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SUMMARY OF THE WORKING GROUP ON DYNAMIC APERTURE

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The state of the art on definition of dynamic aperture, analysis of nonlinear systems, indicators of weak instabilities, nonlinear diffusion, noise effects in nonlinear systems is reviewed and discussed and compared with experiments on particle accelerators, Most remarkable is that novel tools based on Taylor maps are now widely used for accelerator applications. The quantitative understanding of dynamic aperture by particle tracking has been further improved and discrepencies between calculations and observations are now well within a factor of two.

Keywords: Dynamic aperture.

1 OVERVIEW

The working group on dynamic aperture consisted of the following members: Y. Alexahin, R. Bartolini, J.R. Cary, J. Ellison, W. Fischer, O. Brüning, J. Gareyte, N. Gelfant, M. Giovanozzi, G. Hoffstätter, B. Holzer, J. Jowett, J. Lascar, C. Luettge, D. Robin, W. Scandale, T. Toyama, E. Todesco, and F. Willeke. The following topics were discussed.

- The evaluation of the dynamic aperture may be ambiguous if an over-simplified definition is used. The working group discussed various proposals for a satisfactory and unambiguous definition of the dynamic aperture.
- New tools for analysing nonlinear systems have been developed over the last years, the benefit of which is now becoming evident. The discussions were focused on the question to what extent it is possible to systematically design a large dynamic aperture by identifying and manipulating the most relevant parameters of nonlinear accelerator systems. Especially methods based on the analysis of truncated maps are very efficient and promises considerable progress in the understanding of nonlinear

problems in accelerators. One of the application discussed is magnet sorting to optimize the dynamic aperture.

- Early indicators of instability are supposed to speed up considerably the calculation of dynamic aperture by computer simulations. The dynamic aperture limit is given by a weak instability of the motion which is associated with chaotic behaviour. Thus, early indicators are indicators of chaotic orbits. In the last few years, the criterion of the nonvanishing Lyapunov exponent has been applied successfully to accelerators. More recently, the method of frequency map analysis has been proposed as a new powerful indicator of chaoticity. The different methods have been reviewed and compared in the discussion of the working group.
- A complementary tool to the early indication of instability is the estimate of survival times. Methods in use in the field of accelerator physics are based on Nekhoroshevs theorem.[1] Considerable progress was reported upon in the working group. The new method provides rigorous lower bound for the survival times.
- The concept of dynamic aperture is not sufficient to characterize the stability of orbits in accelerators. A weak instability which manifests itself by a very slow growth of oscillation amplitude is best characterized by a quasi-diffusion process. The quasi-diffusion as a consequence of chaotic orbits in nonlinear systems is in practice indistinguishable from the effect of external noise which may be enhanced by the nonlinearity. New approaches to describe diffusion and noise processes have been discussed.
- The question of how well the border of stability of a real accelerator can be predicted by a computer model is most relevant for the design of new accelerators. A series of experiments have been carried out at the CERN SPS to address this question. More recently, the dynamic aperture of the complex superconducting HERA proton ring has been studied systematically and has been compared with simulations. The discussion of the working group concentrated on these two cases.

2 DEFINITION OF THE DYNAMIC APERTURE

The dynamic aperture may be defined as the size of the amplitude of a particle which is close to the border of stability. The problem of measuring dynamic aperture in a computer simulation or in a beam dynamics experiment is the problem of measuring amplitude. To measure amplitude in multidimensional nonlinear dynamical systems in a practically feasible way is quite delicate.

Several ways of defining amplitude in the context of dynamic aperture evaluation have been discussed.

The initial distance $z - z_0$ of a particle from its central orbit, z_0, is only a sufficiently good measure of amplitude if the motion is approximately harmonic. This is usually not the case for particles near the border of stability. A complication arises also from the fact the motion takes place in three, weakly coupled oscillation planes. Particles may depart considerably from their initial value of z. If too simple a definition is used the result of dynamic aperture evaluation may depend on the initial conditions, or on the observation point in the lattice.

Another way to characterize amplitude which is often used is the maximum, minimum or average oscillation amplitude of the particle coordinates on successive turns. In this way, the dependence from the initial conditions and the location of the starting and of the observation point can be removed. However, these pragmatic approaches are in general unsatisfactory for judging on small improvements and subtle differences of accelerator lattices in the design stage.

The proper way to define amplitude is to use the Poincaré invariants of motion $I_i = \oint p_i dq_i$ where p_i and q_i are canonical variables. In the linear, uncoupled theory of particle dynamics in accelerators, these invariants are closely related to the Courant-Snyder emittance ε_i, namely $I_i = \frac{1}{2}(z_i^2\gamma_i + 2\alpha_i z_i p_i + p_i^2\beta_i) = \frac{1}{2}\varepsilon_i$. In the nonlinear case, the invariants may be written as $I_i = \oint d\phi_i \varepsilon(z_i, p_i)$. These invariants however cease to exist if the motion becomes chaotic and thus becomes unstable. Near unstable regions, in the vicinity of resonances, particle trajectories constitute a complex topology in phase space. Therefore it is difficult to evaluate the invariants from a sample of coordinate data obtained from tracking.

Several methods were discussed and compared. Some methods are based on averaging particle coordinates or carrying out the integral over the angle variable numerically by using the tracking data (see for example [3]). If the oscillation amplitude is just averaged, the amplitude values near the unstable fixed points are over-emphasized when the phase variables propagate only slowly. The attempt to use tracking data for numerical integration of the phase space volume suffers from the limited amount of tracking data, especially if motion with three degrees of freedom is considered. A new approach has been presented by M. Giovanozzi[2] which is based on normal forms analysis.[a] Even in the limit of a chaotic orbit, the normal form analysis provides a mean to evaluate the effective invariants. Even though the normal form analysis, which will in practice never be complete but be truncated, will not provide an exact measure of amplitude, it is expected to provide in most cases sufficiently accurate numbers. Giovanozzi has proposed to characterize amplitude by the corresponding four-dimensional phase space volume

$$A = \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} d\alpha d\phi_x d\phi_y r^4(\phi_x, \phi_y, \alpha) \sin(\alpha) \tag{1}$$

where $\sin(\alpha)$ determines the ratio between horizontal and vertical amplitude. In order to evaluate the dynamic aperture, the amplitude parameter r can be obtained as the radius of a hypersphere formed by the transformed tracking coordinates. The object which is so defined is only an approximate sphere depending on the accuracy of the normal form transformation. If $\mathcal{F}$ generates the normal form transformation of the original map $\mathcal{M}$ of the tracking

$$\mathcal{N} = \mathcal{F}\mathcal{M}\mathcal{F}^{-1} \tag{2}$$

($\mathcal{N}$ generates a rotation), then the dynamic aperture may be obtained by

$$A = \frac{4\pi^2}{N} \sum_{n=1}^{N} \sqrt{\sum_{i=1}^{2} (\mathcal{F}z_{ni})^2 + (\mathcal{F}p_{ni})^2}. \tag{3}$$

from coordinate data $z_i = x, y;\; p_i = p_x, p_y$. This measure of aperture may be considered as a rather unambiguous way to define and evaluate dynamic aperture. It can be extended in a straightforward way to higher degree of freedom. However, it is also not completely unproblematic. If resonances occur a low amplitudes, the normal form analysis does not converge. Then, the value of the phase space volume becomes arbitrary or, alternatively, the normal form transformation has to be truncated at low order. Thus resonances inside the dynamic apertures cannot be handled very well by this method.

[a] By transformation to normal form one understands a coordinate transformation of a nonlinear system. In the new coordinate system, the motion is described by a rotation in phase space with an amplitude dependent rotation angle. Exact normal form transformations however do not exist in general since the nonlinear accelerator systems are nonintegrable. If one speaks about normal forms, one means approximated normal forms which are truncated in some order of the coordinates.

In summary, there is no unique way to define dynamic aperture. In many cases, very simple definitions may serve the purpose. One has to carefully examine from case to case which definition to use. It is important to be alert for problems with over-simplified approaches.

3 DESIGN OF LARGE DYNAMIC APERTURE

In order to minimize the detrimental effects of nonlinear fields, systematically driven nonlinear resonances of low order and large amplitude dependence of the tunes are to be avoided in the design of an accelerator. This is — at least conceptually — relatively easy and many design recipies have been developed to achieve low resonance strength and minimum amplitude dependence of the tunes. If the result of such a procedure is not satisfactory any further optimization becomes unfortunately very complicated. However new tools have been developed recently which may provide considerable progress.

An approach to dramatically reduce the effect of nonlinear fields was proposed and presented by D. Cary.[4] Since the border of stability is produced by the chaotic break-down of regular trajectories and since chaotic behaviour is caused by nonlinear resonances, the dynamic aperture should be dramatically improved by systematic canceling of the resonance driving terms. The particle motion in the accelerator may be described by a map $\mathcal{M}$. The first step in this procedure is to have a fast and reliable way to detect resonances and to determine their strengths. This is accomplished by finding the periodic orbits $\vec{z}_{cl.o.} = \mathcal{M}^n \vec{z}_{cl.o.}$ which are associated with each resonance. There are powerful numerical tools available to find the fixed points of a given map. The strength of the corresponding resonance is then characterized by the trace of the Jacobian map $J_n = \partial \vec{z}_n / \partial \vec{z}_0$ of the periodic orbit. For a system with one degree of freedom (2 dimensional phase space), Cary proposes that the strength of a resonance may be characterized by the residue

$$R_n = (2 - \mathbf{tr}\{J_n\})/4 \tag{4}$$

which tells whether the nearby motion of the periodic orbit is linearly stable. R_n is related to the eigenvalue λ of the Jacobian or tangent map and for one degree of freedom is given by

$$\lambda_n = 1 - 2\left(R_n \pm 2\sqrt{R_n^2 + R_n}\right). \tag{5}$$

$R = 1$ may serve as a threshold value for chaos and instability. This criterion is easily and quickly calculated in systems with one degree of freedom. In order to cancel or reduce high order resonance driving terms, one needs high order nonlinear elements as correctors. Given a set of multipole lenses, the strength of the resonances R can in principle be reduced below threshold. Corrector strength may be optimized for example by using numerical minimizers. This procedure has been tested for betatron motion in one degree of freedom for a test lattice based on a cell of the ALS storage ring. In this case the dynamic aperture could be increased by a large factor by means of additional octupole and decapole lenses. The extension to higher dimensions, and thus to more realistic models of the accelerator is unfortunately not trivial. Already with two degrees of freedom, there is the conceptual problem of how to define the criterion for stability since for each resonance one has a whole set of periodic orbits. Last but not least there is an explosion of resonance driving terms in higher dimensional phase space. More work has to be done to demonstrate whether the proposed method can be useful.

The evaluation and the analysis of truncated accelerator maps is used as a design tool for the PEPII B-factory collider rings. This has been reported by Y. Yan and J. Irwin.[12,13] The combination of automated differentiation[14] (differential algebra) and the Lie algebra tools[18,19] has provided us with a powerful analysis of the effect of nonlinear field. An important example is the examination of the truncated map of an accelerator. Based on the work of Dragt and Finn[18] and using the tools which have been developed recently[19] this map can be written in the form

$$\mathcal{M} = \mathcal{R} \exp(:L:) \tag{6}$$

where $\mathcal{R}$ is a linear map. The exponential operator $\exp(:L:) = \sum_{n=1}^{\infty} 1/n! : L :^n$ describes the nonlinear part of the map where $:L:$ denotes a poisson bracket operator

$$\sum_i \frac{\partial}{\partial x_i}\frac{\partial L(x_i, p_i)}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial L(x_i, p_i)}{\partial x_i} \tag{7}$$

and L is a polynominal in the coordinates x_i, p_i. The coefficients of this polynominal characterize the degree of nonlinearity of the accelerator lattice. This becomes particularly apparent, if L is expressed in terms of angle and action variables $J_i = x_i^2 + p_i^2$ (using for example normalized accelerator coordinates $x \rightarrow x/\sqrt{\beta}$; $x' \rightarrow p = (x\alpha + x'\beta)/\sqrt{\beta}$). Then the coefficients are associated with the strength of nonlinear resonances and the strength of the amplitude dependence of the tunes. These coefficients may be compared for different lattice options and are helpful in the interpretation of tracking results. Poor stability of a lattice may be attributed to the magnitude of leading terms. By means of perturbation theory, each of the terms may be recognized as a result of the interference between nonlinear elements and linear focusing properties. In this way, the design can be systematically improved. This has been demonstrated in the PEPII design work. It should be pointed out, that this procedure is of course not a conceptional innovation. What is important is that as far as computing speed and computability of high order effects is concerned, the new tools are much superior to the "classical" methods such as evaluation of low order resonance driving terms by perturbation theory. These new tools thus allow a qualitative step forward in the understanding of nonlinear effects in accelerators. The discussion in the working group concluded, that the tools should be further exploited. It would be most useful if general rules and threshold values could be developed. Further work is encouraged.

A systematic improvement of accelerator lattices which are distorted by nonlinear field errors is magnet sorting. Sorting means that individual magnets, which carry the unavoidable nonlinear field imperfections, are installed in a sequence which minimizes the all-over nonlinear effect. Sorting consists of two problems: The first problem is to define a criterion for the sorting. The second problem is to define a sorting procedure which satisfies the quality criterion. For practical reasons, both, criterion and procedure must work for a successive magnet installation procedure as necessary for large accelerators.

The discussions in the working group was concerned mainly with the first part. It was based on an investigation by W. Scandale on sorting criteria for the LHC.[17] Three criteria have been proposed. They are based on the evaluation of truncated accelerator maps

$$z_i = \sum_n \sum_{j_1+..+j_m=n} F_{j_1 \ldots j_m} \Pi_{k=1}^{m} z_k^{j_k} \tag{8}$$

These are

- The norm of the map: $Q_1 = \sum_n A^n \sum_{\vec{j}} |F_{\vec{j}}$
- The norm of the tune: $Q_2 = \int d\phi(\Delta\nu_x^2(A,\phi) + \Delta\nu_y^2(A,\phi))$ where $\Delta\nu_{x,y}$ are the tune shift with amplitude
- The norm of the resonance: $Q_3 = \sum_n A^n \sum_m |h_{nm}|$ where h_{nm} are the resonance driving terms of resonances $m\nu_x + p\nu_y = q$, which are obtained by a normal form analysis.

All three criteria seem to correlate clearly with the dynamic aperture of a LHC model. The most reliable correlation is the one between the tune criterion and the dynamic aperture. The working group concluded that considerable improvement can be achieved for the dynamic aperture if sorting procedures as described are applied.

4 EARLY INDICATORS

The growth of betatron oscillation amplitude near the border of instability is very weak. This means that the time until a particle leaves the aperture of the accelerator is very long compared to the revolution time. Typically, the growth time is between 10^3 and 10^8 revolutions. This is a problem for numerical studies of stability by computer models. Even with present day computing capacity this remains a difficult problem.

Early indicators of instability are a tool to overcome this difficulty. Since single particle instability is always related to chaotic motion, early indicators are indicators of chaotic, quasi-stochastic motion.

The Lyapunov exponent

$$\sigma = \lim_{d_0 \to 0} \lim_{n \to \infty} \frac{1}{n} \log \frac{|\vec{d}_n|}{|\vec{d}_0|} \tag{9}$$

which characterizes the exponential divergence $\vec{d}_n = \vec{z}_{n1} - \vec{z}_{n2}$ of initially close (distance $\vec{d}_0$) trajectories in time (see for example [9]) is a well established tool in accelerator physics. There is quite some experience available from dynamic aperture calculation for HERA and for the SPS,[8] which demonstrate, that Lyapunov exponents are reliable early indicators with a tendency to somewhat pessimistic predictions since not all chaotic particles will be lost.

Recently, Laskar has proposed to consider changes of the tune as a criterion for chaotic motion. Since a variation of the action I is always correlated with a variation of the tune, $\Delta\nu(I) = \partial^2 H(I,\phi)/\partial I^2 \Delta I$ appropriate criteria can be evaluated by a frequency map. Laskar has demonstrated how to derive such a frequency map in a rigorous way[5] by inverting the relationship $\nu = \nu(I) \to I = I(\nu)$. Here, $\nu(I)$ is the amplitude dependent tune of a nonlinear system. For practical purposes, it is sufficient to detect and examine changes in the tune of a particle moving in an accelerator. A conceptionally simple procedure has been implemented and applied to ALS stability analysis by D. Robin.[6] The procedure consists in calculating the tune from tracking data over a certain number of turns at two different times. The use of sophisticated procedures to improve on precision of the tunes to be evaluated from a limited number of turns such as interpolation between peaks and filtering techniques is essential. Tune diffusion constants may be defined as

$$D_{x,y} = \frac{\nu_{x,y}(t_2) - \nu_{x,y}(t_1)}{t_2 - t_1} \tag{10}$$

where t_1, t_2 are two different times at which the tunes are evaluated. These parameters can be plotted as a function of the initial tune. The plot exhibits a fingerprint of the stability as a function of the tunes in the working diagram. One can clearly recognize the important resonances by enhanced tune diffusion constants. The tune diffusion parameters correspond very well to diffusion parameters in action space and to survival times. Areas in tunes space which have large tune drift coefficients also exhibit large excursions and diffusion in amplitude.

Todesco[7] has compared the predictive power of a tune diffusion criterion with one of the Lyapunov exponent. The question of interest is whether each of the two criteria can be automated by defining a threshold value. The threshold can be defined as value

$$\sigma(n) = \lim_{d_0 \to 0} \frac{1}{n} \log \frac{|\vec{d}_n|}{|\vec{d}_0|} \tag{11}$$

obtained for regular motion. In regular motion, there is only a linear divergence of trajectories in time caused by the amplitude dependent tune

$$\sigma(n) = \frac{1}{n} \log \left(\frac{\partial \nu(J)}{\partial J} \cdot J \cdot n \right). \tag{12}$$

This threshold-predict marks the transition from stable to unstable motion in a simulation study for a four dimensional Hénon map with remarkable precision. In this model, the threshold becomes clearly visible after approximately 10^3 turns. Only a few obviously chaotic particles survive a large number of turns (10^6 in the model used) and there are only a few particles whose chaotic behaviour does not become evident within 10^3 turns. Thus, the Lyapunov exponent can be considered as a rather precise, reasonably reliable early indicator of instability. Todesco verifies the earlier observed trend, that the method of Lyapunov exponent has a tendency to be too pessimistic.

If the variation of the tunes is considered one does not observe a sharp threshold value. A threshold value in tune diffusion

$$D_{x,y} = \frac{\nu_{x,y}(N_1) - \nu_{x,y}(N_2)}{N_2 - N_1} \tag{13}$$

marks a softer transition between stable and unstable motion. But the tune criterion already reaches its final predictive power after a surprisingly short number of turns, in the order of a few hundred.

This discussion may be concluded by the statement that both methods are useful early indicators which may be considered complementary.

5 LONG-TERM BOUNDS

Early indicators of instability are indicators of chaotic orbits. They provide pessimistic estimates of dynamic aperture. Since even weakly chaotic particles may survive sufficiently long in the accelerator, one would like to have an estimate of the survival times of particles near the border of stability. This is the purpose of evaluating long term bounds. The approaches used in accelerator physics are based on a scaling law proposed by Nekhoroshev[11]

$$\|p(t) - p(t = 0)\| < \varepsilon^b \quad \text{for} \quad \|t\| \le T \exp(\varepsilon^*/\varepsilon)^a \tag{14}$$

where a, b, T, ε^* are parameters to be determined for each dynamical system characterized by a Hamiltonian

$$H_0(J) + \varepsilon H(J, \phi); \;\; \varepsilon \ll 1 \tag{15}$$

(see also for example [10]). To make use of this scaling law, it may be interpreted in the following way. For a chaotic trajectory near the border of stability, there exists an approximate torus from which the particle coordinates deviate only slightly within a certain time T. Provided that within a domain of size ΔJ, one has found the largest excursion δJ of a trajectory from its approximate torus,the time the particle needs to leave the domain is at least $\Delta J/\delta J \cdot T$. Numerical approaches based on fitted phase space tori have been proposed to estimate long term bounds for particles traveling in an accelerator.[11] G. Hofstätter recently presented a rigorous estimate of long terms bounds[16] which was discussed in the working group. The method is based on normal form analysis and an interval arithmetic. The normal form analysis is used to determine the approximate torus. Consider a mapping $z_{n+1} = \mathcal{M} z_n$ with the normal form $\mathcal{N} = \mathcal{F}^{-1}\mathcal{M}\mathcal{F}$ and with approximate tori $\vec{z}_{\text{appr}} = \mathcal{F}\vec{r}$. $\vec{r}$ is the radius vector of a hypersphere in normalized phase space. The excursions from the torus are then simply found by the 'one-turn-mapping'

$$\delta r = |\mathcal{F}^{-1}\mathcal{M}\vec{z}| - \mathcal{F}^{-1}\vec{z}|. \tag{16}$$

In order to provide a rigorous bound, one has to make sure, that one has found the largest excursion from all tori in a given domain $\{\vec{z}, \vec{z} + \Delta\vec{z}\}$. This is achieved by introducing an interval arithmetic in the same spirit as automated differentiation or differential algebra.[14] All numerical operations are carried out as vector operations to propagate intervals in place of numbers. Each observable obtains an extension with the values of the mimimum $\underline{I}$ and maximum values $\bar{I}$ of an interval $\{\underline{I}, \bar{I}\}$ in this variable. The connections between these new variables are defined by:

$$\begin{aligned} I &= \{\underline{I}, \bar{I}\} \\ I + J &= \{\underline{I} + \underline{J}, \bar{I} + \bar{J}\} \\ I - J &= \{\underline{I} - \bar{J}, \bar{I} - \underline{J}\} \\ I \cdot J &= \{\min(\underline{I}\cdot\underline{J}, \bar{I}\cdot\underline{J}, \underline{I}\cdot\bar{J}, \bar{I}\cdot\bar{J}), \max(\underline{I}\cdot\underline{J}, \bar{I}\cdot\underline{J}, \underline{I}\cdot\bar{J}, \bar{I}\cdot\bar{J})\} \end{aligned} \tag{17}$$

Using these tools, the largest excursion from the approximate torus is given by

$$\begin{aligned} \delta r &= \{\underline{r}, \bar{r}\} \\ &= \{((|\mathcal{F}^{-}1\mathcal{M} - 1)\vec{z}| - |\mathcal{F}^{-}1\vec{z}|)_{\min}, ((|\mathcal{F}^{-}1\mathcal{M} - 1)\vec{z}| - |\mathcal{F}^{-}1\vec{z}|)_{\max}\} \end{aligned} \tag{18}$$

The benefit of this procedure for practical accelerator design work was discussed. It is obvious, that if a course mesh is chosen to partition phase space into intervals the outcome of the stability analysis will be pessimistic. If the phase space is segmented into finer domains, the analysis will become more realistic. Unfortunately, in large accelerators, the transition between quick losses and metastability is rather sharp, which means that the segmentation has to be rather fine. On the other hand, the method includes the effect of any weak instability in phase space regions which are well within the dynamic aperture. Thus any tiny unstable region can be detected which is not possible by any other method. On the other hand, such regions will always determine the result of the stability analysis, even if they are tiny and unimportant. The conclusion is, that the benefit of the method for stability analysis must be tested for a large accelerator. Long term bound studies for the HERA proton ring have been suggested.

6 EFFECT OF NOISE AND DIFFUSION

Beam losses due to limited dynamic aperture are rather fast losses. They occur typically within a minute or within the first 10^6 turns after the beam has been injected. However, there are still particle losses after much larger times. In the HERA proton ring for example, beam losses after injection extend over hours. The initial beam lifetime $\tau = \frac{1}{N}\frac{\partial N}{\partial t}$ is usually about 15 min. After one minute, τ is about 1 h, after 5 minutes $\tau \simeq 3$ h and after 60 min $\tau \simeq 10$ h. The lifetime expected from scattering at the rest gas in the vacuum vessel is (20–100)h.

It is not useful, to attribute such slow beam losses to a limited dynamic aperture. These losses are the result of several effects. There is scattering at the rest gas or noise in the rf system transferred to the beam. These lead to emittance growth and beam losses also in perfectly linear systems. In a nonlinear system, the scattering amplitudes may be enhanced by the distorted topology of phase space, especially near resonances. Furthermore, there is some slow growth of particle amplitude due to small chaotic regions in phase space which are created by tune modulation.

Some aspects of these phenomena have been discussed in the working group. A. Bazzani[22] presented an analysis of stochastic perturbation of a nonlinear map which he applied to a model of the SPS with strong sextupoles. In the case of weak noise, it has been demonstrated (see for example the contribution of Ellison to these proceedings[23]) that it is justified to average over the betatron phases. Then the diffusion is only considered in the action variable. (An effective approximate action variable is used since nonintegrable systems are considered.) The dynamic aperture is considered as an absorbing boundary. The corresponding Fokker-Plank equation has to be solved numerically. With this model, good agreement between the simulated and analytically calculated time dependent distribution function for the SPS model with strong sextupoles could be achieved.

T. Sen presented a numerical study of noise effects in HERA beam-beam interaction.[25] The HERA proton beam suffers from noise transmitted by the electron- (positron-) beam via the beam-beam interaction. This leads to fluctuation of the tune and fluctuation of the nonlinearity due to beam size fluctuation of the electron beam. There is also a stochastic beam-beam offset which leads to coherent excitations. It turns out, that the direct effect of beam size fluctuation, which changes the nonlinearity of the system in a stochastic manner, is the most detrimental effect for the HERA proton beam.

Some new results were also reported on the effect of tune modulation. The dependence of proton beam losses in HERA from tune modulation of a depth of 10^{-4} and modulation frequencies of up to 1.2 kHz is theoretically well understood and has been experimentally verified at the HERA proton ring.[20] In the HERA proton ring, tune modulation compensation is meanwhile a well established method to reduce proton beam losses in colliding beam operation. Brüning has summarized the continuous effort at HERA to gain control of beam losses.

The classical method of evaluating the diffusion due to tune modulation has been proposed by Chirikov.[26] The motion of a particle near a resonance is strongly distorted by nearby resonances. The size of the distortion of the orbit near the separatrix can be evaluated. The diffusion constant can be computed under the assumption that the phase of the distortion is random since the motion is chaotic. A. Bazzani presented a statistical approach to calculate the diffusion coefficients which seems to agree well with simulations.[24]

7 DYNAMIC APERTURE EXPERIMENTS

How reliable are the predictions of dynamic aperture by numerical simulation? This is one of the crucial questions in the design of a new accelerator. This is the motivation behind a series of experiments which has been carried out in the SPS accelerator at CERN which extended over more than ten years. The result of this large effort is that the dynamic aperture of the SPS with eight strong sextupoles can be predicted to within a few percent by simulations, if the effect of power supply ripples is correctly modeled.

The SPS with strong sextupoles and artificial strong tune modulation can be modeled quite well since all the relevant parameters are known with great precision. Much less certain is the knowledge of the relevant parameters of the HERA proton ring with 416 superconducting dipole magnets with strong but slowly decaying nonlinear field distortions from persistent currents. Although, a large effort has been made to measure the relevant field errors and their time behaviour, the exact distribution of nonlinearities around the machine can never be completely certain. This is due to the dependence of persistent currents on even slight changes in the operation procedures. Therefore it was a very interesting test of the computer models to compare the measured dynamic aperture of HERA with simulations. A series of measurements were performed which have been presented during the workshop.[21] The result is that the dynamic aperture of HERA has been overestimated by 30% by the simulations. Considering the uncertainties, this can be considered as a good success for the tracking procedures.

8 CONCLUSION

The presentations and discussions in the working groups demonstrated that considerable progress has been made in the understanding of dynamic aperture. A few years ago only a few experts developed and used the methods of maps which are now widely used. The benefit of these new tools is becoming evident. Considerable progress has also been made on the important topics of early indicators and long term bounds. They must be integrated now in the design procedures. The long term effort of refining the tracking models by carefully describing the magnetic field errors and including more subtle effects like tune modulation has paid-off meanwhile. The result of dynamic aperture experiments differes from the predictions by tracking calculations only by a 10%–30%. This provides confidence for specifying magnet field quality for future accelerators.

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