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A STRAIGHTFORWARD METHOD FOR CAVITY EXTERNAL Q COMPUTATION

PASCAL BALLEYGUIER*

Commissariat à l'Energie Atomique Département de Physique Théorique et Appliquée, B.P. 12, 91680 Bruyères-le-Châtel, France

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A method for external quality factor computation with a lossless cavity simulation code is derived from a theoretical analysis. The calculation is based on a single code run and the process is suitable for a very large scale of coupling, excepted for very low Q_{ext} values. The method has been validated experimentally on a simple case offering one degree of freedom which permitted to vary the Q_{ext} value over a six decade dynamic range. The error was less than 15% as Q_{ext} varies from 6 millions down to 12 units. It also yields very good agreement when applied to the existing ELSA photo-injector cavity.

Keywords: Cavity; Quality factor; External Q; Impedance matching

1 INTRODUCTION

Impedance matching of an accelerating cavity is important in order to maximize the power transfer from the line to the cavity and to maximize the accelerator efficiency. Codes exist to compute cavities in term of resonance frequencies, quality factors, shunt impedance, peak field level, etc. But computing the impedance matching (or mismatching) of a cavity fed by a line or a waveguide is a different problem. One can show that it is linked with the external quality factor of the cavity: matching occurs when the internal Q (including "losses" due to the beam loading) is equal to Q_{ext}.

* Tel.: +33 (0) 1 69 26 60 61. Fax: +33 (0) 1 69 26 70 24.
E-mail: balleyg@bruyeres.cea.fr.

In 1990, Kroll and Yu proposed a method for computing the external quality factor of a cavity coupled to a waveguide with codes capable to compute resonance frequencies of lossless cavities.[1] The principle is to get the frequency of the cavity coupled to a waveguide for different positions of the terminating short circuit, and to fit the resulting values to an analytical model of the system. As this method is based on frequency resonance differences, it becomes poorly accurate in case of high external Q. Practically, Q_{ext} values seem limited to a few hundreds which is too low for our purpose. In order to improve this limitation, we developed another method in which the Q_{ext} formula is based on a single code run.

Another method, called *current–voltage* has been used by Hartung and Haebel.[2] The authors derive formulas involving the computation of voltages and currents from a Thévenin equivalent circuit of the cavity. The Q_{ext} value is obtained from two quantities (Q_I and Q_V) resulting from different boundary conditions. This method is close to ours, though the calculated integrals are different.

2 DEFINITION OF EXTERNAL Q

Let us consider a lossless cavity coupled to an infinite line (Figure 1). If the cavity initially contains some energy W at its resonance frequency ω, this energy will gradually be driven out of the cavity. We assume that the coupling between the cavity and the line is not too high, so that we can consider the energy W as slowly decreasing with respect to the RF period. We also assume that only one mode can propagate along the line at ω. The power P transported by the

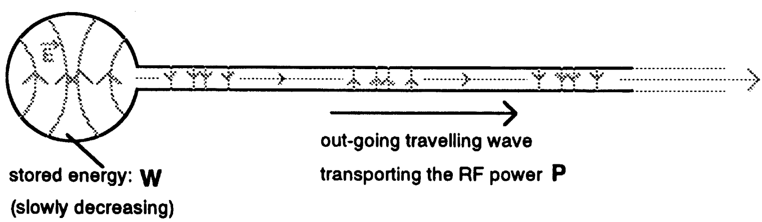

FIGURE 1 A lossless cavity slowly loosing its energy via an infinite line.

travelling wave is then also slowly varying, and the external quality factor coefficient is

$$Q_{\text{ext}} = \frac{\omega W}{P}. \tag{1}$$

The power transported by the travelling wave is given by

$$P = \iint \langle S \rangle \mathrm{d}s,$$

in which the Poynting vector (averaged over one RF period)

$$\langle S \rangle = \frac{|E||H|}{2}$$

is integrated over the cross section of the line. Here, E and H represent the electric and magnetic field phasors, respectively, the factor $\exp(\mathrm{j}\omega t)$ being dropped. Since electric and magnetic field amplitudes are proportional, one gets

$$P = \frac{1}{2\eta_{\text{mode}}} \iint_{\text{line} \times \text{sect}} |E|^2 \mathrm{d}s = \frac{\eta_{\text{mode}}}{2} \iint_{\text{line} \times \text{sect}} |H|^2 \mathrm{d}s,$$

in which η_{mode} is the impedance of the propagating mode, i.e. the ratio between electric and magnetic field amplitudes. The expression for the stored energy is

$$W = \frac{1}{2} \iiint_{\text{cavity}} \varepsilon |E|^2 \mathrm{d}v = \frac{1}{2} \iiint_{\text{cavity}} \mu |H|^2 \mathrm{d}v,$$

and, Eq. (1) results in

$$Q_{\text{ext}} = \frac{\omega \iiint_{\text{cav}} \varepsilon |E|^2 \mathrm{d}v}{(1/\eta_{\text{mode}}) \iint_{\text{line}} |E|^2 \mathrm{d}s}, \tag{2}$$

or

$$Q_{\text{ext}} = \frac{\omega \iiint_{\text{cav}} \mu |H|^2 \mathrm{d}v}{\eta_{\text{mode}} \iint_{\text{line}} |H|^2 \mathrm{d}s}. \tag{3}$$

Unfortunately, these formulas are not helpful in the present form for computing the external Q: a one-way travelling wave can exist either in an infinitely long structure or in a match-loaded line, which are both non-suitable for a finite-geometry and non-dissipative code.

3 FROM TRAVELLING TO STANDING WAVES

The situation described in the previous section is a solution of Maxwell's equations which are symmetric with respect to time. As a consequence, reversing the sign of time gives another possible solution (Figure 2). A wave transporting the power P (slowly increasing) at the same frequency ω, comes from the line and feeds the same lossless cavity. The energy W stored in the cavity is then slowly increasing.

According to the superposition principle, the two solutions mentioned above can be added in terms of electric and magnetic fields. Along the line, the two travelling waves interfere into a standing wave. Within the cavity, the result depends on the relative phases between the two original solutions.

Firstly, let us consider the case in which the two added fields are in phase within the cavity (we consider here either the electric or magnetic field). Then, the resulting field within the cavity is twice higher. Along the line, the maximum field amplitude is located at the antinodes and is twice the amplitude of each of the original travelling wave fields (Figure 3). One can interrupt the line at any antinode with the appropriate reflection condition, without changing the fields at the left-hand side of that reflection plane. Thus, the ratio between the cavity field and the field on the reflection plane is the same as in

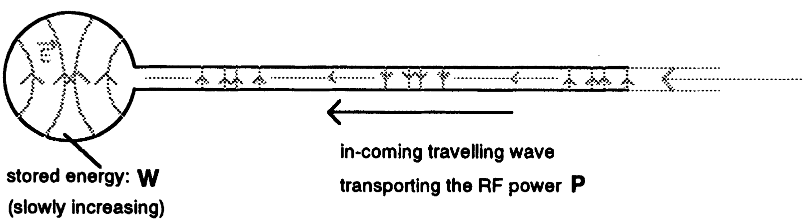

FIGURE 2 A lossless cavity slowly gaining energy from an infinite line.

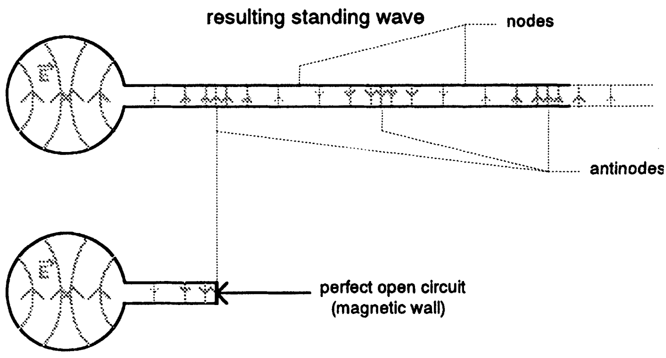

FIGURE 3 A lossless cavity coupled to a line with a standing wave: equivalent to a cavity coupled with a finite line with the convenient termination.

the case of a travelling wave (Figure 1). So, one can apply formulas (2) or (3) with the field resulting from superposition, the line cross section being located at the reflection plane. In case of an open circuit, the electric field yields there an antinode and formula (2) is adequate. In case of a short circuit, formula (3) should be used.

Thus, a lossless code can compute directly the external Q, provided that the line is ended by a convenient reflection condition at the appropriate location. The next section gives a practical way to check this condition.

4 CHECKING THE LINE LENGTH

Let us now consider the case in which the two cavity internal fields (see Figures 1 and 2) are in opposite phase (Figure 4). In a first approximation, the resulting field after addition vanishes within the cavity. (In a more careful analysis, the fact that the energy stored slowly decreases in the case of the out-going travelling wave and slowly increases in the in-coming case implies that the stored energy just passes through zero but is not constant.) Within the line, the nodes/antinodes locations are inverted with respect to the case of Section 3: the line itself becomes resonant at ω.

Inversely, for Q_{ext} computation, the line should be antiresonant at ω. A simple way to insure this condition is (for instance) to have a quarter guide-wavelength with the same type of termination at each

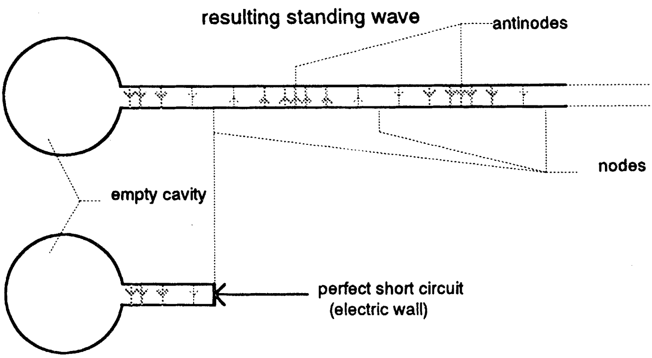

FIGURE 4 A lossless cavity coupled to a finite line ended with the opposite termination: the line resonates at the same frequency ω.

extremity. Approximately, the cavity entrance is seen from the line as an open-circuit (in case of electric antenna or radiating slot) or as a short-circuit (in case of magnetic loop). But that is only an approximation. Moreover, the precise location of the limit between the line and the cavity is rather arbitrary. So, the antiresonance condition should be checked out.

A simple way to verify the antiresonance condition is to simulate exactly the same geometry with just an inverted boundary condition at the line end, and to check the resonance frequencies of the system. If the line length is properly chosen, the global system (cavity plus line) behaves as two coupled resonators owning the same resonant frequency ω. The theory of low-coupled circuits says that the global system yields two resonance frequencies ω_0 and ω_π approximately verifying

$$\omega = \frac{\omega_0 + \omega_\pi}{2}.$$

If the line length is not correct (Figure 5), its own resonant frequency ω_1 differs from ω, and the global system frequencies tend to be ω and ω_1. Then, the average value $\bar{\omega}$ of the two frequencies obtained differs from ω and the line length should be corrected according to

$$\frac{\Delta l}{l} = \frac{\bar{\omega} - \omega}{\omega}. \tag{4}$$

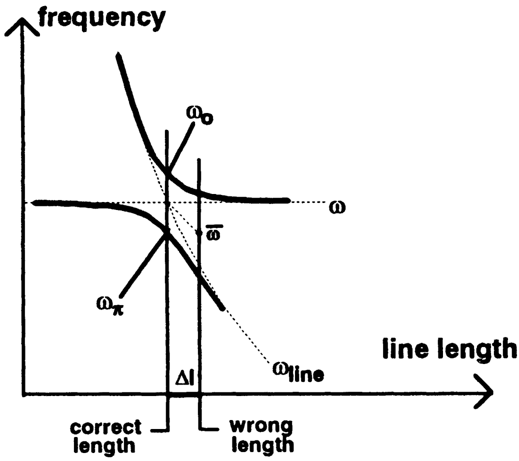

FIGURE 5 Resonance frequencies of a cavity coupled to a resonant line.

5 LINK WITH THE KROLL–YU METHOD

For more clarity, the link with the Kroll–Yu method will be established in a particular case, but the same argumentation could be applied to any other case. We consider here a cavity coupled electrically to a half a wavelength coaxial line terminated by a short circuit. We assume that both of cavity and line are under vacuum. In this case, Eq. (3) gives

$$Q_{\text{ext}} = \frac{\omega \iiint_{\text{cav}} |H|^2 \mathrm{d}v}{c \iint_{\text{line}} |H|^2 \mathrm{d}s}, \tag{5}$$

and the line is antiresonant at ω.

As the electric field vanishes on the short circuit, Eq. (5) can be written as

$$\frac{1}{Q_{\text{ext}}} = \frac{2c}{\omega^2} \frac{\omega \iint_{\text{line}} (\mu_0 |H|^2 - \varepsilon_0 |E|^2) \mathrm{d}s}{2 \iiint_{\text{cav}} \mu_0 |H|^2 \mathrm{d}v}. \tag{6}$$

According to Slater's perturbation theorem,[3] the right-hand term second factor of Eq. (6), is linked to the frequency variation associated with a small displacement $\mathrm{d}x$ of the short circuit

$$\frac{\mathrm{d}\omega}{\mathrm{d}x} = -\frac{\omega \iint_{\text{line}} (\mu_0 |H|^2 - \varepsilon_0 |E|^2)\mathrm{d}s}{2 \iiint_{\text{cav}} \mu_0 |H|^2 \,\mathrm{d}v}.$$

Equation (6) becomes then

$$Q_{\text{ext}} = -\frac{1}{2}\omega\frac{\mathrm{d}\psi}{\mathrm{d}\omega}, \tag{7}$$

in which the short circuit displacement is expressed in phase unit (i.e. $\mathrm{d}\psi = \mathrm{d}x\omega/c$), the antiresonance condition fixing $\psi = \pi$.

One can recognize in Eq. (7) the formula given by Kroll–Yu, valid when $-\mathrm{d}\psi/\mathrm{d}\omega$ reaches its maximum value, which is precisely true (in the present case) for a half wavelength line. We can conclude that our method is equivalent to a precise computation of the maximum slope of the ω versus ψ curve. The antiresonance condition checks that ψ is actually just in the middle between two consecutive crossing points between the cavity and the line resonance frequencies (Figure 6). Comparatively, the Kroll–Yu method is based on a fit of

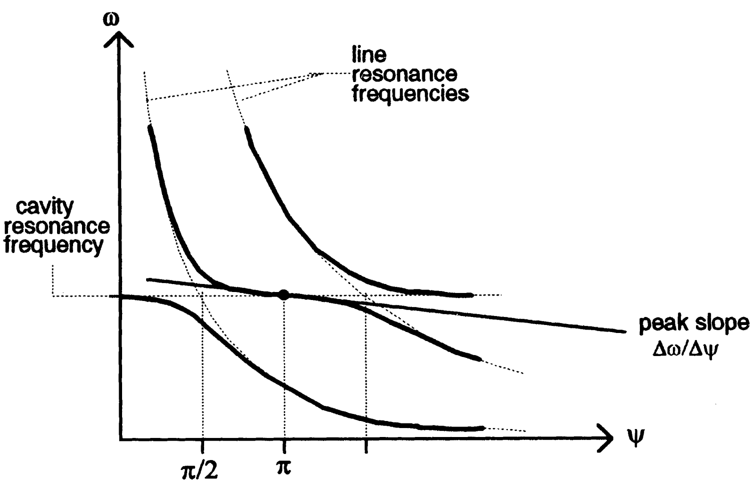

FIGURE 6 Typical resonance frequencies of the coupled system (cavity plus line) versus position of the short circuit termination.

one branch of the ω versus ψ curve. This explains why it is more precise in case of very low coupling ($Q_{ext} < 20$) for which all our previous approximations become irrelevant.

6 EXPERIMENTAL VALIDATION WITH A PILLBOX CAVITY

To validate the method, we used a simple pillbox cavity coupled to a semi-rigid coaxial cable on its axis (Figure 7). There were several reasons for this choice. First of all, the cavity was available: it had been used some years ago to calibrate the perturbing beads used for characterizing the ELSA cavities.[4] Secondly, as the antenna is constituted by the prominent central conductor of the semi-rigid cable, it is easy to vary its length. Thus, the external Q can vary over several orders of magnitude. Finally, as the geometry is completely axisymmetrical, a 2D code can be used, which makes the computations fast and efficient.

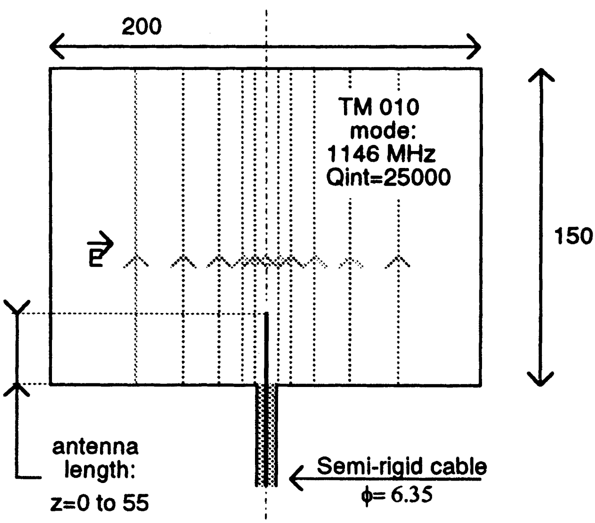

FIGURE 7 The pillbox cavity used for experimental validation of the method for Q_{ext} computing.

Another identical semi-rigid cable situated on the upper plate axis was used for transmission measurement. Its antenna length was fixed to zero (non-prominent) in order to minimize the induced perturbation, and this second antenna was not represented in the code simulation. We used a HP8753 network analyzer to measure the frequency response in reflection and transmission. We measured the global Q as the inverse of the 3 dB relative bandwidth in transmission, and the coupling coefficient β. This coupling coefficient is equal to the voltage standing wave ratio in case of over-coupling, or its inverse in case of under-coupling. The Q_{ext} was deduced by

$$Q_{\text{ext}} = Q(1 + 1/\beta).$$

The input antenna length was varied from 55 mm to 0 mm by cutting it step by step. The step was 5 mm above 20 mm length, and 2.5 mm below. The semi-rigid cable dielectric had a relative permittivity $\varepsilon_r = 2.0$, and its external/internal diameters were 5.35/1.65 mm, respectively.

We used MAFIA for code computations, with the same variations in antenna length. With about 18 000 mesh points and a non-uniform

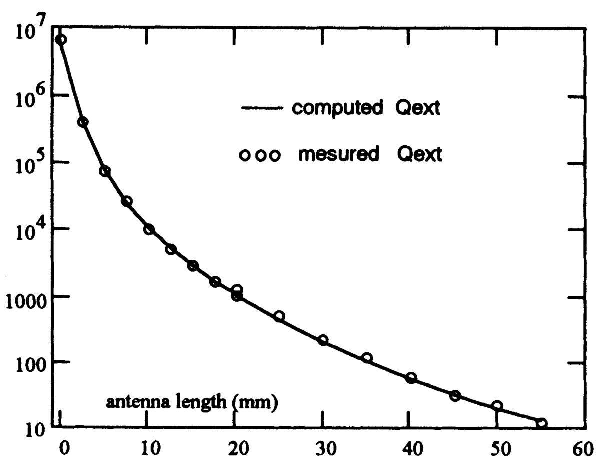

FIGURE 8 Experimental validation of the method for Q_{ext} computation in a pillbox cavity.

mesh, the resolution in the most critical area (the vicinity of the antenna) was below 0.4 mm. The line length was originally fixed to 92.5 mm ($\lambda/2$), but checking the antiresonance condition suggested to reduce it to 89 mm (for short antenna lengths: below 20 mm), down to 83 mm (for the longest antenna). Nevertheless, the antiresonance condition has not to be tightly respected. A 5% error in line length leads to a less than 1% error in Q_{ext} value.

The results (Figure 8) yields an excellent agreement between measured and computed values over six decades. The relative difference is typically between 5 and 10% (in any case less than 15%).

7 3D VALIDATION WITH THE ELSA PHOTO-INJECTOR

In order to check the method against a case which is more representative of accelerators (and in a 3D geometry), we applied it on the ELSA new photo-injector.[5] This 144 MHz cavity is coupled to a ∅230 mm coaxial line via an inductive loop (Figure 9). The internal Q and the coupling coefficient β had already been measured with a network analyzer for two orientations of the loop: $\beta = 1.5$ for $\theta = 27°$, and $\beta = 1.24$ for $\theta = 36°$, θ being the angle between the loop plane and the cavity axis. These values confirm the $\cos^2\theta$ theoretical dependence of β and permit to extrapolate the value $\beta = 1.89$ for $\theta = 0°$. With $Q_{int} = 38\,750$, we deduced the external Q: $Q_{ext}(\theta = 0°) = 20\,500$.

For MAFIA computations, the $\theta = 0°$ loop permits to reduce the computation volume to half a cavity, taking the symmetry with respect to the loop plane into account. Moreover, we considered a fictive horizontal symmetry plane, in order to compute just one-quarter of the cavity volume (Figure 10). The consequence is to simulate two injection loops instead of one. But, as these hypothetical loops are diametrically opposed, their direct interaction can be neglected, and the resulting Q_{ext} should be doubled to give the value for a single loop cavity. With 120 000 mesh points, the MAFIA raw result was: $Q_{ext} = 10\,270$, which is almost the expected value ($20\,500/2 = 10\,250$).

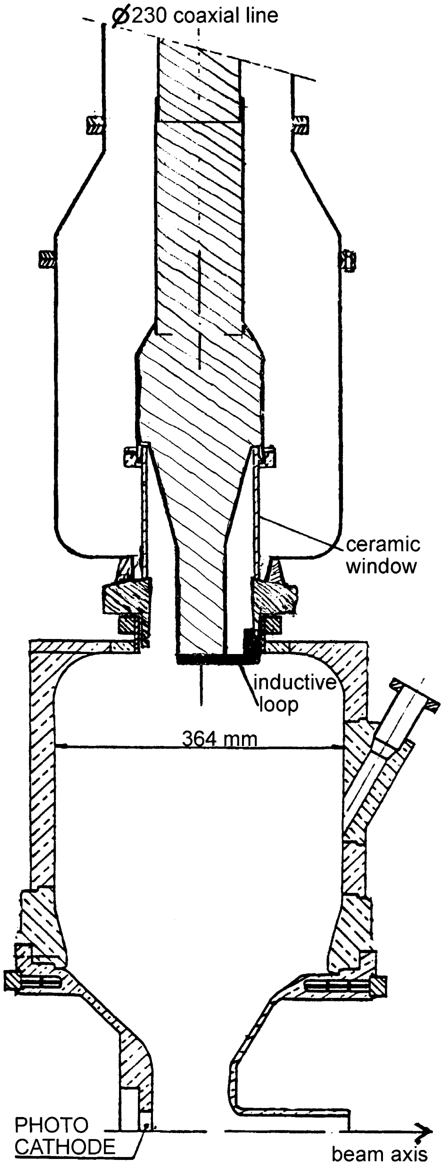

FIGURE 9 The RF power injection loop of the ELSA photo-injector cavity.

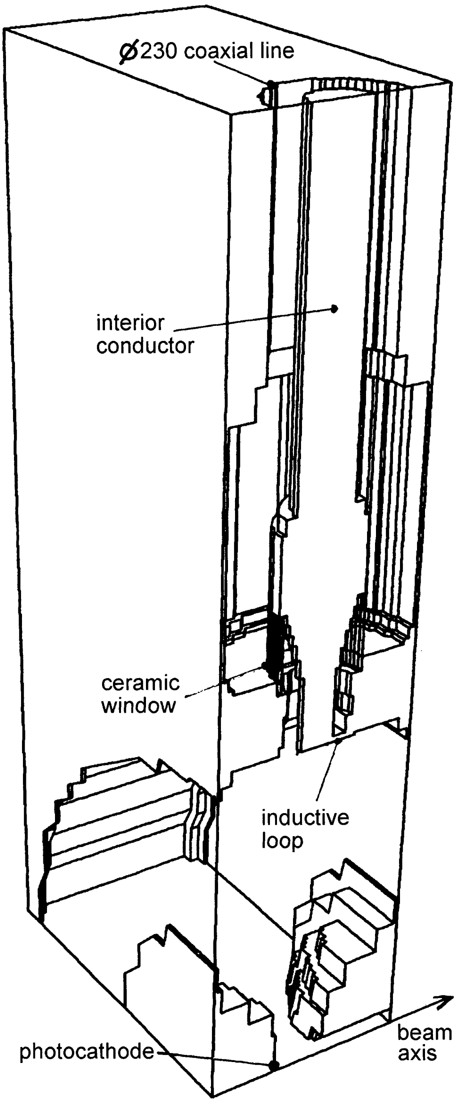

FIGURE 10 The ELSA photo-injector RF power injection loop as simulated by MAFIA.

8 CONCLUSION

In summary the method proposed to compute the external Q is:

- Enter the geometry of the cavity coupled to the line in the code. Let the line be a quarter guide-wavelength long (or multiple), and choose the end of line boundary condition for an antiresonance of the line.
- Check this by inverting the termination type (end of line boundary condition) and computing the two resonant frequencies. If needed, correct the line length according to Eq. (4).
- Run the code with the initial line termination and apply the following formula:

$$Q_{\text{ext}} = K \frac{\omega \iiint_{\text{cav}} |F|^2 \, \mathrm{d}v}{c \iint_{\text{line}} |F|^2 \, \mathrm{d}s}, \tag{8}$$

in which F is the magnetic or electric field, depending on the line termination (short circuit or open circuit, respectively), and K is set (for example) as follows:

- In the case of a cavity coupled to a coaxial line (or any TEM line) both under vacuum, $K = 1$.
- In case of dielectric isolator with a relative dielectric constant ε_{r} in the coaxial line, $K = \varepsilon_r^{\pm 1/2}$ ($+1/2$ or $-1/2$ exponent in case of short circuit or open circuit, respectively).
- In case of a TM mode waveguide, $K = [1 - (\omega/\omega_{\mathrm{c}})^2]^{-1/2}$, ω_{c} being the waveguide cutoff frequency.

This method for external Q computation is efficient and easy to use. Its main advantage is that the formula for Q_{ext} is straightforward and requires a single code run (after checking the line length). Since the results of the code are not used in a differential way, the calculated value is weakly sensitive to the mesh. It is then reliable even in case of very low coupling. On the other hand, the method supposes that the coupling is not too high, and is then not suitable for very low external Q values.

The good result of the validation process has increased our confidence in the method. Right now, it is used for studies of the RF power feeding ports on the future TRISPAL accelerating cavities.[6]

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References

[1] N. Kroll and D. Yu, Computer determination of the external Q and resonant frequency of waveguide loaded cavities, *Particle Accelerators* **34**, 231 (1990).
[2] W. Hartung and E. Haebel, In search of trapped modes in the single-cell cavity prototype for CESR-B, *1993 Particle Accelerator Conference*, Washington, p. 898.
[3] J. Slater, *Microwave Electronics* (D. Van Nostrand Company Inc., New Jersey, 1950).
[4] P. Balleyguier, Field distribution analysis in a multicell cavity: A new method for field equalization, *Nuclear Instruments and Methods A* **294**, 669 (1990).
[5] J.P. Debrion *et al.*, Photo-injector for a FEL and related problems, *Nuclear Instruments and Methods A* **340**, 214 (1994).
[6] P. Balleyguier, Coupling slots without shunt impedance drop, *1996 Linear Accelerator Conference*, Geneva, p. 414.