \quad \omega_{\text{res}} \approx \Omega - \alpha A\sin\theta. \tag{11}$$

4 THE PROBLEM OF DELAY

By delay, we mean that the value of some quantity at time t is related to the value of some other quantity at an earlier time $t - T$, where T is the delay interval. We must indicate why we should expect the method of Nyquist to be applicable to this type of system; after all we have only shown how to elucidate the stability of transfer functions having integer order poles and zeros, that is consisting of the quotient of polynomials.

It is simple to show from the integral definition that the Laplace transform of $F(t - T)$ is $\exp(-sT)F(s)$ provided that $F(t) = 0$ for $t < T$. At first sight, the exponential function does not look like a polynomial. However, it is the limit of a polynomial:

$$\exp(-sT) = (1 - sT/N)^N \quad \text{as } N \to \infty, \tag{12}$$

where N is an integer. One can see that $\exp(-sT)$ has an infinite number of roots; and moreover, these roots are in the right-hand plane at $s = N/T \to +\infty$ and correspond to growing solutions. From Eq. (12) it is clear that we are dealing with a countable infinity of solutions, and so there is some hope of finding them all when the time comes to perform a stability analysis.

4.1 A Simple Example

Suppose that a system is governed by the equation

$$\dot{V} + BV(t - \tau) = I(t). \tag{13}$$

We introduce a dimensionless time $u = t/\tau$, and define $b = B\tau$, then the equation becomes

$$V' + bV(u - 1) = \tau I(t). \tag{14}$$

We form the Laplace transform w.r.t. dimensionless frequency s, to obtain:

$$[s + b\mathrm{e}^{-s}]V(s) = \tau I(s). \tag{15}$$

The natural frequencies of the system are solutions of the characteristic equation obtained by setting $I(s) = 0$. Hence, the system will be stable provided that $s + b\mathrm{e}^{-s} = 0$ has no roots in the right half complex plane.

We substitute $s = \mathrm{j}\omega$ in the characteristic Eq. (15) and form real and imaginary parts to find:

$$\mathcal{A} = b\cos\omega \quad \text{and} \quad \mathcal{B} = \omega - b\sin\omega. \tag{16}$$

Clearly, $\mathcal{A}$ has roots at $\omega_n = (2n + 1)\pi/2$ for integer $n = 0, 1, 2, \ldots$ The condition for no zeros $\mathcal{A}'(\omega_n)\mathcal{B}(\omega_n) < 0$ implies

$$[\omega_n(-1)^n - b]b > 0. \tag{17}$$

This cannot be satisfied for all n; however, it does not need to be. As can be seen from the Nyquist plot, Figure 4, the locus of F at most of the roots of $\mathcal{A} = 0$ have no possibility to encircle the origin. A single encirclement could occur where $\mathcal{B}$ is changing sign, and that is at $n = 0$. Now $\mathcal{A}'(\pi/2)\mathcal{B}(\pi/2) < 0$ implies $b < \pi/2$ and $B < \pi/(2\tau)$; and this is a sufficient condition for stability. Figure 4 shows the Nyquist plot, i.e. the locus of Eq. (16) for three values of b as ω varies from -20 to $+20$. For the case $b = 1.0$ the origin is not encircled, whereas for the case $b = 2.0$ the origin is encircled in a clockwise sense indicating a single root in the right half complex plane. Though it is not drawn, each of the Nyquist plots must be finished with a semi-circle of infinite radius in the right half plane.

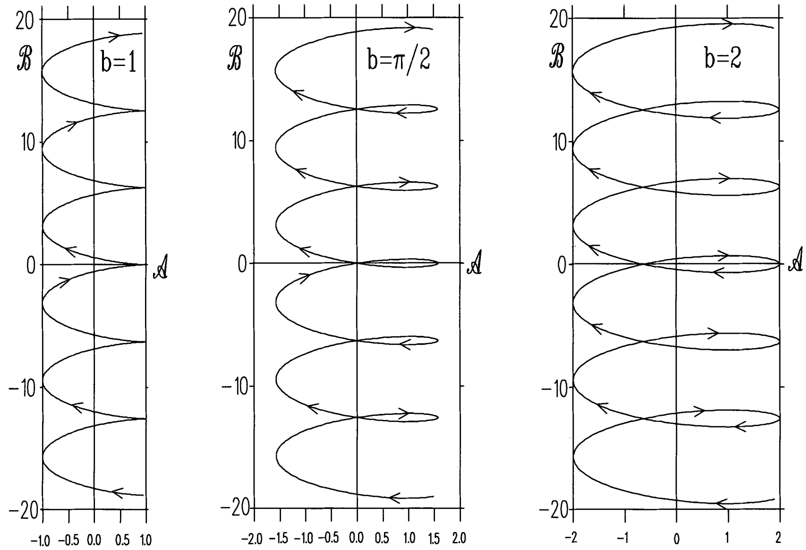

FIGURE 4 Nyquist plot of $b\cos\omega + \mathrm{j}(\omega - b\sin\omega)$.

4.1.1 Not all Roots are Important

We have just encountered an important result: the stability criterion $\mathcal{AB}' - \mathcal{BA}' > 0$ does not have to be satisfied at all the roots ω_A and ω_B; but it must be satisfied at those which can cause an encirclement of the origin. If condition (2) or (3) is used, it should be applied only to those consecutive roots which cause $\mathcal{A}$ or $\mathcal{B}$ to change sign; that is to say we should apply the stability criteria where the roots *alternate*; that is where $\omega_B(n+1) > \omega_A > \omega_B(n)$ for condition (2), or where $\omega_A(n+1) > \omega_B > \omega_A(n)$ for condition (3).

Suppose we set the delay $\tau = 0$ and find a set of roots ω_A and ω_B. Those roots which satisfy $\omega\tau \ll 1$ will not be much shifted when we allow τ to become finite, and we shall call them the 'low frequency roots.' In addition to these low frequency roots, there will also be an infinite set of high frequency roots; and these are usually periodic or approximately so. Of these high frequency roots, it is usual that only a very small subset can actually cause encirclement of the origin, and so we typically have a finite task to ascertain conditions for stability of the system.

5 RESONATOR WITH DELAYED DAMPING

Let us suppose the resonator is equipped with a delayed, voltage-proportional feedback that is intended to reduce the apparent cavity impedance at the drive frequency. We suppose the delay interval is τ. The system is governed by the equation

$$\ddot{V} + 2\alpha[\dot{V}(t) + A\dot{V}(t-\tau)] + \Omega^2 V = 2\alpha R_s \dot{I}. \tag{18}$$

Initially, for simplicity, we shall suppose that $|A| \gg 1$ and that A does not cause any phase-shift. Also, for brevity, let us write $2\alpha A = B$ and $2\alpha R = C$. Hence the differential equation

$$\ddot{V} + B\dot{V}(t-\tau) + \Omega^2 V = C\dot{I}. \tag{19}$$

We introduce a dimensionless 'time' $u = \Omega t$, and dimensionless variables $b = B/\Omega$, $T = \Omega\tau$ and $c = C/\Omega$. Let primes denote derivatives with respect to u; then Eq. (19) becomes

$$V'' + bV'(u-T) + V(u) = cI'. \tag{20}$$

Using the scaled time, if $b = 0$ then the period of oscillations is 2π. We form the Laplace transform with respect to the dimensionless complex frequency s, to find:

$$[s^2 + bs\mathrm{e}^{-sT} + 1]V(s) = csI(s). \tag{21}$$

The eigenvalues of the equation obtained by setting $I(s) = 0$ are the natural free oscillation frequencies of the system.

5.1 Criteria for Stability

To find the stability of this system, we set $s = \mathrm{j}\omega$ and divide the equation into real and imaginary parts $F = \mathcal{A} + \mathrm{j}\mathcal{B}$,

$$\mathcal{A} = 1 - \omega^2 + b\omega\sin(\omega T), \tag{22}$$

$$\mathcal{A}' = -2\omega + bT\omega\cos(\omega T), \tag{23}$$

$$\mathcal{B} = +b\omega \cos(\omega T), \tag{24}$$

$$\mathcal{B}' = +b[\cos(\omega T) - \omega T \sin(\omega T)]. \tag{25}$$

We must investigate the angular rotation rate, given by Eq. (1), at places where the locus of F crosses the axes which divide the complex plane into four quadrants, that is at the roots $\omega_{\mathcal{A}}$ of $\mathcal{A}=0$ and/or the roots ω_B of $\mathcal{B}=0$. It is easiest to locate the roots ω_B of $\mathcal{B}=0$. We need to find one low frequency root, and an infinite, periodic set of high frequency roots.

5.1.1 Low Frequency Root

If $\omega_B=0$ is the only root for which $\mathcal{A}(\omega_B)>0$ then we find the condition $\mathcal{A}\mathcal{B}'>0$ implies $b>0$. However, if $T>\pi/2$ then there will be two or more roots ω_B for which $\mathcal{A}(\omega_B)>0$; and so $b>0$ is not an essential condition. In fact, we shall find that b must change sign periodically as T increases. If we substitute the zero gain condition ($b=0$) into Eq. (26) then we find the maximum allowed gain $b_{\max}$ passes through zero at $\omega_n^2=1$ or $T=(2n+1)(\pi/2)$.

5.1.2 High Frequency Roots

The other roots of $\mathcal{B}$ occur at $\omega_n=(2n+1)\pi/(2T)$ with $n=0,1,2,\ldots$ a positive integer. Substituting we find the condition:

$$\mathcal{A}\mathcal{B}' = [1-\omega_n^2 + b\omega_n(-1)^n][-b\omega_n T(-1)^n] > 0. \tag{26}$$

This condition does not need to be satisfied for all n. However, it must be satisfied for the two adjacent n's which cause $\mathcal{A}$ to change sign; that is $\mathcal{A}(\omega_n)>0$ and $\mathcal{A}(\omega_{n+1})<0$. If condition (26) is not satisfied for these particular n's, then the origin is encircled in a clockwise sense and F has a zero in the right half plane. Because $\mathcal{B}'(\omega_n)$ changes sign with n, stability is possible in principle; and so it is worth proceeding with the analysis.

The exact conditions depend on the delay T. The inequality $\Omega\tau = T\leq\pi/2$ is a special case: a sufficient condition for stability is $\mathcal{A}(\omega_0)\mathcal{B}'(\omega_0)>0$ with $\omega_0=\pi/(2T)$, which implies:

$$0 < b < \frac{(\omega_0^2-1)}{\omega_0} = \frac{\pi}{2T} - \frac{2T}{\pi} \quad \text{or} \quad 0 < B\tau < \left[\frac{\pi}{2} - \frac{2}{\pi}(\Omega\tau)^2\right]. \tag{27}$$

The more general case $T = \Omega\tau > \pi/2$ is a little more complicated. Let us suppose that T is in the neighbourhood of an integer multiple of π, that is $T \approx m\pi$ where m is the nearest integer. We must find the values of b and T which mutually satisfy condition (26) for both $n = (m-1)$ and $n = m$. The two conditions $\mathcal{A}_{m-1}\mathcal{B}'_{m-1} = 0$ and $\mathcal{A}_m\mathcal{B}'_m = 0$ each delineate a curve in the b,T-plane which bounds the stable region. The point of intersection of the two curves is the place where the precedence of the conditions swaps over, and is the solution of $\mathcal{A}(\omega_{m-1}) = \mathcal{A}(\omega_m) = 0$. Substituting $\omega_{m-1} = \omega_m - (\pi/T)$ we find:

$$[2\omega_m - \pi/T] \times [b(-1)^m - \pi/T] = 0. \tag{28}$$

Because $(2\omega_m - \pi/T) = 2m\pi/T$ is always positive, this is an equation for the maximum gain. We substitute the gain $b(-1)^m = \pi/T$ into $\mathcal{A}(\omega_m) = 0$ to establish the quantities $T_{\rm crt}$ and $b_{\max}$,

$$T_{\rm crt} = m\pi\sqrt{1 - \frac{1}{4m^2}} \;\rightarrow\; m\pi, \tag{29}$$

$$(-1)^m b_{\max} = \frac{\pi}{T_{\rm crt}} \;\rightarrow\; \frac{1}{m} \text{ for large } m. \tag{30}$$

$T_{\rm crt}$ is a 'critical' value of the delay. Below $T_{\rm crt}$, the limiting gain is the solution of $\mathcal{A}_{m-1} = 0$. Above $T_{\rm crt}$, the limiting gain is the solution of $\mathcal{A}_m = 0$. Exactly at $T_{\rm crt}$ the limiting stable gain is equal to $b_{\max}$. Now we may state the extremal gain conditions:

$$\text{if } (2m-1)\pi/2 \le T \le T_{\rm crt} \text{ then } b = (-1)^m[1 - \omega_{m-1}^2]/\omega_{m-1}; \tag{31}$$

$$\text{if } T_{\rm crt} \le T \le (2m+1)\pi/2 \text{ then } b = (-1)^m[\omega_m^2 - 1]/\omega_m. \tag{32}$$

The gain stability boundary is sketched in Figure 5. If the gain b (with the appropriate sign) is smaller than given in conditions (31) and (32) then all natural oscillations are self-damped. Examples of the locus $F(\omega)$ in the vicinity of the origin are sketched in Figures 6–11 and the case of $m = 3$ is given in Ref. [9]. Alongside we show the corresponding time domain response when excited by a Dirac δ-function impulse applied at $t = 0$.

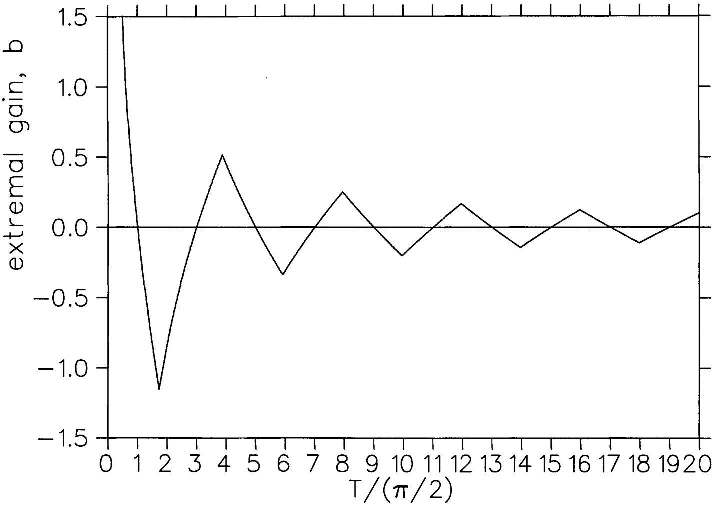

FIGURE 5 Extremal values of gain, b compatible with stability versus 'delay' $T = \Omega\tau$.

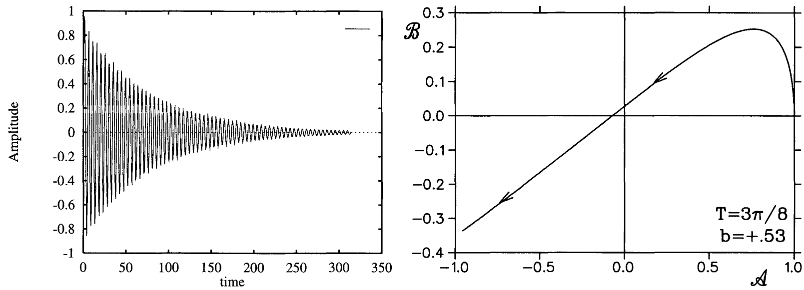

FIGURE 6 Stable; nearest integer $m = 0$.

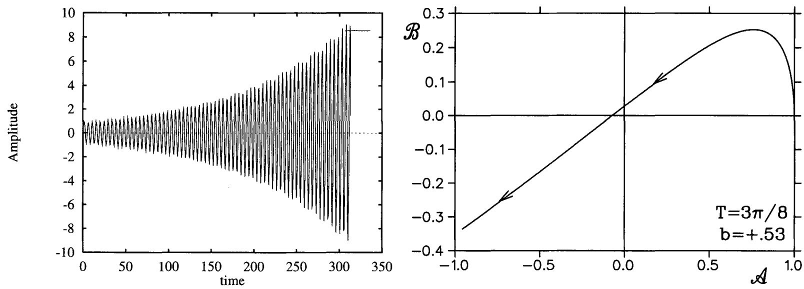

FIGURE 7 Unstable; nearest integer $m = 0$.

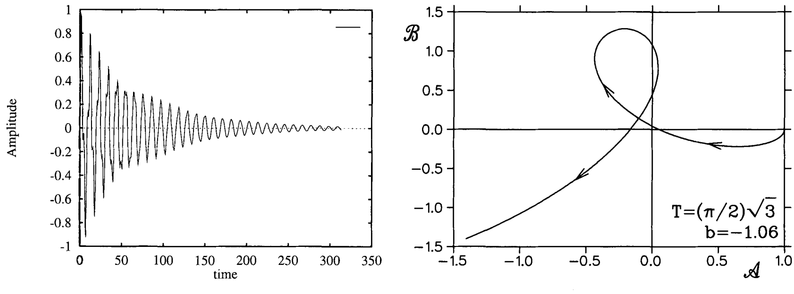

FIGURE 8 Stable; nearest integer $m = 1$.

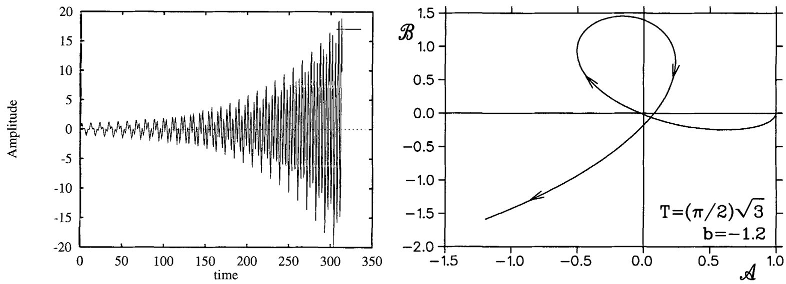

FIGURE 9 Unstable; nearest integer $m = 1$.

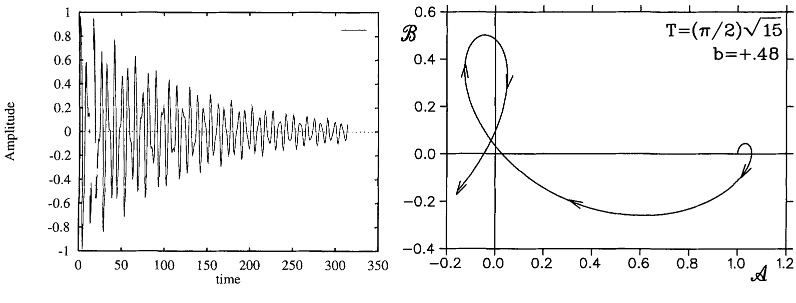

FIGURE 10 Stable; nearest integer $m = 2$.

5.2 Radio-Frequency System

If one desires order that the cavity resonator appears as a real load when driven at its original resonance frequency, Ω, then one must take the

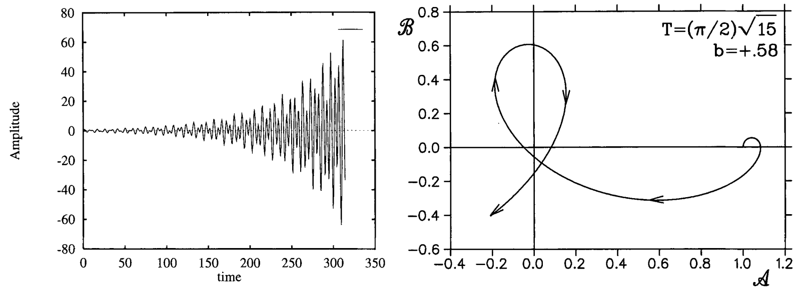

FIGURE 11 Unstable; nearest integer $m=2$.

cases $T=\Omega\tau=m\pi>T_{\mathrm{crt}}$. Hence the maximum gain condition is

$$b=\frac{(-1)^m}{m}\left[1-\frac{1}{2(2m+1)}\right]\ \rightarrow\ \frac{(-1)^m}{m}\ \text{for large } m. \tag{33}$$

Let us convert from dimensionless to real units for the case of a cavity resonator. Let us suppose $\Omega\tau=T=m\pi$ and $B=b\Omega=2\alpha A$. Hence the gain limit is given by $A=(\Omega/2\alpha)((-1)^m/m)$ but $\Omega/(2\alpha)=Q$ and so the maximum possible stable gain is

$$A=(-1)^m Q/m. \tag{34}$$

Because this is at the limit of stability, it is usual to 'back off' from this extreme and take $A=(-1)^m Q/(2m)$; this latter finding agrees with the stability limit given by Ries[6] and Boussard.[5] Given that $m=\Omega\tau/\pi$ and $Q/\Omega=\tau_c/2$, Eq. (34) can also be written as

$$A\times\tau=(-1)^m\tau_c\times(\pi/2). \tag{35}$$

Of course, other choices are possible for the delay interval, τ. One could take the condition $\omega_{\mathrm{rf}}\tau=m\pi$ where $\omega_{\mathrm{rf}}\neq\Omega$ is the desired drive radio-frequency. Below transition, $\omega_{\mathrm{rf}}<\Omega$ and so $T>m\pi$ is to be inserted in the maximum gain condition (32). Let $T=m\pi+(\Omega-\omega_{\mathrm{rf}})\tau\approx m\pi+(\tau/\tau_c)\tan\Psi$ and substitute in expression (32). In the limit of large m, the extremal gain is approximately

$$A\approx(-1)^m\frac{Q}{m}\left[1-\frac{2}{\pi}\frac{\tau}{\tau_c}\tan\Psi\right]\approx(-1)^m\left[\frac{\pi}{2}\frac{\tau_c}{\tau}-\tan\Psi\right], \tag{36}$$

where $\Psi>0$ because $\omega_{\mathrm{rf}}<\Omega$.

5.3 With Phase-shift

Suppose that we now consider the case that there is an additional phase advance in the feedback, θ (hence the gain becomes $b^{-j\theta}$ where $\theta = \theta_0 \text{sign}(\omega)$). Most of the previous results carry over as before.

$$F(\omega) = 1 - \omega^2 + b\sin(\omega T + \theta) + jb\omega\cos(\omega T + \theta). \tag{37}$$

$\mathcal{B}$ has roots at $\omega_n T = [(2n+1)(\pi/2) - \theta]$. Consider the case that the delay is in the neighbourhood of $m\pi$. The intersection of the bounding curves is again given by (28) and leads to

$$(-1)^m b_{\max} = \frac{\pi}{T_{\text{crt}}} \tag{38}$$

and

$$T_{\text{crt}} = (m\pi - \theta)\sqrt{1 - \frac{1}{4(m - \theta/\pi)^2}}. \tag{39}$$

The gain conditions are then formally identical with those given above ((31) and (32)), but with the revised definition $\omega_n(\theta)$.

5.4 Refined Models

The constraint $A \gg 1$ is removed in Ref. [9], but leads to rather lengthy expressions and we shall not reproduce them here. Stability of the case $0 < A \leq 1$ is also treated in Ref. [9].

6 SMALL-SIGNAL ANALYSIS OF CAVITY RESONATOR

When performing the stability analysis of a cavity resonator interacting with a charged particle beam, we shall consider small perturbations about the steady state, and develop the analysis in terms of the transfer functions for small phase and amplitude modulations of the carrier wave. We shall now derive these functions under the assumption that the modulation frequency is much smaller than the drive and resonance frequencies of the cavity. The carrier is nolonger arbitrary, but rather $\omega_c = \omega_{\text{rf}}$ is the frequency needed to maintain synchronism with the particle beam.

6.1 Steady-state

Let us suppose that the gain, A (including its sign) has been chosen consistent with the delay τ, according to the results of Section 5. Let us suppose that in addition to the phase changes introduced by the delay, the gain has an intrinsic phase-shift θ. The correct model for the feed-back gain is $Ae^{j\omega_c\tau}e^{j\theta}$ where $|\theta| \leq \pi$ but $\omega_c\tau$ is unbounded. Assuming an $\exp(+j\omega_c t)$ excitation, for the carrier wave, the system (steady-state) is governed by the equation

$$[(\Omega^2 - \omega_c^2) + 2j\alpha\omega_c]\mathbf{V}^0 = 2j\alpha\omega_c R_s[\mathbf{I}_T^0 - (A/R)e^{-j(\omega_c\tau+\theta)}\mathbf{V}^0], \qquad (40)$$

which may be written in terms of the cavity time constant, τ_c, and detuning angle, Ψ, as

$$[1 - j\tan\Psi]\mathbf{V}^0 = R_s\mathbf{I}_T^0 - Ae^{-j(\omega_c\tau+\theta)}\mathbf{V}^0. \qquad (41)$$

Note, if $\Omega\tau + \theta \neq m\pi$ then cavity does not appear pure real at the resonance frequency. Thus although the amplitude response is largest, the current drive and voltage response might not be in-phase at the original resonance frequency, Ω.

6.2 Non-steady-state

Let us now consider small perturbations, such that $\mathbf{V}(t) = \mathbf{V}^0 e^{j\omega_c t}[1 + \mathbf{e}_v(t)]$ and $\mathbf{I}_T(t) = I_T^0 e^{j\omega_c t}[1 + \mathbf{e}_T(t)]$. Here $\mathbf{e}_r = a_r + j\phi_r$ models small amplitude-and-phase modulation signals. If the modulations are low frequency, the system is governed (approximately) by

$$\tau_c\mathbf{V}^0\dot{\mathbf{e}}_v + [1 - j\tan\Psi]\mathbf{V}^0(1 + \mathbf{e}_v) = R_s\mathbf{I}_T^0(1 + \mathbf{e}_T) - Ae^{-j(\omega_c\tau+\theta)}\mathbf{V}^0[1 + \mathbf{e}_v(t - \tau)]. \qquad (42)$$

After subtracting the steady-state equation, the small signals obey:

$$\tau_c\dot{\mathbf{e}}_v + [1 - j\tan\Psi]\mathbf{e}_v + Ae^{-j(\omega_c\tau+\theta)}\mathbf{e}_v(t - \tau) = R_s[\mathbf{I}_T^0/\mathbf{V}^0]\mathbf{e}_T. \qquad (43)$$

It shall prove prudent to examine whether use of these approximations leads to any results which differ from or contradict the exact results presented in Section 5. To use the resonator to accelerate a charged particle beam it is usual to take $\omega_c\tau+\theta\approx m\pi$. For brevity, we write $\omega_c\tau+\theta=w$.

Though it is *not* essential to our arguments, one possibility is to adjust $\Omega\tau=m\pi$ and adjust $\theta=(\Omega-\omega_{rf})\tau$ so that $w=m\pi$ exactly, in which case the impedance is not pure real when driven at the angular frequency Ω.

6.2.1 *Characteristic Equation*

We compare real and imaginary parts of $\mathbf{e}_V$ and $\mathbf{e}_T$ to obtain the system matrix

$$\begin{bmatrix} 1+s\tau_c+Ae^{-s\tau}\cos w & Ae^{-s\tau}\sin w+\tan\Psi \\ -Ae^{-s\tau}\sin w-\tan\Psi & 1+s\tau_c+Ae^{-s\tau}\cos w \end{bmatrix}\begin{bmatrix} a_V \\ \phi_V \end{bmatrix}=\dots. \tag{44}$$

Here A is pure real and we do not assume $A\gg 1$. The natural frequencies are obtained by setting the determinant equal to zero, leading to the characteristic equation $c_2s^2+c_1s+c_0=0$ where the polynomial coefficients are:

$$c_0=(1+Ae^{-s\tau}\cos w)^2+(Ae^{-s\tau}\sin w+\tan\Psi)^2, \tag{45}$$

$$c_1=2\tau_c(1+Ae^{-s\tau}\cos w), \tag{46}$$

$$c_2=\tau_c^2. \tag{47}$$

We set $s=+j\omega$ to find $F=\mathcal{A}+j\mathcal{B}$. Notice that whereas in Section 5 'ω' was a dimensionless variable and $\omega\times\Omega$ indicated an absolute frequency with respect to d.c.; in this section 'ω' is a relative frequency and the true excitation frequency is $\omega_c+\omega$ with respect to d.c. The function $\mathcal{B}$ has a single low frequency root, located at $\omega_B=0$ and a periodic set of high frequency roots.

6.2.2 *Low Frequency Root*

We must study the quantity $\mathcal{A}(0)\times\mathcal{B}'(0)$. Now $\mathcal{A}(0)=c_0$ $(s=0)$ and being the sum of real squares is always positive,

$$\mathcal{B}'(0)=[\tau_c(1+A\cos w)-A\tau(A+\cos w+\sin w\tan\Psi)] \tag{48}$$

and from the stability condition $\mathcal{A}\mathcal{B}' > 0$ we conclude:

$$A\tau < \tau_c \frac{(1 + A\cos w)}{(A + \cos w + \sin w \tan \Psi)}. \tag{49}$$

This condition is a little mysterious in as much as it is not an obvious counterpart to predictions by the analysis of Section 5. Further, it should be noted that the AM/PM transfer functions are exact at $\omega = 0$. However, Eq. (34) was derived for the case $\theta = 0$, $w = m\pi$ and $A \gg 1$; so we find (49) reduces to $A\tau < \tau_c(-1)^m$ which implies $A < 2Q/(\Omega\tau)$ assuming m even. But Eq. (34) assumes $\Omega\tau = m\pi$ and so $A < 2Q/(m\pi)$ which differs little from the condition $A < Q/m$ established earlier. The condition (49) also guarantees that $\mathcal{B}$ does not have a multiple root at $\omega = 0$.

6.2.3 *High Frequency Roots*

Under the conditions $|A| \gg 1$ and $w = m\pi$ the high frequency roots of $\mathcal{B}$ occur at approximately $\omega_n = (2n+1)\pi/(2\tau)$ with $n = 0, 1, 2, \ldots$ Now, $\Omega\tau \approx m\pi$ and so $\omega_n \approx \Omega(2n+1)/(2m)$. Given the conditions under which the the AM and PM transfer functions were derived (that is $|\omega| \ll \omega_c$), we expect stability conditions founded on the high frequency roots only to be reliable when $m \gg (n+1)$ which implies long delays. Certainly any result obtained for $n > 0$ should be treated with extreme scepticism. Having stated these warnings, let us now proceed with the analysis.

The quantity $\mathcal{A}(\omega_n) \times \mathcal{B}'(\omega_n)$ is:

$$[\tan^2 \Psi - (A - (-1)^{n+m}\tau_c\omega_n)^2] \times 2A\tau[A - (-1)^{n+m}\tau_c\omega_n]. \tag{50}$$

This condition does not have to be satisfied for all n, just for those consecutive values which cause $\mathcal{A}$ to change sign. When $\tan \Psi = 0$ (that is $\Omega = \omega_c$) we notice that $\mathcal{A}(\omega_n) < 0$ for all n; and, consequently, we infer that $\mathcal{A}$ changes sign between $\omega = 0$ and $\omega = \pi/(2\tau)$. If this is so, then a sufficient condition for stability is that $\mathcal{A}(\omega_0) < 0$ and $\mathcal{B}'(\omega_0) < 0$. When $\tan \Psi = 0$ the stability condition is $A < (-1)^{n+m}\tau_c\omega_n$; and substituting $n = 0$ and setting m even gives $A < \pi\tau_c/(2\tau) = Q\pi/(\Omega\tau)$ or $A < Q/m$ when $\Omega\tau = m\pi$. Hence, when $\tan \Psi = 0$, Eq. (50) is a re-statement of condition (34).

When the detuning angle is non-zero, and we substitute $n = 0$, we find the condition

$$[(\tau_c\omega_0 - A)^2 - \tan^2\Psi] \times (\tau_c\omega_0 - A) > 0, \tag{51}$$

from which we conclude (by solving the quadratic)

$$A \leq \pi\tau_c/(2\tau) \pm \tan\Psi \text{ or approximately } A < \frac{\pi}{2}\frac{\tau_c}{\tau} - (\Omega - \omega_c)\tau_c \tag{52}$$

because $\tan\Psi \approx \tau_c(\Omega - \omega_c)$ for small detunings. Expression (52) is a re-statement of (36).

7 SACHERER FORMALISM

In Section 2 we introduced a formalism to determine when the transfer function has poles/zeros in the right half complex plane. The sheer power of this formalism will generate stability criteria, but without giving much insight as to the nature of the instability. We shall explain the origin of the criteria by applying Sacherer's theory[4] for computing longitudinal bunched beam instability. The nature of the instability will become clear as a consequence of considering the resistive and reactive parts of the impedance for various 'key' frequencies predicted by the algebraic Nyquist criterion.

It is widely known that Robinson's first criterion (for the a.c. instability) and second criterion (for the d.c. instability) can be obtained from Sacherer's equations for bunched beam longitudinal instability. The first criterion comes from considering the growth rate which is proportional to the resistive part of the impedance, while the second criterion comes from considering the coherent frequency shift which is proportional to the reactive part of the impedance. We can anticipate finding analogues by considering the real and imaginary parts of the cavity impedance with delayed feedback.

Let ω_s be the synchrotron frequency, and Z be an impedance. Let $\xi = (I_{ac}^{b}/2V_{rf})$ be the ratio of beam current component at the RF to the peak cavity voltage. According to the two-mode-theory for a narrow-band impedance, the angular oscillation frequency s of a dipole density perturbation is proportional to the impedance:

$$j(s^2 + \omega_s^2)/\omega_s^2 \approx [Z(+\omega_{rf} + \omega; \sigma) - Z(-\omega_{rf} + \omega; \sigma)]\xi. \tag{53}$$

Let us consider the case of small or zero growth rate, and substitute $\sigma = 0$ in the right-hand side. Let R and X be the real (resistive) and imaginary (reactive) parts of the impedance. We find the growth rate

$$\begin{aligned} \omega\sigma/\omega_s^2 &\approx -[R(+\omega_{rf} + \omega) - R(-\omega_{rf} + \omega)]\xi \\ &\approx -[R(\omega_{rf} + \omega) - R(\omega_{rf} - \omega)]\xi \end{aligned} \tag{54}$$

and the coherent frequency

$$\begin{aligned} (\omega^2 - \omega_s^2)/\omega_s^2 &\approx -[X(+\omega_{rf} + \omega) - X(-\omega_{rf} + \omega)]\xi \\ &\approx -[X(\omega_{rf} + \omega) + X(\omega_{rf} - \omega)]\xi. \end{aligned} \tag{55}$$

Before we can evaluate these expressions for mode frequency, we must find an explicit form for the impedance.

7.1 Robinson Criteria

For the case of the parallel LCR resonator, which is used to model the cavity fundamental resonance, the impedance can be calculated analytically. We suppose the drive or carrier frequency is ω_{rf} and substitute a trial time dependence $\exp(j\omega_{rf} + s)t$ into Eq. (4). Then one finds the impedance

$$Z = \frac{V}{I} = \frac{R_s}{1 + (j\omega_{rf} + s)/(2\alpha) + \Omega^2/(2\alpha(j\omega_{rf} + s))}. \tag{56}$$

Let us suppose that $|s| \ll \omega_{rf}$. From the definition of the detuning angle (7) it is clear that $\tan\Psi(-\omega_{rf}) = -\tan\Psi(+\omega_{rf})$. Thus we may write:

$$Z(\pm\omega_{rf} + \omega; \sigma) \equiv Z(\pm\omega_{rf}, s) \approx \frac{R_s}{1 \mp j\tan\Psi(+\omega_{rf}) + s\tau_c}. \tag{57}$$

The impedance at the carrier frequency is simply $Z(\omega_{rf}, 0)$. The difference of the impedance at the sideband of positive and negative carrier frequencies is:

$$Z(+\omega_{\rm rf}, s) - Z(-\omega_{\rm rf}, s) \approx \frac{R_{\rm s}\, 2{\rm j} \tan \Psi(+\omega_{\rm rf})}{(1 + s\tau_{\rm c})^2 + \tan^2 \Psi}. \tag{58}$$

To find the impedance at the upper (positive) and lower (negative) sidebands we substitute $s = \pm {\rm j}\omega$ to find:

$$\begin{aligned} Z(+\omega_{\rm rf} \pm \omega) &= \frac{R_{\rm s}}{1 - {\rm j}(\tan \Psi \mp \omega\tau_{\rm c})} \\ &\approx \frac{R_{\rm s}}{\sec^2 \Psi} [1 + {\rm j} \tan \Psi(+\omega_{\rm rf}) + {\rm j}(\mp\omega\tau_{\rm c}) \exp(+{\rm j}2\Psi)]. \end{aligned} \tag{59}$$

The approximation is valid provided $|\omega\tau_{\rm c}| \ll \sec^2 \Psi$. In what follows, we shall assume the product $\omega_{\rm s} \times \tau_{\rm c} < 1$ as is most often the case for normal conducting cavities.

7.1.1 Growth Rate, σ

To find the growth rate, we substitute $\omega = \omega_{\rm s}$ and consider the resistive part

$$R(\omega_{\rm rf} - \omega_{\rm s}) - R(\omega_{\rm rf} + \omega_{\rm s}) = -2R_{\rm s}\omega_{\rm s}\tau_{\rm c} \cos^2 \Psi \sin 2\Psi. \tag{60}$$

Here $\Psi = \Psi(+\omega_{\rm rf})$. Provided that the detuning is positive ($\Psi > 0$), then the resistance is larger at the upper synchrotron sideband than at the lower, and (according to Eq. (54)) the beam is stable below transition energy.

7.1.2 Coherent Frequency, ω

We consider the reactive part at upper and lower sidebands:

$$\frac{\omega^2 - \omega_{\rm s}^2}{\omega_{\rm s}^2} = -\xi[X(\omega_{\rm rf} - \omega_{\rm s}) + X(\omega_{\rm rf} + \omega_{\rm s})] \approx (I_{\rm ac}^{\rm b}/2V_{\rm rf})R_{\rm s} \sin(2\Psi). \tag{61}$$

This equation predicts that the coherent oscillation frequency will approach zero when

$$\frac{I_{\rm ac}^{\rm b} R_{\rm s}}{V_{\rm rf}} = \frac{2}{\sin(2\Psi)}. \tag{62}$$

(However, in the case that $\omega_s\tau_c > 1$, the beam frequency increases and the instability limit is reached when the cavity mode frequency acquires a real part.)

8 ANALOGUES OF THE ROBINSON CRITERIA

The total, steady-state drive current $\mathbf{I}_T$ is the sum of a generator component $\mathbf{I}_g^0 = I_g^0 e^{j\Psi_g}$ and the fundamental beam current component $\mathbf{I}_b^0 = jI_b^0 e^{j\Phi_b}$. It is customary[8] to define $I_V^0 = V^0/R$ and introduce the current ratios $Y_b = I_b^0/I_V^0$ and $Y_g = I_g^0/I_V^0$. Let $w = \omega_{rf}\tau + \theta$ as before. From the steady-state condition (41) we find the working curves,

$$1 + A\cos w = Y_g \cos\Psi_g - Y_b \sin\Phi_b, \tag{63}$$

$$\tan\Psi + A\sin w = Y_b \cos\Phi_b - Y_g \sin\Psi_g. \tag{64}$$

Let us now consider small dipole oscillations of the particle beam about the steady-state phase, Φ_b. We define the synchrotron frequency *sans* the usual trigonometric factor to be:

$$\Omega_s^2 = f_\infty^2 \left| \frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2} \right| \frac{eV^0}{2\pi h E_s}, \tag{65}$$

where the meanings of the symbols follows that used by Bovet.[7] We also define $\omega_s^2 = \Omega_s^2 \cos\Phi_b$. Using the beam response equations of Ref. [8], the system matrix is given by:

$$\begin{bmatrix} 1 + s\tau_c + Ae^{-s\tau}\cos w & Ae^{-s\tau}\sin w + \tan\Psi & -Y_b\cos\Phi_b \\ -Ae^{-s\tau}\sin w - \tan\Psi & 1 + s\tau_c + Ae^{-s\tau}\cos w & Y_b \sin\Phi_b \\ -\Omega_s^2 \sin\Phi_b & -\Omega_s^2\cos\Phi_b & s^2 + \Omega_s^2\cos\Phi_b \end{bmatrix} \times \begin{bmatrix} a_V \\ \phi_V \\ \phi_b \end{bmatrix} = \cdots. \tag{66}$$

The natural frequencies are obtained by setting the determinant equal to zero, leading to $c_4 s^4 + c_3 s^3 + c_2 s^2 + c_1 s + c_0 = 0$ where the polynomial coefficients are:

$$c_0 = \Omega_s^2 \cos\Phi_b[(1 + A\mathrm{e}^{-s\tau}\cos w)^2 + (\tan\Psi + A\mathrm{e}^{-s\tau}\sin w)^2] - \Omega_s^2 Y_b(\tan\Psi + A\mathrm{e}^{-s\tau}\sin w), \tag{67}$$

$$c_1 = 2\Omega_s^2\tau_c\cos\Phi_b(1 + A\mathrm{e}^{-s\tau}\cos w), \tag{68}$$

$$c_2 = (1 + A\mathrm{e}^{-s\tau}\cos w)^2 + (A\mathrm{e}^{-s\tau}\sin w + \tan\Psi)^2 + (\Omega_s\tau_c)^2\cos\Phi_b, \tag{69}$$

$$c_3 = 2\tau_c(1 + A\mathrm{e}^{-s\tau}\cos w), \tag{70}$$

$$c_4 = \tau_c^2. \tag{71}$$

We set $s = +\mathrm{j}\omega$ to find $F = \mathcal{A} + \mathrm{j}\mathcal{B}$.

8.1 Low Frequency Roots

If we set the delay to zero, we find that $\mathcal{B}$ has three low frequency roots: one located at $\omega_{\mathcal{B}} = 0$ and a pair at $\omega_{\mathcal{B}}^2 = \Omega_s^2\cos\Phi_b$. By good fortune, provided that $w = m\pi$, these low frequency roots do not move when the delay, τ, is made non-zero.

8.1.1 Root at $\omega = 0$

The quantity $\mathcal{A}(0) \times \mathcal{B}'(0)$ is equal to:

$$c_0(s = 0) \times \Omega_s^2\{2\cos\Phi_b[\tau_c(1 + A\cos w) - A\tau(A + \cos w + \sin w\tan\Psi)] + Y_b A\tau\sin w\}. \tag{72}$$

This is easiest to interpret when $w = m\pi$ and m is even, in which case the stability condition is:

$$[(1 + A)^2 + \tan^2\Psi - Y_b\tan\Psi] \times [\tau_c - A\tau] > 0. \tag{73}$$

Now, from the conditions for stability of the cavity resonator without beam, we know $A\tau < \tau_c$; and so it follows that $Y_b \tan\Psi < (1+A)^2 + \tan^2\Psi$ is a necessary condition for stability. This is a satisfying result because it is identical with Robinson's second, or power limited, stability criterion; and this was anticipated because delay cannot change the nature of a d.c. instability – because for d.c. signals, an arbitrarily long delay does not change the value of the signal.

Now, let us generalize to the case of arbitrary w. The stability condition $\mathcal{B}'(0) > 0$ leads to

$$A\tau < \frac{\tau_c(1 + A\cos w) + (Y_b A\tau \sin w / 2\cos\Phi_b)}{(A + \cos w + \sin w \tan\Psi)}, \tag{74}$$

which is also the condition for avoiding a double root at $\omega = 0$. Condition (74) is the counterpart of (49). It is important to appreciate that this condition could supersede the simple condition $A\tau < \tau_c$.

For mathematical compactness we define:

$$\tan\nu = (\tan\Psi + A\sin w)/(1 + A\cos w). \tag{75}$$

Then, assuming $(1 + A\cos w) > 0$ the condition $\mathcal{A}(0) > 0$ becomes:

$$\begin{aligned} Y_b \tan\nu &< \cos\Phi_b(1 + A\cos w)\sec^2\nu \\ \text{or}\quad Y_b &< 2\cos\Phi_b(1 + A\cos w)/\sin 2\nu. \end{aligned} \tag{76}$$

This is the anticipated analogue of Robinson's second criterion, Eq. (62), and (apart from the terms in w) the result is independent of whether the feedback is delayed or not.

8.1.2 Roots at $\pm\omega_s$

Let us now consider $\mathcal{A}$ and $\mathcal{B}$ evaluated at $\omega_s^2 = \Omega_s^2 \cos\Phi_b$:

$$\mathcal{A}(\pm\omega_s) = -\Omega_s^2 Y_b[A\cos(\omega_s\tau)\sin w + \tan\Psi], \tag{77}$$

$$\mathcal{B}(\pm\omega_s) = \Omega_s^2 Y_b A \sin(\omega_s\tau)\sin w \tag{78}$$

and so ω_s is an exact root of $\mathcal{B}$ when $w = m\pi$. Let us evaluate $\mathcal{A}(\omega_s)\mathcal{B}'(\omega_s)$:

$$4\Omega_s^2 Y_b[1 + (-1)^m A\cos(\omega_s\tau)]\omega_s[\omega_s\tau_c - (-1)^m A\sin(\omega_s\tau)]\tan\Psi. \tag{79}$$

This quantity can become negative, indicating instability, in a variety of ways. For simplicity, suppose m is even. From the stability of the resonator without the particle beam we know that

$$\tau_c > A\tau \geq A \sin(\omega_s \tau)/\omega_s. \tag{80}$$

Hence, from (79) we conclude the stability conditions:

$$\tan \Psi > 0, \tag{81}$$

$$1 + A \cos(\omega_s \tau) > 0. \tag{82}$$

The inequality (81) is the first Robinson criterion and it tells us to detune the cavity in the correct sense: $\Omega > \omega_{rf}$ when below transition energy. The extra condition (82) implies the possibility of instability when the synchrotron frequency is high and the delay is very long; from which we conclude a sufficient condition $\omega_s \tau < \pi/2$ or $\omega_s < \omega_0$.

Let us attempt to understand the physics behind Eq. (79). Consider an RF cavity with no (or ideal, delayless) feedback. Detuning the cavity (with the resonance frequency greater than the drive frequency, below transition energy) has the effect that the real part of the cavity impedance damps dipole motion of the beam. If driven on-resonance, the real part of the impedance is of course the cavity shunt resistance, which, being positive, is dissipative. To be exact, for the classic Robinson-type instability it is the *difference* of the impedance at the lower and upper synchrotron sidebands which counts, and it is $\tan \Psi$ which controls the sign of this difference. This is the reason for the appearance of $\tan \Psi$ in Eq. (79).

Suppose now there is a feedback with a long delay, and the phase of the feedback is adjusted so that the real part of the cavity impedance at the drive frequency is positive. With feedback, the real part of the impedance at upper and lower synchrotron sidebands becomes proportional to $[\omega_s \tau_c - (-1)^m A \sin(\omega_s \tau)]$, and this is the reason for the appearance of this term in Eq. (79). For a sufficiently long delay, one finds that at the upper and lower synchrotron sidebands of the drive frequency, the real part of the cavity impedance looks like a *negative* resistance. This is the reason for the term $[1 + (-1)^m A \cos(\omega_s \tau)]$ appearing in Eq. (79).

8.2 High Frequency Roots

Now, time has come to find a periodic set of high frequency roots of $\mathcal{B}=0$. To simplify matters we shall consider the case of very large gain, that is $|A|\gg 1$. We shall consider $\mathcal{B}$ evaluated at $\omega_n=(2n+1)\pi/(2\tau)$ where n is an integer,

$$\mathcal{B}(\omega_n) = (-1)^n A \sin w[\Omega_s^2 Y_b + 2\omega_n^2 \tan\Psi - 2\Omega_s^2 \cos\Phi_b \tan\Psi], \quad (83)$$

from which we conclude the ω_n are exact roots of $\mathcal{B}=0$ if $w=\omega_{rf}\tau+\theta=m\pi$,

$$\mathcal{A}(\omega_n; w=m\pi) = [A^2 - 2(-1)^{m+n} A\tau_c\omega_n + (\tau_c\omega_n)^2 - \tan^2\Psi] \times (\omega_n^2 - \Omega_s^2\cos\Phi_b) - \Omega_s^2 Y_b \tan\Psi, \quad (84)$$

$$\mathcal{B}'(\omega_n; w=m\pi) = 2A\tau(-A + (-1)^{m+n}\tau_c\omega_n)(\omega_n^2 - \Omega_s^2\cos\Phi_b). \quad (85)$$

To simplify matters let m be even. We note that the transfer functions are only valid for small modulation frequencies and so we set $n=0$. Now $\omega_0>\omega_s$ and so the condition $\mathcal{AB}'>0$ becomes:

$$\left\{[(\tau_c\omega_0 - A)^2 - \tan^2\Psi] - \frac{\Omega_s^2 Y_b \tan\Psi}{(\omega_0^2 - \omega_s^2)}\right\} \times (\tau_c\omega_0 - A) > 0. \quad (86)$$

This condition is very similar to Eq. (51). Now $A<\omega_0\tau_c=(\pi/2)(\tau_c/\tau)$ from the criterion for cavity stability in the absence of the particle beam; and so we conclude:

$$\left(\frac{Q\pi}{\Omega\tau} - A\right)^2 > \tan^2\Psi + \frac{(\Omega_s\tau)^2 Y_b \tan\Psi}{(\pi/2)^2 - (\Omega_s\tau)^2\cos\Phi_b}. \quad (87)$$

This condition implies that the maximum, stable feedback gain is reduced under conditions of heavy beam loading. Of course, the condition is only accurate under the condition of long delay: $\Omega\tau\gg\pi/2$.

If the relation (86) becomes equal to zero because of the term in braces $\{\cdots\}$, then a coherent oscillation occurs at the high frequency

$\omega_0 = \pi/(2\tau)$. The dipole mode frequency is shifted away from the synchrotron frequency by the very large reactance at the sidebands of the carrier that occurs if $|\tau_c\omega_0 - A| \approx \tan\Psi$. Essentially, the reactive impedance raises the coherent frequency to a point high enough that it can oscillate in synchronism with a spontaneous high frequency oscillation of the resonator-with-feedback. Despite the fact that the feedback is in-phase at the carrier, at this sideband frequency ω_0 the feedback is in quadrature so making the effective impedance look very reactive.

9 SACHERER FORMALISM APPLIED TO RESONATOR WITH DELAYED FEEDBACK

Let us suppose the feedback gain is pure real, and so the 'extra' phase advance $\theta = 0$. Again, we substitute the time dependence $\exp(\mathrm{j}\omega_{\mathrm{rf}} + s)t$ into the voltage and current relation (18) and so find the impedance:

$$Z = \frac{V}{I} = \frac{R_s}{[1 + A\mathrm{e}^{-(\mathrm{j}\omega_{\mathrm{rf}}+s)\tau}] + (\mathrm{j}\omega_{\mathrm{rf}} + s)/(2\alpha) + \Omega^2/(2\alpha(\mathrm{j}\omega_{\mathrm{rf}} + s))}. \quad (88)$$

Since $\theta = 0$ the quantity $w = \omega_{\mathrm{rf}}\tau$. It is best‡ to adjust $w = m\pi$ with m a positive integer; this implies $\Omega\tau \neq m\pi$, but provided $|A| < Q/(2m)$ there is no possibility of the resonator becoming unstable. We are soon to find stability conditions for the charged particle beam. If $w \neq m\pi$ then the beam will become unstable before the threshold given in (94). For simplicity, we shall set m even. After some algebraic manipulation, the impedance becomes:

$$Z(\pm\omega_{\mathrm{rf}}, s) = \frac{R_s}{(1 + A\mathrm{e}^{-s\tau}) \mp \mathrm{j}\tan\Psi(+\omega_{\mathrm{rf}}) + s\tau_c}. \quad (89)$$

When driven at the original resonance frequency Ω the cavity shunt impedance is reduced by a factor $(1 + A\mathrm{e}^{-\mathrm{j}\Omega\tau})$. The difference of the impedance at positive and negative carrier frequencies is:

$$Z(+\omega_{\mathrm{rf}}, s) - Z(-\omega_{\mathrm{rf}}, s) \approx \frac{R_s\, 2\mathrm{j}\tan\Psi(+\omega_{\mathrm{rf}})}{(1 + A\mathrm{e}^{-s\tau} + s\tau_c)^2 + \tan^2\Psi}. \quad (90)$$

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‡See Section 9.3.1.

Again, let us evaluate the impedance at the upper and lower sidebands; we substitute $s = \pm j\omega$ to find:

$$Z(+\omega_{\rm rf} \pm \omega) = \frac{R_{\rm s}}{[1 + A\cos(\omega\tau)] - {\rm j}[\tan\Psi + A\sin(\pm\omega\tau) \mp \omega\tau_{\rm c}]}. \quad (91)$$

9.1 Short Delay

Initially, let us consider a short delay, so that $(1 + A\cos\omega_{\rm s}\tau) > 0$. Hence we consider $\omega_{\rm s}\tau < \pi/2$. We define a 'reduced' detuning angle by $\tan\mu = \tan\Psi/(1 + A\cos\omega\tau)$. Then the impedance can be written:

$$Z(\omega_{\rm rf} \pm \omega) \approx \frac{R_{\rm s}}{(1 + A\cos\omega\tau)\sec^2\mu} \times \left[1 + {\rm j}\tan\mu + {\rm j}\frac{[\mp\omega\tau_{\rm c} + A\sin(\pm\omega\tau)]\exp(+{\rm j}2\mu)}{(1 + A\cos\omega\tau)}\right]. \quad (92)$$

The approximation is valid for $|-\omega\tau_{\rm c} + A\sin\omega\tau| \ll (1 + A\cos\omega\tau)\sec^2\mu$.

9.1.1 Growth Rate

From Eq. (54), the growth rate is proportional to the resistive part of the impedance evaluated at upper and lower synchrotron sidebands of the radio-frequency:

$$R(\omega_{\rm rf} - \omega_{\rm s}) - R(\omega_{\rm rf} + \omega_{\rm s}) = -2R_{\rm s}[\omega_{\rm s}\tau_{\rm c} - A\sin(\omega_{\rm s}\tau)] \times \sin 2\mu \frac{\cos^2\mu}{(1 + A\cos\omega_{\rm s}\tau)^2}. \quad (93)$$

From Eq. (93) the meaning of the second parenthesis of Eq. (79) becomes clear. In order that the resistance at the upper synchrotron sideband be greater than at the lower sideband, it is essential that

$$\omega_{\rm s}\tau_{\rm c} - A\sin(\omega_{\rm s}\tau) > 0. \quad (94)$$

If this condition (which is (80)) is violated, then the $\Re[Z]$ is positive at both sidebands, but the difference of $\Re[Z]$ evaluated at the sidebands is negative, leading to a Robinson-type instability.

Notice, in the case of zero delay one finds:

$$R(\omega_{\rm rf} - \omega_{\rm s}) - R(\omega_{\rm rf} + \omega_{\rm s}) = -2R_{\rm s}\omega_{\rm s}\tau_{\rm c} \sin 2\mu \frac{\cos^2\mu}{(1+A)^2}, \tag{95}$$

with $\tan\mu = \tan\Psi/(1+A)$. Equation (95) implies the natural damping of a dipole oscillation is reduced compared with a cavity that has no feedback. This is because the feedback has the effect of 'flattening' the impedance as a function of the excitation frequency. Hence the slope of R in the neighbourhood of $\omega_{\rm rf}$ is reduced, and the difference of the resistive parts at upper and lower synchrotron sidebands is smaller.

9.1.2 *Low Coherent Frequency*

We consider the reactive part of the impedance at upper and lower sidebands,

$$\frac{\omega^2 - \omega_{\rm s}^2}{\omega_{\rm s}^2} \approx \frac{I_{\rm ac}^{\rm b} R_{\rm s}}{2V_{\rm rf}} \frac{\sin(2\mu)}{(1 + A\cos\omega\tau)}. \tag{96}$$

This equation predicts that the coherent oscillation frequency will approach zero when

$$\frac{I_{\rm ac}^{\rm b} R_{\rm s}}{V_{\rm rf}} = \frac{2(1+A)}{\sin(2\mu)} \quad \text{and} \quad \tan\mu = \frac{\tan\Psi}{(1+A)}. \tag{97}$$

This is identical with the form that criterion (76) takes when $w = m\pi$ because in this case $\tan\nu = \tan\mu$.

9.1.3 *High Coherent Frequency*

The imaginary part of the impedance is given by:

$$\Im[Z(\omega_{\rm rf} + \omega)] = \frac{R_{\rm s}[\tan\Psi - \omega\tau_{\rm c} + A\sin\omega\tau]}{(1 + A\cos\omega\tau)^2 + (\tan\Psi - \omega\tau_{\rm c} + A\sin\omega\tau)^2}. \tag{98}$$

In the presence of a long delay, one can anticipate a high frequency oscillation mode of the system. From Eq. (55), the coherent frequency shift is proportional to the reactive part of the impedance. Let us evaluate Eq. (55) at the sideband frequencies $\omega_0 = \pm\pi/(2\tau)$. We note that $A\cos(\omega_0\tau) = 0$ and $A\sin(\omega_0\tau) = A \times \mathrm{sign}(\omega_0)$, thus

$$\frac{\omega_0^2 - \omega_s^2}{\omega_s^2} \approx \frac{-2\xi R_s \tan\Psi}{\tan^2\Psi - [A\ \mathrm{sign}(\omega_0) - \omega_0\tau_c]^2}. \tag{99}$$

Essentially, this expression (which is equivalent to (86)) tells us that if the reactive impedance is very large at the sidebands of the carrier frequency, then the coherent frequency of the dipole mode can be lifted to a very high frequency equal to ω_0. The condition for this to occur is approximately

$$\frac{\omega_0^2}{\omega_s^2} \approx \frac{(I_{ac}^b R_s/V_{rf})\tan\Psi}{[\omega_0\tau_c - A\,\mathrm{sign}(\omega_0)]^2 - \tan^2\Psi}. \tag{100}$$

It is the right hand side which is constrained by this equation, the left side is a given ratio of angular frequencies. One may usually avoid this kind of instability by the condition

$$[(\pi/2)(\tau_c/\tau) - A] \gg \tan\Psi \geq 0, \tag{101}$$

which has the effect of making the reactance small at the sidebands of the carrier.

Thus we must take a stronger constraint on the feedback gain. In the absence of beam loading, the stability of the cavity-with-feedback depends on the condition $\omega_0\tau_c > A$ or $\pi\tau_c/2 > A\tau$. However, in the presence of the beam one must use the condition $\omega_0\tau_c - A > \tan\Psi$ or $(\pi/2)(\tau_c/\tau) > (A + \tan\Psi)$.

9.2 Long Delay

The real part of the impedance evaluated at synchrotron sidebands is given by:

$$\Re[Z] = \frac{R(1 + A\cos\omega_s\tau)}{(1 + A\cos\omega_s\tau)^2 + (\tan\Psi \mp \omega_s\tau_c + A\sin(\pm\omega_s\tau))^2}. \tag{102}$$

Let us explain the stability condition (82). When the delay is long enough that $(1 + A\cos\omega_s\tau) = 0$, then there is zero 'resistance' at the sidebands, and the damping effect of the cavity resonator on the beam dipole mode is lost. Thus we define 'long delay' by the relation $\omega_s\tau > \pi/2$.

For still longer delays, $(1 + A\cos\omega_s\tau) < 0$, the resistance at both sidebands becomes negative, even though the resistance is positive at the carrier frequency ω_{rf}. The resistance is given by the real part of Eq. (92), but of course the angle μ (introduced in Section 9.1) is negative if Ψ is positive. If $\mu < 0$ and $\omega_s\tau_c > A\sin(\omega_s\tau)$ then the difference of $\Re[Z]$ evaluated at upper and lower sidebands is negative and there is a Robinson-type dipole-mode instability. This explains condition (82).

9.2.1 How to Operate the Cavity

The requirement that the cavity-with-feedback should be stable (in the absence of beam) guarantees that the condition $\omega_s\tau_c > A\sin(\omega_s\tau)$ will always be fulfilled. For short delays $\omega_s\tau < \pi/2$ we should detune the cavity so $\Psi > 0$, so as to avoid the classic Robinson-type instability. For long delays, $\pi/2 < \omega_s\tau < \pi$, we should detune the cavity so $\Psi < 0$; that is the reverse of what is normally expected. In other words, when the delay is long, $\Psi > 0$ has become the condition to achieve the Robinson-type instability.

Of course, one must also consider the reactive compensation of the steady-state beam loading, which leads to the expression (64). The minimum reflected power condition, $\Psi_g = 0$, will usually prevent one from detuning the cavity in the opposite sense to normal, and so prevent one from operating with $\omega_s\tau > \pi/2$.

9.3 Feedback Mis-phased

In Section 9, we have so far assumed $\omega_{rf}\tau = m\pi$ and $\theta = 0$ so that the impedance is *not* pure real when $\omega_c = \Omega$. Sometimes, however, the delay is adjusted so that the impedance is pure real at the original resonance frequency, that is $\Omega\tau = m\pi$.

Let us write $w = \omega_{rf}\tau + \theta = m\pi + \vartheta$, where $\vartheta = (\omega_{rf} - \Omega)\tau$ and $\vartheta(-\omega_{rf}) = -\vartheta(+\omega_{rf})$. Then we can consider two cases: (i) $\theta = 0$, $\omega_{rf}\tau \neq m\pi$ and the impedance is real when $\omega_c = \Omega$; and (ii) $\omega_{rf}\tau = m\pi$ but there is an additional phase advance in the feedback of value $\theta = \vartheta$.

When the feedback is mis-phased, the impedance at the upper and lower sidebands is given by:

$$Z(+\omega_{\mathrm{rf}} \pm \omega) = \frac{R_s}{[1 + A\cos(+\vartheta \pm \omega\tau)] - \mathrm{j}[\tan\Psi + A\sin(+\vartheta \pm \omega\tau) \mp \omega\tau_c]}, \quad (103)$$

$$Z(-\omega_{\mathrm{rf}} \pm \omega) = \frac{R_s}{[1 + A\cos(-\vartheta \pm \omega\tau)] - \mathrm{j}[\tan\Psi + A\sin(-\vartheta \pm \omega\tau) \mp \omega\tau_c]}. \quad (104)$$

We define ϕ by the expression

$$\tan\phi = \frac{\tan\Psi + A\sin\vartheta}{1 + A\cos\vartheta}. \quad (105)$$

Then an approximate expression for the impedance is:

$$\begin{aligned} &Z(\omega_{\mathrm{rf}} \pm \omega)(1 + A\cos\vartheta)\sec^2\phi \\ &\approx 1 + \mathrm{j}\tan\phi + \frac{1}{1 + A\cos\vartheta}[-A(\cos(\vartheta \pm \omega\tau) - \cos\vartheta) \\ &\quad + \mathrm{j}(A\sin(\vartheta \pm \omega\tau) - A\sin\vartheta \mp \omega\tau_c)]\exp(+\mathrm{j}2\phi). \end{aligned} \quad (106)$$

9.3.1 Growth Rate

The growth rate is proportional to the difference of the resistive part of the impedance at upper and lower synchrotron sidebands:

$$R(\omega_{\mathrm{rf}} - \omega_s) - R(\omega_{\mathrm{rf}} + \omega_s) = -2R_s[\omega_s\tau_c \sin 2\phi - A\sin(\omega_s\tau)\sin(2\phi - \vartheta)]\frac{\cos^2\phi}{(1 + A\cos\vartheta)^2}. \quad (107)$$

Depending on the sign of ϑ, the system could become unstable before the threshold given by expression (94) is reached. Essentially, if $w \neq m\pi$ then one of the sidebands will find a reduced resistance before condition (94) occurs. Of course, it would be advantageous if this were the lower synchrotron sideband,¶ in which case the system could actually be

¶Below transition, the upper sideband damps and the lower sideband anti-damps.

more stable. By inspection, we desire $\sin(\vartheta - 2\phi) > 0$ or $\vartheta > 0$. Now, if the cavity is detuned to compensate the reactive beam loading below transition energy then $\tan\Psi > 0$ and $\Omega > \omega_{\mathrm{rf}}$. Thus $\vartheta = (\omega_{\mathrm{rf}} - \Omega)\tau \approx -(m\pi/2Q)\tan\Psi$ is negative, and the resistance at the upper (i.e. damping) sideband will reduce before that at the lower sideband, and so the limit (94) cannot be reached unless $\omega_{\mathrm{rf}}\tau = m\pi$ and $\vartheta = 0$.

9.3.2 *Coherent Frequency*

The coherent frequency is proportional to the sum of the reactive part of the impedance at upper and lower sidebands:

$$\frac{\omega^2 - \omega_s^2}{\omega_s^2} = -\frac{I_{\mathrm{ac}}^{\mathrm{b}} R_s}{V_{\mathrm{rf}}} \frac{\cos^2\phi}{(1 + A\cos\vartheta)} \times \left[\tan\phi + \frac{A \times (1 - \cos\omega\tau)\sin(2\phi - \vartheta)}{(1 + A\cos\vartheta)}\right]. \tag{108}$$

This equation predicts that the coherent oscillation frequency will approach zero when

$$\frac{I_{\mathrm{ac}}^{\mathrm{b}} R_s}{V_{\mathrm{rf}}} = \frac{2(1 + A\cos\vartheta)}{\sin(2\phi)}. \tag{109}$$

10 CONCLUSION

We have introduced a new and powerful method for determining whether the roots of the characteristic function (in powers of s and $e^{-s\tau}$) lie in the left or right half complex plane. The method has been applied to the stability analysis of a resonator with delayed feedback interacting with a particle beam, leading to analogues of the Robinson stability criteria. Though the criteria for a resonator with feedback are unchanged by the delay, they are supplemented by three new conditions (74), (82) and (87)which clearly depend on the delay.

By using the Sacherer formalism, we have explained a number of stability criteria that were previously derived by 'blindfold' application of the Nyquist criteria to a beam and cavity equipped with delayed feedback. It might be thought, that the Sacherer-type approach makes the previous work redundant, but we must remind the reader that without the earlier work we should not have known at which sideband frequencies to evaluate the impedances. Take, for example, the high-frequency coherent oscillation; without prior knowledge it would have been a true leap of the imagination to evaluate the impedances at the frequency $\omega_0 = \pi/(2\tau)$.

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