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MEASURING HAMILTONIAN COEFFICIENTS WITH A WOBBLING METHOD

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We propose a method to determine the coefficients of the hamiltonian that represents the cumulative effect of all sextupoles in a circular accelerator. The method is based on the low-frequency sinusoidal excitation of orbit corrector magnets and detecting BPM signals at mixed harmonics of the exciting signals. The method is tested using a realistic model of LEP and the influence of BPM errors is investigated.

Keywords: Dynamic aperture; particle dynamics; lattices.

1 INTRODUCTION

In a recent report[1] we discussed a method that allows to determine both phases of the sum and difference coupling resonances non-invasively by exciting orbit correction magnets and observing the response of the beam at selected beam position monitors (BPM) at the exciting frequencies. The beam line used in Ref. 1 was purely linear which entails that only the exciting signals are visible in the BPMs. In the presence of non-linear elements such as sextupoles, octupoles, and decapoles this will no longer be the case. The non-linearities will produce a mixture of the exciting frequencies such as twice or thrice the exciting frequency and sums of different frequencies. We will develop an algorithm that reconstructs the hamiltonian from observed frequencies at a given set of BPM and then diagnose aberrations which are generated by the cumulative effect of all sextupoles in the machine. The wobbling frequencies are far away from the tune and we thus perform measurements which are non-resonant with the beam's betatronic motion. Therefore we are not limited to the measurement of aberrations which cause resonances that are close to that of the beam such as discussed in Ref. 2. Since the frequency response of the system is very flat, all aberrations are detected with similar accuracy, whether they are resonant with the beam or not.

In order to describe the non-linearities in the beam line we employ a hamiltonian framework and assume that we have a linear beam line in which point-like non-linear elements are placed. Each non-linearity is represented by its hamiltonian, e.g. an upright sextupole is described by $H_3 = k_2 L(x^3 - 3xy^2)/6$ which describes the kick caused by that element with the aid of the Poisson-bracket.[3] The hamiltonian of each non-linear element is then mapped to the end of the beam line (the reference point). This procedure only requires the linear map between the position of the element and the reference point. Having accumulated all elements at the reference point we concatenate them using the Campbell-Baker-Haussdorff (CBH) formula[3] and obtain the following representation of the map $\mathcal{M}$ through the beam line

$$\mathcal{M} = e^{:-H:} R \tag{1}$$

where R is the linear transfer matrix through the beam line and H is a polynomial in the variable (x, x', y, y') that describes the cumulative effect of all non-linear elements. In what follows we assume that the reference point is chosen to be in normalized phase space, i.e. to be at $\beta_x = \beta_y = 1\,\mathrm{m}$ and $\alpha_x = \alpha_y = 0$. In that case R is the direct sum of two rotation matrices. In the presence of sextupolar aberrations only H is given by $H = h_1 x^3 + h_2 x^2 x' + h_3 x^2 y + h_4 x^2 y' + h_5 x x'^2 + h_6 x x' y + h_7 x x' y' + h_8 x y^2 + h_9 x y y' + h_{10} x y'^2 + h_{11} x'^3 + h_{12} x'^2 y + h_{13} x'^2 y' + h_{14} x' y^2 + h_{15} x' y y' + h_{16} x' y'^2 + h_{17} y^3 + h_{18} y^2 y' + h_{19} y y'^2 + h_{20} y'^3$ with the implied ordering of the monomials. In the remainder of this report we will show that all h_j can be determined experimentally with good accuracy. Moreover, the h_j can be converted to physically more accessible parameters that describe the strengths of resonances.[4]

2 DIAGNOSTIC SETUP AND FILTERING

In order to measure the 20 coefficients of the hamiltonian we need a sufficiently large number of frequencies to observe. In the simulation used in this note we use two horizontal and two vertical orbit correctors, all oscillating at different frequencies. The frequencies were chosen such that the smallest difference between two mixed frequencies is maximum. Fixing one frequency then determines the other three. We choose the following four $f_1 = 0.0073722\, f_0$, $f_2 = 0.0131550\, f_0$, $f_3 = 0.0161366\, f_0$, and $f_4 = 0.0175514\, f_0$ where f_0 is the revolution frequency which is 11 245 kHz in LEP or LHC. We then choose two horizontal and two vertical BPM which are used to detect the response of the beam. The BPM need to record the position of the beam on 32 768 consecutive turns which may require special hardware. For simplicity the wobbling correctors, the BPMs and the reference point need to lie in a section of beam line that contains no non-linear elements.

The four wobbling frequencies $f_1, \ldots, f_4$ can mix to yield the following 17 different frequencies: 0, $2f_1$, $f_1 + f_2$, $f_1 - f_2$, $f_1 + f_3$, $f_1 - f_3$, $f_1 + f_4$, $f_1 - f_4$, $2f_2$, $f_2 + f_3$, $f_2 - f_3$, $f_2 + f_4$, $f_2 - f_4$, $2f_3$, $f_3 + f_4$, $f_3 - f_4$, $2f_4$. Observing all frequencies at all BPM we obtain 68 different observables of which we use all, except the constant term for a total of 64 observables, which turn out to be sufficient to determine the 20 terms in the hamiltonian.

The order of magnitude of the signals can be easily assessed by the following analysis. Assume that the reference point is in normalized phase space and that the BPM and orbit correctors are also situated at the same location. Then the normalized kick effected to the beam is $\sqrt{\beta}\varepsilon$, where ε is the kick and the mixing signal is proportional to the square of that, with the coefficient h of the hamiltonian being part of the proportionality constant. The observed BPM signal, transformed into normalized phase space, is given by $x/\sqrt{\beta}$ such that we obtain the approximate relation

$$\frac{x}{\sqrt{\beta}} \approx h\beta\varepsilon^2 \tag{2}$$

yielding the simple relation $x \approx h\beta^{3/2}\varepsilon^2$. Inserting typical values for LEP $\beta = 100\,\text{m}$, $h = 10/\sqrt{\text{m}}$ and $\varepsilon = 10\,\mu\text{rad}$ we obtain $x \approx 1\,\mu\text{m}$. We must remember that this is the amplitude of a harmonic oscillation with a known frequency and can thus be measured with high accuracy.

In the simulations reported in this paper we use a detuned LEP optics at high energy labelled K21P46 which contains about 500 sextupoles. The coefficients of the hamiltonian for this beam line are shown in the top left of Figure 2. The correctors and BPM were chosen from the RF section of IR 1 such that the phase advances between respective elements are

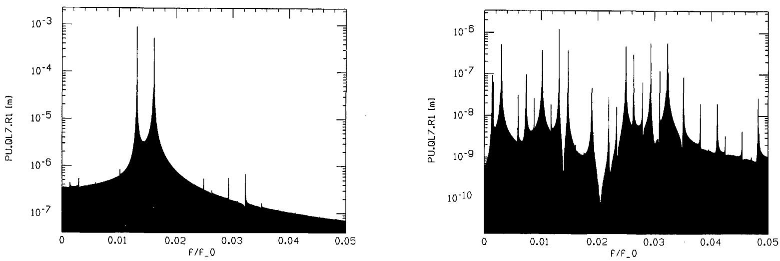

FIGURE 1 The beam's spectrum at position monitor PU.QL7.R1, before and after filtering.

around 90 degrees and the corresponding beta functions are maximum. The correctors are then excited with an amplitude of 10 μrad which generate orbit deviations of ± 1 mm in the arcs.

The left of Figure 1 shows the Fast-Fourier-Transform (FFT) of position of the beam at the horizontal position monitor labelled PU.QL7.R1. We clearly observe the exciting frequencies as large peaks with amplitudes on the order of 1 mm, but also other frequencies which come from the mixing due to sextupoles. In a separate run we turned off all sextupoles and the secondary peaks vanished. We observe that the width of the large peaks swamp the secondary signals which are two to four orders of magnitude weaker than the primary ones. The width of the primary peaks is a consequence of the finite number of turns that the beam position is sampled. We know, however, the exact frequency of the primary signals and can thus construct a notch-filter to remove them from the raw data. We proceed as follows: First we generate time series (sine-like and cosine-like) of the same length as the raw data, namely 32 768 turns, which contain only the primary frequency and Fourier-Transform them. The resulting spectra are then fitted to the data points near the primary peaks (normally 9 frequency bins for the sine- and cosine-like transform) which yield the amplitudes of both phases. The primary frequencies with the proper amplitudes are then removed from the raw data and the resulting time series is Fourier-Transformed which leads to the filtered spectrum shown in Figure 1. There the primary frequencies are removed and the secondary frequencies with sub-micron amplitudes are clearly visible, consistent with the above estimate.

An interesting point to note is that the number of peaks in Figure 1 is larger than that of the secondary frequencies. Most of the observed peaks can be

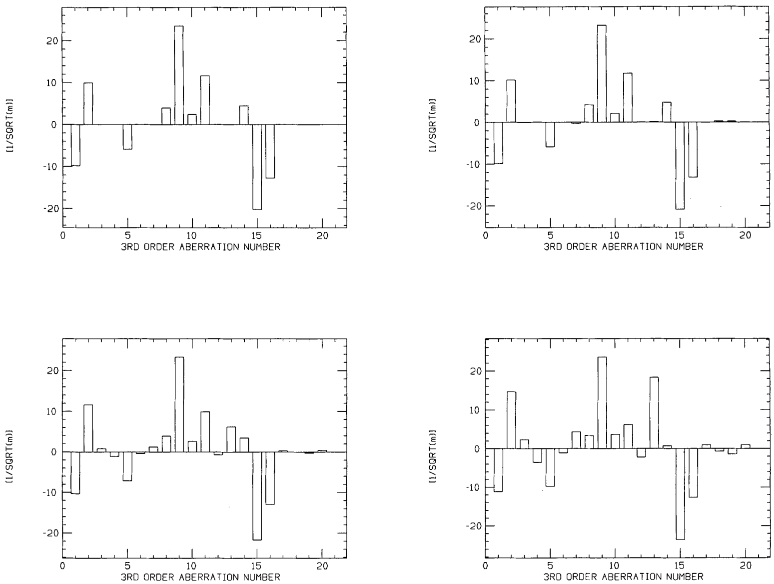

FIGURE 2 The original (top left) and the reconstructed coefficients of the hamiltonian with with 0 μm (top right), 10 μm (bottom left) and 30 μm (bottom right) BPM noise (right).

attributed to the mixing of two frequencies, but there are a few that are left unaccounted, implying that octupolar effects are also visible in the spectrum which is consistent with the effect that the hamiltonian coefficients of the sextupoles in octupolar order are about three orders of magnitude larger than those of sextupolar order which compensates the extra factor of $\sqrt{\beta}\varepsilon$ that comes from mixing three frequencies. The tertiary and higher order peaks will be investigated in a later report.

3 THEORY

We assume that all dipole correctors and BPM are situated in the beginning of the beam line with no non-linear elements between them. Moreover we can back-propagate the kick of the correctors to the start of the beam line and write $\vec{\varepsilon}_i = \hat{R}_i^{-1}(0, \varepsilon_i \sin\omega_i t, 0, 0)$ where $\hat{R}_i$ is the transfer matrix from the start of the beam line to the (here: horizontal) corrector, labelled i, its wobbling amplitude is denoted by ε_i and its frequency by ω_i. If we now choose the reference point $\vec{x} = (x, x', y, y')$ at the start of the beam line, we can write the total map as

$$\vec{x}_{\text{final}} = e^{:-H:} R(\vec{x}_{\text{initial}} + \vec{\varepsilon}) \tag{3}$$

where $\vec{\varepsilon} = \sum_{i=1}^{4} \vec{\varepsilon}_i$ is the sum of back-propagated kicks of the four correctors. Since *the wobbling frequencies ω_i are very small the orbit will follow the corrector kick adiabatically* and we can solve Equation 3 for the quasi-static fixed point $\vec{x}_{\text{final}} = \vec{x}_{\text{initial}}$ with the result

$$\vec{x} = (1 - R)^{-1} \left[R\vec{\varepsilon} - :H: (R(\vec{x} + \vec{\varepsilon})) \right] \tag{4}$$

where we expand the exponential to first order. The Lie-operator $:H:$ operates on the quantity $R(\vec{x} + \vec{\varepsilon})$ resulting in a non-linear equation, because H is a third order polynomial and its application to a state vector $\vec{x}$ yields an expression that is quadratic in the components of $\vec{x}$. The action of $:H:$ on the elements of the state vector x_α can be written as a matrix equation

$$:H: x_\alpha = [H, x_\alpha] = \sum_{j=1}^{20} \sum_{k=1}^{10} a_{\alpha jk} h_j z_k \tag{5}$$

where we denote the four components of the vector $\vec{x}$ by x_α and define z as a vector that contains the ten quadratic monomials of x, x', y, y' in the following ordering $x^2, xx', xy, xy', x'^2, x'y, x'y', y^2, yy', y'^2$ and $a_{\alpha jk}$ contains the numeric coefficients. Now we are in a position to solve Equation 4 perturbatively by expressing $\vec{x} = (x, x', y, y')$ as a superposition of the wobbling frequencies and their mixing results

$$\vec{x} = \vec{x}_0 + \sum_{i=1}^{4} \vec{x}_{1,i} \sin \omega_i t + \sum_{j=1}^{16} \vec{x}_{2,j} \cos \tilde{\omega}_j t \tag{6}$$

where $\tilde{\omega}_j$ runs over the mixed frequencies as shown above. Subscript i labels the four correctors, and subscripts 0, 1, and 2 label the frequency mixing order. Note that in Equation 6 we assume that all excitation signals are sine-like, implying that their mixing products are only constant terms and cosine-like mixing frequencies, which accounts for the absence of terms with sine-like mixed frequencies. To first order in the wobbling frequencies and the amplitude of $\vec{\varepsilon}$ we obtain

$$\sum_{i=1}^{4} \vec{x}_{1,i} \sin \omega_i t = (1-R)^{-1} R\vec{\varepsilon} \ . \tag{7}$$

Here the left hand side is first order in the wobbling amplitude and also wobbles in phase with the excitation frequencies. The next iteration is obtained by re-inserting the left hand side into Equation 4 which yields a second order equation in the amplitudes that has components at frequency zero ($\vec{x}_0$) and at mixed frequencies ($\vec{x}_{2,j}$)

$$\vec{x}_0 + \sum_{j=1}^{16} \vec{x}_{2,j} \cos \tilde{\omega}_j t = -(1-R)^{-1} :H: \left[(1-R)^{-1} R\vec{\varepsilon}\right] \ . \tag{8}$$

The left hand side describes the response of the beam's position at the start of the beam line to the wobbling correctors due to third order aberrations. In order to see the response at a BPM we propagate (x, x', y, y') to the BPM position by left-multiplying with $\tilde{R}$, the transfer matrix from the start of the beam line to the BPM. The BPM pattern is then contained in the first (for a horizontal BPM) or third (for a vertical BPM) component of the vector $\tilde{R}\left(\vec{x}_0 + \sum_{j=1}^{16} \vec{x}_{2,j} \cos \tilde{\omega}_j t\right)$.

In Equation 8 the Lie-operator $:H:$ operates on $(1-R)^{-1} R\vec{\varepsilon}$ where $\vec{\varepsilon}$ is given as the sum of four back-propagated wobbling correctors $\vec{\varepsilon}_i$. Consequently we define the vector $\vec{v}_i$ that corresponds to a particular corrector by

$$\vec{v}_i \sin \omega_i t = (1-R)^{-1} R\vec{\varepsilon}_i \ . \tag{9}$$

Denoting the four components of $\vec{v}_i$ by greek indices α or β we can evaluate the quadratic terms that arise from applying the Lie-operator $:H:$

$$\sum_{i=1}^{4} v_{i\alpha} \sin \omega_i t \sum_{j=1}^{4} v_{j\beta} \sin \omega_j t = \sum_{m=1}^{17} b_{km} \cos \tilde{\omega}_m t \ . \tag{10}$$

In the previous equation we clearly see how the mixing of two sine-like oscillations generate a constant term and cosine-like mixing terms of the sum and difference frequencies. Equation 10 is used to define the 10×17

matrix b_{km} where k runs over the ten different combinations of α and β which correspond to the ten quadratic monomials alluded to above and m runs over the 17 mixed frequencies.

The presented method can be coded on a computer in a fairly straightforward way. First, the definition of the Poisson Brackets in Equation 5 are stored in an array $a_{\alpha jk}$ of dimension $4 \times 20 \times 10$. The first index corresponds to the $\alpha = 1, \ldots, 4$. The second index denotes the hamiltonian coefficient h_j, and the third index k refers to the ten quadratic monomials. Second, the vectors $\vec{v}_i$ are calculated. Third, we calculate b_{km} of dimension 10×17 which is defined in terms of the $\vec{v}_i$ by Equation 10. The first index k refers again to the ten quadratic monomials and the second index m to the 17 frequencies in the ordering given above. All information about placement and amplitude of the wobbling correctors is contained in b through the definition of the vectors $\vec{v}_i$. Equation 8 can finally be written as matrix multiplication

$$c_{\alpha jm} = \sum_{\beta=1}^{4} (1 - R)^{-1}_{\alpha\beta} \sum_{k=1}^{10} a_{\beta jk} b_{km} \tag{11}$$

where $c_{\alpha jm}$ is a matrix of dimension $4 \times 20 \times 17$ that contains the information by how much coordinate x_α at the start of the beam line wobbles at frequency $\tilde{\omega}_m$ due to the hamiltonian aberration h_j. Having found the beam's response at the start of the beam line we can determine the response at a particular BPM by left-multiplying c with the transfer-matrix to the BPM $\tilde{R}$. For a horizontal BPM we get

$$T(m, j) = \sum_{\alpha=1}^{4} \tilde{R}_{1\alpha} c_{\alpha jm} \tag{12}$$

for the dependence of the amplitude of frequency $\tilde{\omega}_m$ due to aberration h_j. For a vertical BPM we replace the 1 in the previous equation by 3.

The matrix T_{ij} relates the 20 hamiltonian coefficients h_j to the $68 = 4 \times 17$ signals s_i at different frequencies at the four BPM which allows us to write $s_i = \sum_j T_{ij} h_j$. The signals s_i can be measured using the filtering technique described in the previous section and the hamiltonian coefficients can be inferred by solving for the h_j in the least-squares sense $h_j = \left((T^t T)^{-1} T^t\right)_{ji} s_i$. Errors in the BPM response pattern can be taken into account by left-multiplying T by a weight matrix $\Lambda_i = \mathrm{diag}\,(1/\sigma_1, \ldots, 1/\sigma_{68})$ and observing that the diagonal elements of the covariance matrix $(T^t T)^{-1}$ are the squares of the errors on the h_j.

4 LEP

We test the reconstruction of the hamiltonian coefficients with the LEP lattice used for the generation of Figure 1. Raw data for the analysis are generated in tracking runs over $N = 32\,768$ turns. The recorded BPM data are filtered in order to reduce the large peak at the excitation frequencies as described in Section 2. In the next step the amplitudes af all mixed frequencies are extracted from the BPM data x_i, $i = 1, \ldots, N$ by cosine-fourier transformation,

$$s_k = \frac{2}{N} \sum_{i=1}^{N} x_i \cos \tilde{\omega}_k t_i \tag{13}$$

where $\tilde{\omega}_k$ are the mixed frequencies and t_i is the time elapsed since the start of the measurement. This procedure guarantees a phase-synchronous detection of the beam's response to the wobbling excitation. The s_k are stored in a file and read by the analysis program which calculates the hamiltonian coefficients according to the theory presented in the previous section. The top right of Figure 2 shows the reconstructed coefficients which are in excellent agreement to the original ones presented at top left.

In order to investigate the robustness of the method under the influence of finite BPM resolution we use the same tracking data that led to Figure 1 and add random numbers with standard deviation of $10\,\mu$m and $30\,\mu$m, truncated at three standard deviations, and repeat the above analysis. First we remove the peaks of the fundamental, then we extract the BPM signal pattern coefficients s_k according to Equation 13 and then feed the resulting BPM pattern to the analysis program. The hamiltonian coefficients are shown in the bottom row of Figure 2. We clearly see that the coefficients are reproduced rather well.

At first sight the fact that we detect sub-micron signals in the presence of noise that is on the order of $10\,\mu$m looks surprising, but the apparent contradiction is resolved by observing that we detect the small amplitude wobbling signal phase-synchronously over many ($N = 32768$) turns. This detection method is equivalent to averaging and the noise is consequently reduced. From Equation 13 we find a reduction factor of $1/\sqrt{N/2}$. For BPM noise of $10\,\mu$m we obtain an effective error of $0.1\,\mu$m which explains the quality of the reconstruction.

5 CONCLUSIONS AND OUTLOOK

We presented a theory that explains the spectral pattern observed at BPMs that stem from wobbling orbit correction magnets at low frequencies. The theory can be used to deduce all third order geometric aberrations in a storage ring. We tested that method using a standard LEP optics. Using tracking data with wobbling correctors included and filtering the raw data carefully we reproduced the coefficients in the hamiltonian to high accuracy. Including BPM noise of 10 μm in the simulation allows faithful reconstruction of the hamiltonian aberrations.

The ability to detect and correct geometric aberrations rather quickly (32 000 turns in LEP or LHC corresponds to a little over three seconds) may be important during the rather slow ramp in LHC (which takes about 20 minutes). During the ramp the super-conducting magnets produce large sextupolar and higher contributions, which may be diagnosed by the described method and corrected using "knobs" made of linear combinations of sextupoles that affect a single aberration, only.[4,5] Moreover, extending the method to higher order will allow to measure terms that are responsible for amplitude dependent tune shift.

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References

[1] V. Ziemann, *On the Correction of Large Random Skew Quadrupolar Errors During the Ramp in LHC,* Particle Accelerators, 1995, Vol. 51, pp. 155–179.
[2] J. Bengtsson, *Non-linear Transverse Dynamics for Storage Rings with Applications to the Low-Energy Antiproton Ring (LEAR) at CERN,* CERN 88-05, August 1988.
[3] A.J. Dragt, *Lectures on Non-Linear Orbit Dynamics,* in AIP conference proceedings No. 87, Fermilab Summer School, 1981.
[4] V. Ziemann, *Identification of Third Order Hamiltonian Coefficients from Forced Coherent Beam Oscillations,* CERN-SL/95-94(AP) and TSL Note 95–22, October 1995.
[5] W. Scandale and R. Schmidt, *Stopband Correction in the SPS with sets of two orthogonal sextupoles,* CERN SPS/DI-MST/Note/85–4, July 1985.