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EMITTANCE GROWTH DUE TO TUNE FLUCTUATIONS AND THE BEAM-BEAM INTERACTION

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Analytical formulae and computer simulation results are presented for the emittance growth caused by small asymmetries of the beam-beam force, caused by small fluctuations of the phase, small offsets between the beams and fluctuations in the size of the opposite beam.

Keywords: Dynamic aperture.

1 INTRODUCTION

Consider an ensemble of particles such that the phase is uniformly distributed over the whole range $0 \leq \psi_x \leq 2\pi$ amongst this ensemble. Then the phase average of the beam-beam force $\langle F(x) \rangle_{\psi_x}$ over this ensemble vanishes. In this paper we will consider the diffusion and emittance growth that result if we destroy this symmetry a little. This symmetry of the beam-beam force can be destroyed by small fluctuations in (i) the phase, (ii) offsets between the beams, and (iii) size of the opposing beam. The main source of tune fluctations are the quadrupole strength fluctuations, either directly by power supply noise in quadrupoles or indirectly through closed orbit fluctuations through non-linear magnets and mechanical vibrations of non-linear magnets. There are also sources of "pure" tune fluctuations which arise when the chromaticity is non-zero and there is momentum diffusion either due to Intra-beam scattering, RF noise or beam gas scattering. Offset fluctuations between the beams at the interaction points can occur due to power supply noise either in the dipoles or in misaligned higher order multipole magnets, especially the high β quadrupoles in the interaction regions. Beam size fluctuations can occur for example due to RF noise and dispersion in the RF cavities. An example of this occurred during the November 1993 luminosity run in HERA.[1]

The principal source of random variations is noise in the power supplies which can have a fairly complicated spectrum. Analysis shows[2] that the output power noise density falls as $1/f^2$ over a wide range of frequencies for a high quality oscillator which can be modelled as a linear oscillator. At the lowest frequencies the spectrum has a $1/f^3$ dependence due to the intermodulation of carrier noise and flicker noise while at the highest frequencies, the output noise has a flat spectrum. For our analysis we will consider Ornstein-Uhlenbeck noise which has a $1/f^2$ spectrum so as to reasonably model the noise spectrum that particles are subjected to and also because it is the only stationary Gaussian Markov process.[3]

Simulations of the emittance growth with a linear lattice and the beam-beam force in the strong-weak approximation show a dramatic influence of noise, specially near low order resonances. Figure 1 shows the emittance growth of 1000 particles over 10^7 turns with the tune set to $\nu_0 = 1/4$, both without and with noise. Without noise, there is no evidence of any emittance growth. With tune noise of amplitude $\Delta\nu_r = 10^{-4}$ there is a large emittance growth. Most of this emittance growth occurs due to a small fraction of the total number of particles going to large amplitudes. Notice also the fact that the emittance grows only slowly over the first 10^6 turns and increases substantially after that. In one space dimension, the emittance growth seen with noise is significant only for low order resonances (less than 10th order) while in two space dimensions, the emittance growth is significant even for high order resonances such as the 14th.[4]

2 TUNE FLUCTUATIONS OFF RESONANCE

In action angle (J, ψ) variables, the Hamiltonian describing the beam-beam interaction is

$$H = \nu_0 J + U\delta_p(\theta) \tag{1}$$

where ν_0 is the nominal tune, U is the beam-beam potential and δ_p is the periodic delta function. The beam-beam potential experienced by a proton due to a Gaussian bunch with N_b particles of charge e is

$$U(x) = \frac{N_b r_p}{\gamma_p} \int_0^\infty \frac{1}{2\sigma^2 + q} \left\{1 - e^{-x^2/(2\sigma^2+q)}\right\} dq \tag{2}$$

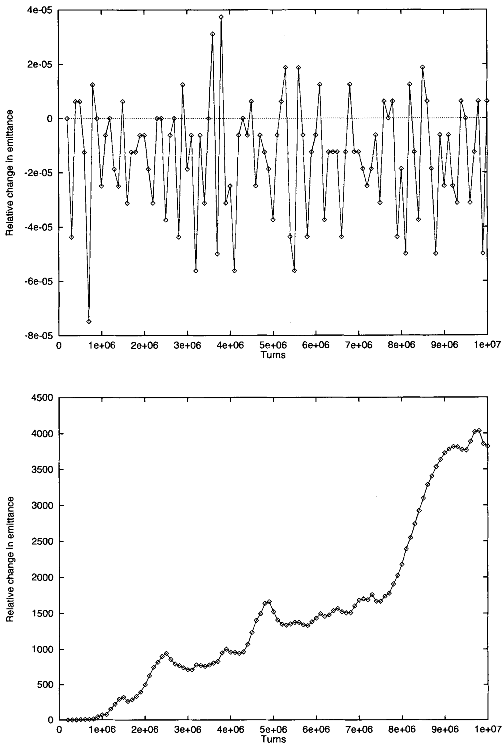

FIGURE 1 Relative change in the emittance of 1000 particles due to the one dimensional beam-beam interaction at the fourth integer resonance $\nu_0 = 0.25$. Top: without external noise. Bottom: with tune noise of amplitude $\Delta\nu_r = 10^{-4}$.

Substituting $x = \sqrt{2\beta^* J}\cos\psi$ where β^* is the beta function at the interaction point, we obtain the Fourier expansion

$$U(J, \psi) = C\left[F_0(\alpha) + 2\sum_{k=1}^{\infty} F_k(\alpha)\cos 2k\psi\right] \tag{3}$$

$\alpha = \beta^* J/(2\sigma^2)$ is a dimensionless measure of the particle amplitude and $C = N_b r_p/\gamma_p$. The Fourier coefficients are $F_0(\alpha) = \int_0^\alpha [1 - e^{-z} I_0(z)]/z\, dz$, $F_k(\alpha) = (-1)^{k+1}\int_0^\alpha e^{-z} I_k(z)/z\ dz$. Integrating the equations of motion over a turn, we obtain the one turn beam-beam map, $\Delta\psi = 2\pi\nu_0 + \partial U/\partial J$, $\Delta J = -\partial U/\partial\psi$. As mentioned earlier, in the absence of time dependent effects and far from resonances, the phase averaged change in action due to the beam-beam force vanishes $\langle \Delta J \rangle = 0$, so that J can be considered a conserved quantity in the averaged sense.

Now we suppose that the phase advance ψ has a fluctuating component from turn to turn. At turn n the phase is $\psi(n) = 2\pi n\nu(J) + \psi_0 + \sum_{i=0}^{n} \Delta\psi_r(i)$ where $\nu(J)$ accounts for the action dependence of the tune due to the beam-beam potential, ψ_0 is the initial phase and $\Delta\psi_r(i)$ is the random contribution at turn i. The change in action to first order in the random phase is

$$\begin{aligned} \Delta J_r(n) &= \frac{d}{d\psi}\Delta J(n)\Delta\psi_r(n) + O(\Delta\psi_r^2) \\ &= -\frac{\partial^2}{\partial\psi^2}U(\psi(n))\Delta\psi_r(n) + O(\Delta\psi_r^2) \end{aligned} \tag{4}$$

We will assume that (i) $\Delta J \ll J$ and (ii) the noise process is stationary so that we can define a correlation function $K_\nu(n)$ as

$$\langle \Delta\psi(l)\Delta\psi(n+l)\rangle = 4\pi^2\Delta\nu_r^2 K_\nu(n) \tag{5}$$

where $\Delta\nu_r$ is the amplitude of the tune noise. The correlation function K_ν is related to the spectral density $S(f)$ by the cosine transform, $K_\nu(n) = 2\int_0^\infty S(f)\cos(2\pi fn/f_0)df$, where f_0 is the revolution frequency. The diffusion coefficient due to the fluctuating tune is defined as $D_\nu(J) = \lim_{N\to\infty}[J(N) - J(0)]^2/N$. We obtain

$$D_\nu(J) = 128 f_0(\pi C\Delta\nu_r)^2 \sum_{k=1}^{\infty} k^4 F_k^2 S(2kf_0\nu) \tag{6}$$

This result states that tune noise at only the even harmonics of the betatron frequency cause diffusion in the action when the nominal tune ν_0 is sufficiently far from resonances.

For the Ornstein-Uhlenbeck process the correlation function for random variables with unit variance is $K_\nu(n) = e^{-n/\tau_c}$, τ_c is the correlation time measured in units of the revolution time of the particle around the ring. The spectral density $S(f) \sim 1/(4\pi^2 f^2/f_0^2 + 1/\tau_c^2)$, falling to half its maximum value at a frequency $f_{1/2} = f_0/(2\pi\tau_c)$. Substituting in this correlation function leads to

$$D_\nu(J) = 128(\pi C \Delta\nu_r)^2 \sinh(1/\tau_c) \sum_{k=1}^{\infty} \frac{k^4 F_k^2}{\cosh(1/\tau_c) - \cos 4\pi k\nu} . \qquad (7)$$

The diffusion coefficient can also be calculated by a simulation. The simulation model is a linear lattice with the beam-beam force at a single interaction point. 100 amplitudes are chosen from an exponential distribution and at each amplitude 10 particles are chosen with different initial phases. The diffusion coefficient at each amplitude is obtained by averaging the ten diffusion coefficients of these particles. Chirikov's method[5] is used to calculate "diffusion coefficients" D_1, D_2 over two intervals $\Delta N_1, \Delta N_2$ as follows:

$$D_k = \frac{2}{N_k(N_k - 1)} \sum_{m}^{N_k} \sum_{\substack{n \\ m>n}}^{N_k} \frac{[\bar{J}_k(m) - \bar{J}_k(n)]^2}{\Delta N_k(m - n)}, \qquad k = 1, 2 \qquad (8)$$

where $N_1 \Delta N_1 = N_2 \Delta N_2 = N$, N being the total number of turns. $\bar{J}_k(m)$ is the average of J over the mth interval of ΔN_k. For motion that is only oscillatory and bounded, these coefficients differ by two or three orders of magnitude while for true diffusive motion these coefficients are fairly close.

With an external noise source present, one expects the motion to be diffusive and indeed the two coefficients D_1, D_2 we calculate with $\Delta N_1 = 10^5$, $\Delta N_2 = 10^5$, $N = 10^7$ are within 10% of each other. Figure 2 shows a comparison of D_2 with the analytically calculated diffusion coefficient from Equation (7) at a correlation time of $\tau_c = 1.1$ (corresponding to nearly white noise). The agreement is very good. The simulations have been repeated for larger correlation times of 10, 100 and 10000. The agreement between the analytical and numerical diffusion coefficients is reasonably good in all cases, with the analytical coefficient larger at all amplitudes.

3 EMITTANCE GROWTH DUE TO TUNE FLUCTUATIONS

Simulations show that with tune noise for example, there is emittance growth of say 1000 particles over a few minutes. The emittance growth of a bunch with 10^{10} protons over the storage time of a day cannot however be estimated by simulations. If we make certain assumptions about the noise process,

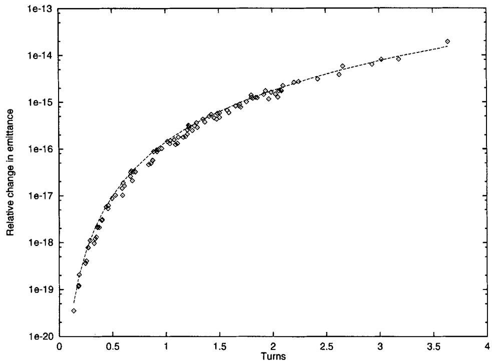

FIGURE 2 Comparison of the diffusion coefficient from Equation (7) and the numerically calculated Chirikov diffusion coefficient due to tune flucutations with amplitude $\Delta\nu_r = 10^{-4}$. Correlation time of the Ornstein-Uhlenbeck noise process is $\tau_c = 1.1$, the tune is $\nu_0 = 0.291$.

then the evolution of the beam density and hence moments of the beam can be followed by integrating the Fokker-Planck equation. Assuming then that the diffusive growth in the action J is Markovian, the evolution of the density ρ is given by

$$\frac{\partial}{\partial t}\rho = \frac{1}{2}\frac{\partial}{\partial J}\left[D(J)\frac{\partial}{\partial J}\rho\right] \tag{9}$$

where for $D(J)$ the diffusion coefficient we use the analytic expression given by Equation (7). We will use the method of lines to numerically integrate this 1D Fokker-Planck equation.[6] The boundary condition is that particles are lost at an action J_b corresponding to the position of the beam pipe, i.e. $\rho(J_b, t) = 0$ at all times. The diffusion coefficient $D_\nu(J)$ and its first derivative (from Equation (7)) both vanish at the origin. From these properties it follows that the density at the origin obeys

$$\rho(J=0,t) = \rho_0(J=0), \qquad \frac{\partial}{\partial J}\rho(J=0,t) = e^{D''t/2}\frac{\partial}{\partial J}\rho_0(J=0) \tag{10}$$

where ρ_0 is the initial density. We introduce dimensionless time, action and density variables (τ, X, U) respectively defined as $\tau = t/t_0$, $X = J/J_b$, $U = J_b\rho$. The parameter t_0 is interpreted as the number of revolutions around the ring per time step of the numerical integration. The scaled diffusion

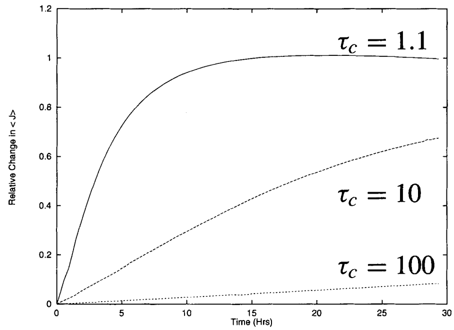

FIGURE 3 Change in the relative action $\langle J \rangle$ due to tune fluctuations with nominal tune $\nu_0 == 0.291$ over a time of 30 hours obtained by integrating the Fokker-Planck equation. Shown are the changes due to noise at three correlation times $\tau_c = 1.1$ (nearly white noise), 10 ($f_{1/2}$=750 Hz) and 100 ($f_{1/2}$=75 Hz). Tune noise amplitude is $\Delta\nu_r = 10^{-4}$.

coefficient is $\mathcal{D}(X) = t_0 D(J)/J_b^2$. The Fokker-Planck equation in the dimensionless variables is numerically integrated for the required number of turns by appropriately choosing t_0.

Once the density $\rho(J, t)$ is known, the average action of the distribution at any time is found from

$$\langle J \rangle = \frac{1}{\int_0^{J_b} \rho(J_b, t) dJ} \int_0^{J_b} J\rho(J, t) dJ. \tag{11}$$

Figure 3 shows the relative change in the average action $\langle J \rangle$ due to tune noise at three different correlation times with the higher correlation times corresponding to noise with less high frequency content. With $\Delta\nu_r = 10^{-4}$ and nearly white noise ($\tau_c = 1.1$) acting on the beam we see that the emittance doubles over a period of 10 hours before decreasing due to particles being lost at the beam-pipe. Diffusion is slower for the other two noise realizations so particles do not reach the beam-pipe in 30 hours and the emittance grows nearly linearly to a maximum of about 70% for $\tau_c = 10$ and 10% for $\tau_c = 100$ respectively. Due to both filtering of power supplies and attenuation in the metallic lining of beam pipes, a correlation time of $\tau_c = 100$ is likely to be the most realistic model of the noise spectrum experienced by the beam. At $\tau_c = 100$, the spectral density at twice the betatron frequency (the part of the noise spectrum responsible for the diffusion) is smaller by a factor of 7×10^{-6}

than the spectral density near zero frequency. The emittance growth increases nearly linearly with the tune noise amplitude $\Delta\nu_r$ since $D_\nu(J) \propto (\Delta\nu_r)^2$. At $\tau_c = 100$ and $\Delta\nu_r = 10^{-3}$, the emittance increases initially by 90% in about 7 hours before decreasing and reaching a value about 70% greater than the initial value at the end of 30 hours.

4 TUNE FLUCTUATIONS NEAR RESONANCE

The dynamics with noise near low order resonances are significantly different than that far away from such resonances. A detailed analytical treatment of the effects of noise near resonances will appear elsewhere.[7] The phase space structure is specially important when the noise is sufficiently weak that the details of the resonance islands have not been completely smeared out. For noise strength below a certain value, particles can be trapped within these islands and be thereby transported in phase space as the islands move, leading to emittance growth. Trapping occurs when the time for a resonance to diffuse a distance equal to its width is greater than the time period of motion around the resonance island. Let $\Delta\nu_r$ be the rms amplitude of tune fluctuations, then trapping occurs for $\Delta\nu_r < \Delta\nu_r^T$, where the trapping limit amplitude is given by

$$(\Delta\nu_r^T)^2 = \frac{1}{2}\nu_{isl}(\Delta\nu_W)^2 . \tag{12}$$

ν_{isl} is the island tune and $\Delta\nu_W$ is the island width in tune space. For the beam-beam interaction, these parameters for a resonance island of order $2k$ are given by

$$\Delta\nu_W = \frac{C}{\pi}\sqrt{|\, 2F_0''(J_s)F_k(J_s)\,|}, \quad \nu_{isl} = k\Delta\nu_W \tag{13}$$

where J_s is the action at the stable fixed point. Putting these back we obtain the trapping condition

$$(\Delta\nu_r)^2 \le (\Delta\nu_r^T)^2 = \frac{\sqrt{2}kC^3}{\pi^3}[|\, F_0''(J_s)F_k(J_s)\,|]^{3/2} \tag{14}$$

For typical HERA parameters, $\Delta\nu_r^T \sim 2.5 \times 10^{-4}$ for the 4th integer resonance ($k = 2$).

In the other extreme limit, the tune fluctuations may be strong enough to wash out the resonance structure. When the phase is completely random from turn to turn, the random contribution is of the same magnitude or more than the nominal tune, $\Delta\nu_r^{CR} > 1/(2k)$. In this situation the random tune can move a particle from one resonance island to the neighbouring resonance island and beyond in one turn. In the not so extreme limit but still fast diffusion is the case when the random tune causes the particle to diffuse from one resonance island to the next in the time it takes for a particle to go around the first island. This occurs for $\Delta\nu_r > \Delta\nu_r^f$ where

$$(\Delta\nu_r^f)^2 = \frac{1}{2}(\Delta\nu_{sep})^2\nu_{isl} = \frac{1}{8k^2}\nu_{isl} = \frac{\sqrt{2}C}{8\pi k}\sqrt{| F_0''(J_s)F_k(J_s) |} \qquad (15)$$

For typical HERA parameters, the maximum value of the island tune is 0.008 at the 4th integer resonance. The corresponding lower limit for fast tune diffusion is $\Delta\nu_r^f \sim 0.016$.

The frequencies of the tune fluctuation are also important in determining the dynamical response. When resonance islands are present, motion in the vicinity of these islands slows down and the island tunes are much smaller than the nominal tune. Exactly at the resonance tune the islands disappear (because the fixed points are at infinity) but the periods keep increasing with amplitude. Figure 4 shows the phase portrait of two particles at the 4th integer resonance. The emittance growth seen in Figure 1 is due to particles diffusing from the inner regions onto the curves with long extensions along the axes in Figure 4. Since the motion on these latter curves has a long period, it can take a long time before the emittance growth is observed, as is the case in Figure 1. A detailed analysis[7] shows that near resonances, low frequency noise which is resonant with the slow motion of the linear invariant is responsible for diffusion. This is in complete contrast to the off-resonance case. A similar topological reason should also explain the emittance growth in the two degree of freedom case.

5 SUMMARY

We have analysed the effects of tune fluctuations on the emittance growth of a hadron beam colliding with another beam. Off low order resonances, tune noise at even harmonics of the betatron frequency is responsible for

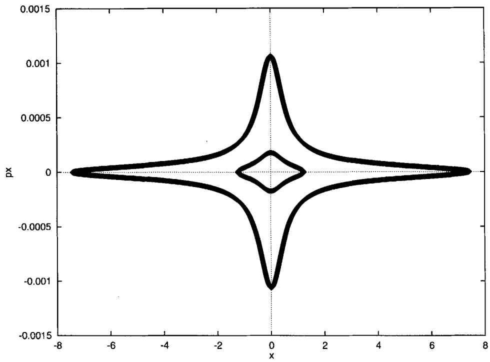

FIGURE 4 Phase portrait of two particles at the tune $\nu_0 = 0.25$ without external fluctuations.

particle diffusion and emittance growth. Near low order resonances, low frequency noise causes diffusion. Other fluctuations which lead to emittance growth include beam offset fluctuations and fluctuations in the size of a beam. A detailed analysis can be found in a forthcoming publication.[7] Briefly stated, we find that off resonances, offset fluctuations at odd harmonics of the betatron tune and beam size fluctuations at even harmonics of the betatron tune lead to diffusion. Both of these fluctuations lead to greater emittance growth than tune fluctuations in the off-resonance case for reasonable values of the fluctuating amplitudes. These three fluctuations which are almost always present might explain a large part of the observed emittance growth. They also underline the importance of choosing high precision power supplies, especially for quadrupoles and dipoles in the interaction regions. Future work will focus on understanding the effects of these fluctuations in the two degree of freedom case.

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