Particle Accelerators, 1997, Vol. 56, pp. 181–193
Reprints available directly from the publisher
Photocopying permitted by license only

SPACE CHARGE EFFECTS ON THE LONGITUDINAL OSCILLATION MODES OF ELECTRON-COOLED BUNCHED BEAMS

SERGEI S. NAGAITSEV[a], MICHAEL J. ELLISON,
TIMOTHY J.P. ELLISON[b], MARK S. BALL,
VLADIMIR DERENCHUK, GARY EAST,
BRETT J. HAMILTON and PETER SCHWANDT

The Indiana University Cyclotron Facility, 2401 Milo B. Sampson Lane, Bloomington, IN 47405 USA

(Received 29 May 1996; in final form 1 November 1996)

The longitudinal time structure and oscillation modes of very cold rf-bunched proton beams were experimentally studied in the IUCF electron cooling ring. Both longitudinal bunch density and quadrupole oscillation frequency measurements indicate a high degree of space charge dominance. The non-linear (sinusoidal) rf focusing field normally causes an incoherent synchrotron frequency spread leading to rapid decoherence of a coherent oscillation. In this regime of very cold beams, however, the small oscillations instead remain coherent and a coherent shift in the frequency of small amplitude dipole synchrotron oscillations is observed. This phenomenon of coherent space-charge tune shift has not previously been observed, nor predicted.

Keywords: Beam cooling; collective effects.

1 INTRODUCTION

An electron-cooling system[1] can reduce the longitudinal and transverse ion beam emittances, or temperatures, to extremely small values. In this regime the ion beam longitudinal self-fields, together with the external radio frequency (rf) focussing field, to first order determine the longitudinal charge

[a] Present address: Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA.

[b] Present address: Energy Conversion Devices, Inc., 1675 West Maple Rd., Troy, MI 48084 USA.

distribution of the stationary bunch.[2] The electric field from the beam space charge acts to cancel the externally-applied rf electric field for particles inside this distribution. In this space-charge dominated regime, the measured small amplitude synchrotron oscillation frequency of the bunch as a whole deviates from a single particle model prediction and is a function of beam intensity. We suggest that the non-linearity of the rf voltage, which would lead to a synchrotron frequency spread inside the bunch in the emittance dominated regime, leads instead to a coherent frequency shift in the space-charge dominated regime. To the authors' knowledge no complete theory describing this effect exists and therefore only a simple model, which fits the measured data extremely well, is presented here.

An existing theory for an arbitrary bunch distribution, developed by F. Sacherer[3] and applied in the past (see Ref. [4], for example), treats space-charge as a perturbation and uses the linear rf voltage approximation. Thus, no coherent dipole frequency shift is expected. In a particular case of a parabolic bunch density and linear rf voltage there exists a theory describing longitudinal bunch oscillations for any degree of space charge dominance.[5] One can show that in this case there exists a simple analytic expression for the quadrupole oscillation frequency:[6]

$$\left(\frac{f_q}{f_{so}}\right)^2 = \frac{V_{\text{eff}}}{V_{\text{rf}}} + 3 \tag{1}$$

where V_{rf} is the amplitude of the rf cavity voltage, V_{eff} is the effective rf voltage amplitude inside the bunch in the presence of space charge repulsion, and f_q is the bunch length (quadrupole) oscillation frequency. The symbol f_{so} is the small-amplitude, stationary, single-particle synchrotron frequency[7] given by:

$$f_{so} = f_o \sqrt{\frac{h|\eta|}{2\pi\beta^2} \frac{eV_{\text{rf}}}{E}}, \tag{2}$$

where f_o is the revolution frequency of the synchronous particle; h is the harmonic of the revolution frequency at which the beam is bunched; e is the proton charge; $E = \gamma Mc^2$ is the synchronous particle energy; γ and β are the usual relativistic parameters; and η is the phase slip factor $\equiv -(df_o/f_o)/(dp/p)$, where p is the synchronous particle momentum. Typical values for these and other parameters in the IUCF Cooler may be

TABLE I Typical values for beam and synchrotron parameters in the IUCF Cooler ring.

	Beam Parameters		
Parameter	*Symbol*	*Value*	*Units*
Kinetic energy	KE	45	MeV
Beam current	I	1–4,000	μA
Momentum spread (rms)	$\Delta p/p$	$(1\text{–}10)\cdot 10^{-5}$	
Revolution frequency	f_o	1.03168	MHz
Relativistic parameters	β	0.3	
	γ	1.05	
	Machine Parameters		
Circumference	$2\pi R$	86.7	m
Phase slip factor	η	–0.86	
rf voltage amplitude	V_{rf}	10–400	V
rf harmonic number	h	1	
Small amplitude synchrotron frequency	f_{so}	128–810	Hz

found in Table I. In the absence of space charge interaction ($V_{\text{eff}} = V_{\text{rf}}$) Equation (1) gives $f_q = 2f_{so}$, which simply corresponds to bunch rotation in longitudinal phase space; in the case of zero momentum spread ($V_{\text{eff}} = 0$) we have $f_q = \sqrt{3}f_{so}$. Thus a measurement of the bunch length (quadrupole) oscillation frequency provides a direct measurement of the degree of space charge dominance, or the degree to which the electric field from the beam space charge cancels the externally applied *linear* rf field within the bunch.

2 LONGITUDINAL BUNCH DISTRIBUTION

It is standard to describe the longitudinal dynamics of electron cooled beams by the Fokker-Planck equation (see Ref. [8] for example):

$$\frac{\partial\Psi}{\partial t} + \dot{\phi}\frac{\partial\Psi}{\partial\phi} + \dot{\delta}\frac{\partial\Psi}{\partial\delta} = \frac{\partial}{\partial\delta}\left(\lambda\delta\Psi + \frac{D}{2}\frac{\partial\Psi}{\partial\delta}\right), \tag{3}$$

where $\Psi(\phi, \delta, t)$ is the longitudinal distribution function, ϕ is the phase coordinate of the particle within the bunch with respect to the rf cavity voltage phase, δ is the relative momentum of the particle, t is the laboratory frame time, λ is the electron cooling rate, and D is the diffusion rate. The equations of motion for a particle experiencing synchrotron oscillations within the stationary bucket should be modified to include space-charge forces:

$$\dot{\phi} = 2\pi h\eta f_o\delta, \tag{4}$$

$$\dot{\delta} = f_o\frac{eV_{\mathrm{rf}}}{\beta^2 E}\sin(\phi) + \frac{Z_o g e^2 N f_o}{\gamma^2\beta^2 E}\frac{ch^2}{R}\frac{\partial\rho(\phi,t)}{\partial\phi}, \tag{5}$$

where c is the speed of light, Z_o is the impedance of free space ($\approx 377\ \Omega$), g is a geometrical factor [= ln(*pipe radius/beam radius*) + 1/2 for a perfectly conductive cylindrical vacuum chamber and a round, uniform density beam], R is the radius of the synchrotron ring, and N number of particles per bunch. The unitless longitudinal linear charge density, $\rho(\delta, t)$, is defined as

$$\rho(\phi,t) = \int_{-\infty}^{+\infty}\Psi(\phi,\delta,t)d\delta. \tag{6}$$

Both Ψ and ρ are normalized to unity.

For a stationary distribution, the time dependence of Ψ vanishes and the solution of Equation (3) can be written as:[9]

$$\Psi_o(\phi,\delta) = \frac{1}{(2\pi)^{1/2}\sigma}e^{-\delta^2/2\sigma^2}\rho_o(\phi), \tag{7}$$

where $\rho_o(\phi)$ is the stationary longitudinal linear charge density given by:

$$\rho_o(\phi)e^{-\alpha\rho_o(\phi)} = \rho_o(0)e^{-\alpha\rho_o(0)}\exp(\kappa[1-\cos(\phi)]). \tag{8}$$

The value of $\rho_o(0)$ must be chosen such that $\rho_o(\phi)$ is normalized to unity. The constants α and κ are given by the expressions:

$$\alpha = \frac{gh}{\gamma^2\beta^3\sigma^2\eta}\frac{e^2 Z_o N f_o}{E}; \quad \kappa = \frac{1}{2\pi\sigma^2\beta^2\eta h}\frac{eV_{\mathrm{rf}}}{E}. \tag{9}$$

For a given number of particles per bunch, N, Equation (8) contains two parameters, which are typically unknown in the experiment: σ and g. The first is a measure of the longitudinal beam temperature, the second is a measure of the potential energy (or longitudinal space-charge impedance). For vanishing impedance

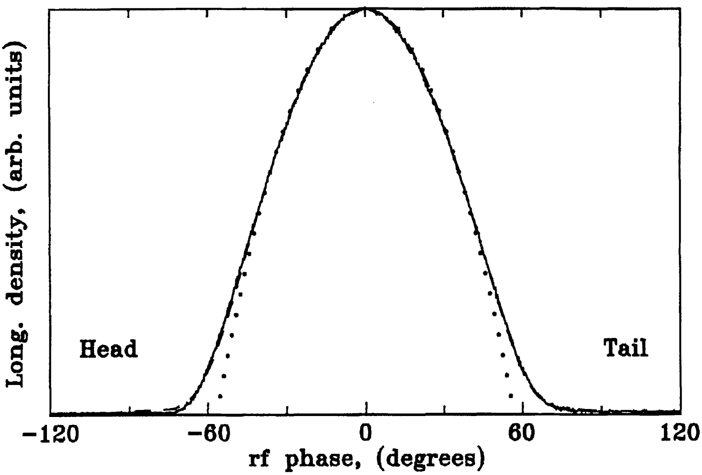

FIGURE 1 Measured (solid) and theoretical (dashed and dotted) longitudinal charge density for $I = 350\ \mu A$, $V_{rf} = 13.4$ V, $h = 1$. Note that the solid and dashed lines coincide almost perfectly.

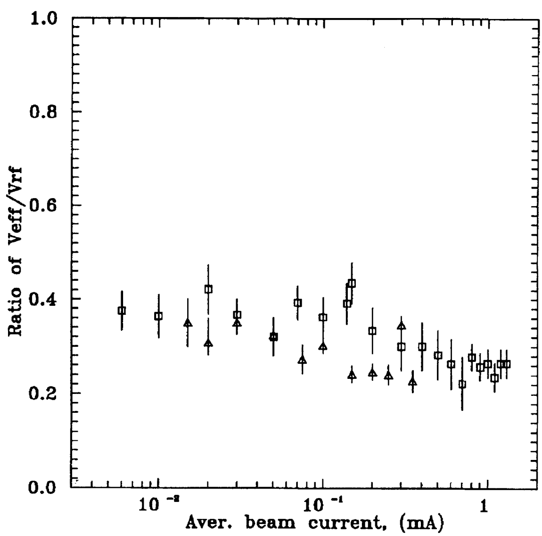

FIGURE 2 Ratio of the effective rf voltage to the applied rf voltage derived from the bunch shape fitting for $V_{rf} = 18$ V ($\triangle$) and $V_{rf} = 126$ V ($\square$).

($\alpha \to 0$) the linear density $\rho_o(\phi)$ is Gaussian-like. In the other limiting case of zero momentum spread ($\sigma \to 0$) the linear density becomes:

$$\rho_o(\phi) = \frac{\gamma^2\beta}{2\pi g h^2}\frac{V_{\mathrm{rf}}}{Z_o e N f_o}(\cos(\phi) - \cos(\phi_o)), \tag{10}$$

where $|\phi| \leq \phi_o$ and $2\phi_o$ is the bunch length, determined by normalization. The stationary distribution function, $\Psi_o(\phi, \delta)$, is then given by:

$$\Psi_o(\phi, \delta) = \rho_o(\phi)\delta(\delta), \tag{11}$$

where $\delta(\,)$ is a delta-function. Note that Equation (10) can also be trivially derived by merely equating rf and space charge forces in Equation (5).[2]

Measurements of the stationary longitudinal bunch profiles of an electron-cooled 45 MeV proton beam were performed in the IUCF Cooler ring as a function of the rf voltage and beam current. The bunch shape was measured using a broad bandwidth (0.002 to 200 MHz) longitudinal pick-up electrode and a 10-bit full scale, 1 GSample/sec digitizing oscilloscope. The data were transferred to a computer for offline analysis. The analysis consisted of fitting the experimental longitudinal bunch density to the theoretical prediction (Equation (8)) using two parameters (σ and g). This analysis is thoroughly described in Ref. [6], where the effective rf voltage within the bunch was shown to be only 20–30% of applied rf voltage. Figure 1 shows a comparison between the measured bunch distribution of a 45 MeV proton beam (solid line) with completely space charge dominated, or zero-momentum spread, beam density (dotted line), given by Equation (10) with $g = 3.1$ and $\sigma = 0$. Also shown in Figure 1 is a theoretical bunch density (dashed line), given by Equation (8) with $g = 2.4$ and $\sigma = 4 \times 10^{-5}$. In fact, by fitting the measured bunched density to the Equation (8) with finite momentum spread and to the Equation (10) with zero-momentum spread one can obtain the ratio of the rf effective voltage to the applied rf voltage:

$$\frac{V_{\mathrm{eff}}}{V_{\mathrm{rf}}} = 1 - \frac{g}{g_o}. \tag{12}$$

Here g_o is a value of g, obtained by fitting the bunch shape with $\sigma = 0$. In Figure 2 this ratio is shown as a function of proton beam current.

3 LONGITUDINAL OSCILLATION MODES

We now turn to the dynamics of space-charge dominated beams. It is obvious that the time-dependent Fokker-Planck equation (Equation (3)) does not have a simple analytic solution and, therefore, certain approximations are required to advance our studies into time-dependent domain. One such approximation is that for the most time-dependent phenomena (such as longitudinal bunch oscillations) the damping and diffusion in the RHS of Equation (3) can be neglected such that the Fokker-Planck equation reduces to the Vlasov equation. Quantitatively, this implies that the angular synchrotron frequency $\omega_{so} = 2\pi f_{so}$ should be much greater than $\lambda/2$, which is typically true in the IUCF Cooler for rf voltages $V_{\mathrm{rf}} > 5$ V.

A general solution of time dependent Vlasov equation, developed in Ref. [3], treats the space charge as a perturbation and uses the linear rf voltage approximation. Figure 2 shows, however, that the effective rf voltage inside the bunch of the well-cooled proton beam rarely exceeds 40% of the applied rf voltage. Thus, the space charge can no longer be treated as a perturbation and a model of a completely space-charge dominated (zero momentum spread) beam can be used instead. First, we will present the time dependent linear bunch density, $\rho(\phi, t)$, as a sum of the stationary density, $\rho_o(\phi)$, and a small time dependent perturbation $\rho_*(\phi, t)$:

$$\rho(\phi, t) = \rho_o(\phi) + \rho_*(\phi, t). \tag{13}$$

Limiting our consideration to completely space-charge dominated beams and with the help of Equation (10) and Equation (5) we obtain:

$$\dot{\delta} = \frac{Z_o g e^2 N f_o}{\gamma^2 \beta^2 E} \frac{c h^2}{R} \frac{\partial \rho_*(\phi, t)}{\partial \phi}. \tag{14}$$

In addition, using continuity equation and recalling Equation (4) one can write:

$$\frac{\partial \rho_*(\phi, t)}{\partial t} + h\eta\omega_o \frac{\partial}{\partial \phi}(\rho_o(\phi)\delta) = 0, \tag{15}$$

where $\omega_o = 2\pi f_o$. Taking another time derivative of Equation (14) and substituting in Equation (15) ($\rho_o(\phi)$ is the time independent stationary distribution given by Equation (10)) one obtains the following equation:

$$\frac{\partial^2 \rho_*}{\partial t^2} + \omega_o^2 \frac{h\eta e V_{\mathrm{rf}}}{2\pi\beta^2 E} \frac{\partial}{\partial\phi}\left(\frac{\partial\rho_*}{\partial\phi}(\cos(\phi) - \cos(\phi_o))\right) = 0. \tag{16}$$

Using a Fourier series expansion

$$\rho_*(\phi, t) = \sum_{n=1}^{\infty} \rho_n(\phi) e^{-i\omega_n t} \tag{17}$$

and recalling Equation (2) (η is negative below transition) we arrive to the final equation:

$$\frac{d}{d\phi}\left(\frac{d\rho_n}{d\phi}(\cos(\phi) - \cos(\phi_o))\right) + \left(\frac{\omega_n}{\omega_{so}}\right)^2 \rho_n = 0. \tag{18}$$

The boundary conditions for Equation (18) are such that the function $\rho_n(\phi)$ must remain finite at $\phi = \pm\phi_o$. In addition, the integral of $\rho_n(\phi)$ on the interval $[-\phi_o; +\phi_o]$ must be zero due to the conservation of total number of particles.

For small bunch length ($\phi_o \ll 1$) Equation (18) becomes the Legendre equation:

$$\frac{d}{d\chi}\left(\frac{d\rho_n(\chi)}{d\chi}(1-\chi^2)\right) + 2\left(\frac{\omega_n}{\omega_{so}}\right)^2 \rho_n(\chi) = 0, \tag{19}$$

where a new variable $\chi = \phi/\phi_o$ was introduced. On the interval $[-1; 1]$ it has a finite solution if and only if

$$\omega_n = \sqrt{\frac{n(n+1)}{2}}\omega_{so}, \tag{20}$$

where n is an integer greater than zero. Thus, the two lowest modes (dipole and quadrupole) are:

$$\begin{aligned} \omega_1 &= \omega_{so}, \\ \omega_2 &= \sqrt{3}\omega_{so}. \end{aligned} \tag{21}$$

These frequencies are exactly what one would expect for a completely space-charge dominated beam with a linear rf voltage (see Equation (1)).

As the bunch length increases, the frequencies ω_n become more complex functions of ϕ_o. Nevertheless, these frequencies can be found numerically by integrating Equation (18). An approximate value of the lowest order ($n = 1$) frequency can also be found using the method of Stodola and Vianello[10] — an iterative procedure, which can be useful for the approximate determination of the eigenvalues and eigenfunctions of a boundary-value problem. To apply this method we will first notice that one can obtain an equation for δ, similar to Equation (18), by taking a derivative over ϕ of Equation (15) and substituting in Equation (14):

$$\frac{d^2 f_n}{d\phi^2} + \left(\frac{\omega_n}{\omega_{so}}\right)^2 \frac{f_n}{\cos(\phi) - \cos(\phi_o)} = 0, \tag{22}$$

where $f_n = \delta_n(\cos(\phi) - \cos(\phi_o))$. This differential equation has the homogeneous boundary conditions ($f_n(-\phi_o) = f_n(\phi_o) = 0$), and thus the method of Stodola and Vianello can be directly applied. By choosing an initial approximation of $f_n \approx (\cos(\phi) - \cos(\phi_o))\cos(\phi)$ this method gives on the first iteration an approximate value for the dipole frequency ω_1:

$$\left(\frac{\omega_1}{\omega_{so}}\right)^2 \approx \frac{\int_{-\phi_o}^{+\phi_o}(\cos(\phi) - \cos(\phi_o))\cos(\phi)d\phi}{\int_{-\phi_o}^{+\phi_o}(\cos(\phi) - \cos(\phi_o))d\phi}. \tag{23}$$

The value of ω_1 given by this expression deviates by less than 1% from the numerically calculated frequency for $\phi_o \leq 90^\circ$.

4 MEASURED FREQUENCIES

Measurements of the longitudinal oscillation frequencies (dipole and quadrupole) of an electron-cooled 45 MeV proton beam were performed in the IUCF Cooler ring as a function of rf voltage and beam current. The synchrotron (dipole) oscillation frequency was measured using a phase detector.[11] The synchrotron oscillations were excited by nonadiabatically shifting the phase of the rf cavity voltage by a fixed value ($\approx 10^\circ$) in less than 40 μs — a small fraction of the synchrotron oscillation period. The beam phase information was stored in a computer together with a measurement of the bunch longitudinal profile just prior to exciting the oscillation.

The phase detector measures the phase of the first harmonic of the bunch charge density, and consequently the bunch centroid, provided that the longitudinal density stays symmetrical during the oscillation. Often small modulations in the phase oscillation amplitude can be observed due to admixture of other (nonsymmetrical) modes. Nevertheless, the frequency of phase oscillations can be accurately measured. We would also like to point out that in this experiment, after initial excitation, the synchrotron oscillations remained coherent for many synchrotron periods (typically 30–40). This allowed an accurate measurement of the oscillation frequency. "Snap-shots" of the longitudinal bunch profile during the synchrotron oscillations showed that no decoherence due to non-zero momentum spread was observed. In fact, an observed slow decrease in the amplitude of these oscillations was solely due to the electron cooling, which by itself introduces small intensity-independent synchrotron frequency shift much like a shift in damped pendulum oscillation frequency.

A comparison between measured synchrotron frequencies and Equation (23) is shown in Figure 3, where the FWHM of the bunch is used instead of ϕ_o because measured bunch shapes have short Gaussian-like tails (see Figure 1). Since the bunch length is, to first order, determined by the beam current, the synchrotron frequency is a function of beam intensity. This dependence of the synchrotron frequency on beam current has not been previously observed.

After considering various known mechanisms (e.g. beam loading, resistive impedance) which could lead to a dependence of the synchrotron frequency on the beam intensity we concluded that neither of these effects can be responsible for the magnitude of frequency change we observed. In fact, any rf voltage amplitude decrease due to beam loading is compensated by the automatic level control (ALC) whose bandwidth is 5 kHz, which is much higher than typical synchrotron frequency. Also, any longitudinal resistive impedance, which could lead to the observed frequency decrease, would have to be so large ($\approx$ 10 kΩ) that the bunch density would be significantly asymmetric.[12] The observed bunches (Figure 1), on the other hand, are symmetric for all attainable currents. We, therefore, suggest that we observed a previously unknown coherent space-charge tune shift. The mechanism of this tune shift can be described as follows: a high degree of space-charge compensation leads to a suppression of decoherence in the rf phase oscillations. Thus the bunch behaves as a single macroparticle and "sees" the rf voltage averaged over the bunch distribution. This averaged voltage can be written as:

$$V_{av} \sin(\alpha) = V_{\mathrm{rf}} \int_{-\phi_o+\alpha}^{\phi_o+\alpha} \rho_o(\phi - \alpha) \sin(\phi) d\phi, \tag{24}$$

where α is the rf phase displacement and V_{av} can be interpreted as an effective amplitude which would be seen by the bunch if all the particles were at α. For completely space-charge dominated beams this averaging can be performed analytically. The dipole oscillation frequency can be then written as:

$$\left(\frac{\omega_1}{\omega_{so}}\right)^2 = \frac{V_{av}}{V_{rf}} = \frac{\int_{-\phi_o}^{\phi_o} \rho_o(\phi) \sin(\phi + \alpha) d\phi}{\int_{-\phi_o}^{\phi_o} \rho_o(\phi) \sin(\alpha) d\phi}. \tag{25}$$

One can prove that this expression is identical to Equation (23) for the bunch densities given by Equation (10).

We would like to point out that the measured data in Figure 3 deviate from the theoretical value for longer bunches. There are a number of reasons for this discrepancy: firstly, the actual bunch is marginally longer than the ideal space-charge dominated bunch due to non-zero momentum spread, though both have the same FWHM. Secondly, Equation (25) assumes a rigid bunch. However, we have observed that the bunch "leans" in the direction of increasing $|\phi|$. All these effects cause a reduction in the effective rf voltage and consequently further reduce the dipole frequency.

Quadrupole oscillations were excited in this study by suddenly incrementing the rf voltage by a fixed value ($\approx$ 6 V, a roughly 50% increase). The frequency of these oscillations was measured by monitoring the power in one of the higher revolution frequency harmonics as a function of time. The harmonic number, typically 20–40, was chosen to be sensitive to the bunch length changes.

The ratio of the measured quadrupole frequency, ω_2, to the single particle synchrotron frequency ω_{so} is shown in Figure 4. Note that the value of quadrupole frequency is sensitive to the proton beam momentum spread. We have already shown that for the linear rf voltage model this frequency varies from $2\omega_{so}$ to $\sqrt{3}\omega_{so}$ depending on the degree of space-charge compensation (see Equation (1)). In the case of sinusoidal rf voltage one has to solve the Vlasov equation for the finite momentum spread beam. In principle, this would allow us to determine the momentum spread from the measured quadrupole frequencies. Such a treatment, to our knowledge, has yet to be developed.

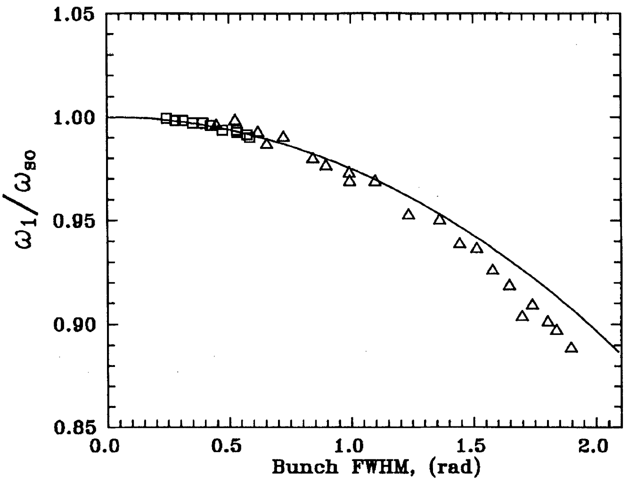

FIGURE 3 Measured synchrotron frequency as a function of bunch length (FWHM) for $V_{rf} = 18$ V ($\triangle$) and $V_{rf} = 126$ V ($\square$). The solid curve is the theoretical prediction of Equation (23).

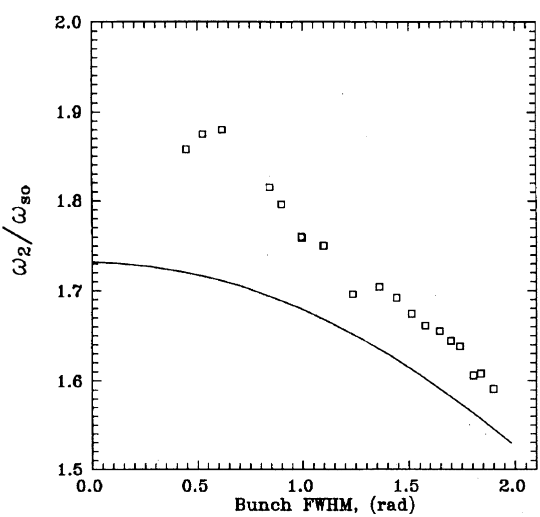

FIGURE 4 Measured quadrupole oscillation frequency ($V_{rf} = 18$ V). Solid line is the zero-momentum spread quadrupole frequency, numerically calculated from Equation (18) for $n = 2$.

5 CONCLUSION

The longitudinal time structure and oscillation modes of very cold rf-bunched proton beams were experimentally studied in the IUCF electron cooling ring. Both longitudinal bunch density and dipole frequency measurements indicate a high degree of space charge dominance. In this regime the nonlinearity of the rf focusing field (which would normally result in synchrotron frequency spread) leads to a coherent shift in the frequency of small amplitude dipole synchrotron oscillations. This phenomenon of coherent space-charge tune shift has not been previously observed.

This new form of beam, in which incoherent synchrotron and possibly betatron oscillations are suppressed, is of great interest for the future high brightness accelerators, where space charge effects could play an important role in limiting beam intensities and beam brightness.

Acknowledgements

This work was supported by the National Science Foundation (Grant No. NSF PHY 93-14783).

References

[1] Budker, G.I. (1967). *Atomnaya Energya*, **22**(5), pp. 346–348; Ellison, T., Friesel, D. and Brown, R. (1989). In *Proc. of the 1989 IEEE Particle Accelerator Conference*, pp. 633–635 (Chicago, IL, March 20–23).
[2] Ellison, T. *et al.* (1993). *Phys. Rev. Lett.*, **70**(6), pp. 790–793.
[3] Sacherer, F. (1972). CERN Int. Rep./SI-BR/72-5.
[4] Hansen, S. *et al.* (1975). In *IEEE Trans. Nucl. Sci.*, **NS-22**(3), pp. 1381–1384.
[5] Neuffer, D. (1980). *Particle Accelerators*, **11**, pp. 23–36.
[6] Nagaitsev, S., Ellison, T., Ellison, M. and Anderson, D. (1994). In *Proceedings of the 1993 Workshop on Beam Cooling (4–8 October 1993, Montreux, Switzerland)*, CERN Report No. 94-03, pp. 405–410.
[7] Veksler, V. (1944). *Comptes Rendus (Doklady), Acad. Sci. USSR*, **43**, 444; (1945). **44**, 393; Humphries, S. (1986). *Principles of Charged Particle Acceleration*, John Wiley & Sons, New York, p. 549.
[8] Ruggiero, A.G. (1994). In *Proceedings of the 1993 Workshop on Beam Cooling (4–8 October 1993, Montreux, Switzerland)*, CERN Report No. 94-03, p. 219.
[9] Chao, A.W. (1993). *Physics of Collective Beam Instabilities in High Energy Accelerators*, John Wiley & Sons, Inc., p. 287.
[10] Hildebrand, F.B. (1976). *Advanced Calculus for Applications*, Prentice-Hall, Inc., 2nd edition, p. 197.
[11] Ball, M., Ellison, T. and Hamilton, B. (1991). In *Proc. of the 1991 IEEE Part. Accel. Confer., 6–9 May 1991*, IEEE, New York, p. 1210–1212.
[12] Ruggiero, A.G. (1977). In *IEEE Trans. Nucl. Sci.*, **NS-24**(3), pp. 1205–1207.