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REDUCING CHAOS IN FOUR DIMENSIONAL SYMPLECTIC MAPS

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A method for finding integrable four-dimensional symplectic maps is outlined. The method relies on solving for parameter values at which the linear stability factors of the fixed points of the map have the values corresponding to integrability. We propose that this method be applied to accelerator lattices in order to increase dynamic aperture.

Keywords: Dynamic aperture; lattice; maps.

1 INTRODUCTION

Much progress has been made in the problem of determining whether a dynamical system is chaotic. Poincaré[1] showed that for a general perturbation of an integrable multidimensional oscillator, there is no invariant analytic in the perturbation parameter. The problem of small denominators in normal form theory[2] prevents one from finding a convergent invariant in the neighborhood of a linearly stable fixed point. The understanding of what happens when an integrable system is perturbed was greatly increased by the KAM theorem,[3–6] which showed that some invariant surfaces remain (provided the linear frequencies in normal form theory are not linearly related with integers smaller than 4). Complementarily, Melnikov[7] and subsequent work on the intersection of stable and unstable manifold showed that chaotic motion was very easy to find. These are the approaches one must take to determine the stability of a given Hamiltonian system, such as the motion of the asteroids or the stability of the solar system; one would like to know whether the motion for given initial conditions is stable, and so one would like methods for determining whether a trajectory found in given region of phase space for a given Hamiltonian is, or is likely, to be integrable or chaotic.

In contrast, there is another class of problems in which one has freedom in choosing the Hamiltonian, and for various reasons one would like to have a system that is either completely chaotic or uniformly integrable. For example, one might desire a chaotic fluid flow so that chemicals are mixed uniformly in a short time. An example of the needing integrability occurs in the design of systems that must confine particles, such as fusion confinement devices or accelerators. The goal of this paper is to outline how one might find symplectic maps with reduced chaos and, specifically, how this might be applied to accelerator lattices to increase dynamic aperture. A more complete discussion is elsewhere.[8]

The restrictions that arise in real systems are what make this problem nontrivial. Without restrictions, one can write down any number of integrable Hamiltonians. One need only ensure that the Hamiltonian have sufficient symmetries such that by use of Noether's theorem one can find a complete set of invariants. However, the accelerator designer does not have a list of integrable Hamiltonians to choose from. Instead he has various kinds of magnets that he must string together in a lattice in such a way that the beam is confined. As a result, no accelerator is integrable, as the current procedure for accelerator design (discussed below) requires the insertion of nonlinear elements, and such elements generically lead to nonintegrability, as noted above. Hence, to appreciate the present discussion, it is important to understand not only dynamics, but dynamics as it is limited in a practical situation, which here is accelerator design.

The basic ideas for finding integrable systems[9] were first applied using fixed-point indicators of integrability[10] in the context of finding three-dimensional toroidal magnetic field with lines lying on nested toroidal surfaces. Such systems are Hamiltonian of one and a half degrees of freedom[11] and, hence, correspond to two dimensional symplectic maps. In this case there are no systems known rigorously to be integrable while having no internal current yet having nonzero winding number. (These latter requirements arise from physics issues.) However, perturbation theory indicated that for systems with the latter properties, approximately integrable systems did exist. Application of methods similar to those discussed in this paper allowed one to find systems with greatly reduced chaotic regions and, therefore, larger confined plasma volume. A series of designs found by using these ideas were published.[12]

Subsequently we applied these ideas to the similar problem of the dynamics in the uncoupled horizontal dynamics of an accelerator lattice.[13] As this is also a system of one and a half degrees of freedom, the method carried over in a straightforward manner. The difference was primarily that the parameters became the amplitudes of the nonlinear multipoles in the accelerator magnets, whereas before the parameters were the fourier harmonics of the winding law of the coils.

The method proved to be successful here too, insofar as only one transverse degreee of freedom was concerned, i.e., for trajectories with initial conditions (momentum and position) entirely in the horizontal plane (an invariant plane for the lattices were considered). However, an examination of the dynamics of the four dimensional system (by considering trajectories with initial conditions out of the horizontal plane) found that the dynamics was worse. That is, the four dimensional volume of confined initial conditions was found to be smaller even though the uncoupled horizontal dynamics was improved.

These results meant that to be able to reduce the chaotic dynamics in Hamiltonian systems of two and a half degrees of freedom, one would have to consider the full dynamics. In this paper we discuss how this could be done. We begin, in the following section, by reviewing why accelerators must be nonlinear. Then we discuss the linear stability of fixed points in four dimensional maps.

Our hope is that such methods will allow those who need to develop stable particle confinement systems such as accelerator lattices to control the chaos in these systems as much as they now control the linear dynamics. With this degree of control, large dynamic apertures may be achievable in strongly nonlinear machines such as synchrotron light sources,[14] allowing them to produce light beams of greater brightness. Alternatively, smaller dynamic apertures may be put in place to, for example, collimate the beam, i.e., reduce its emittance or energy spread.

2 WHY ACCELERATOR LATTICES MUST BE NONLINEAR

In modern high-energy accelerators, particles are confined by the principle of strong, or alternate-gradient, focussing.[15] Quadrupole magnets, which have transverse magnetic field varying linearly with distance from the center, are placed with alternating polarities around the machine (in combination with bending magnets). As this motion is linear, there is a quadratic invariant of the motion, known as the Courant-Snyder invariant, for each of the three degrees of freedom: the horizontal and vertical, or *betatron*, degrees of freedom and the longitudinal, or *synchrotron*, degree of freedom. The magnet strengths and separations are chosen such that the hypersurface of constant Courant-Snyder invariants is closed for particles having energy near the design energy. That is, the motion is analogous to that of harmonic oscillators with a restoring rather than unstable linear force. The particles move on the invariant hypersurface with three characteristic frequencies, known as *tunes*, one for each degree of freedom. We will discuss this in more detail in the following section.

The choice of the tunes is dictated by stability. Linear systems of one and a half degrees of freedom are typically unstable to arbitrarily small perturbations (always present due to magnet imperfections or unavoidable nonlinearity), if the tune is an integer multiple of 1/4, 1/3, or 1/2. The KAM theorem states that in the presence of perturbations, there will remain a region of stable trajectories (more precisely, a region where most of the invariant surfaces are only distorted, not broken) provided the tune is a "sufficiently irrational number". This has a technical meaning[16] that is roughly that good rational approximations of the tune $\nu = m/n$ require the use of large integers m and n. Moreover, the KAM theory indicates that it is better to choose a highly irrational tune to obtain the best stability properties.

Unfortunately, the tune has a natural variation with energy due to the fact that the efficacy of bending of trajectories by magnetic fields decreases with energy; the magnetic rigidity increases with energy. As a result, the tune would naturally vary with energy, and so at some energy the tune would be a low-order rational and highly unstable to perturbations. To eliminate this *chromaticity*, proportional to the derivative of tune with energy at design energy, sextupole magnets, for which the magnetic field varies with the square of the deviation from the design orbit, are inserted. Then, higher-energy particles, which circulate on a larger radius orbit, effectively see a larger quadrupole field so as to counteract the increase in rigidity. By judicious choice of the strength of the sextupole, the chromaticity can be made to vanish.

Thus one comes full circle. In spite of the fact that one would like a stable, predictable linear system, one must introduce nonlinearity to get stability for small oscillations over a range of energies. As a result, one must understand the chaotic dynamics associated with such nonlinearity. Given the present limited understanding of nonlinear systems, much of accelerator design relies on detailed examination of the dynamics for many proposed scenarios. Naturally, the designer begins by restricting himself to systems with good linear dynamics and small chromaticity. However, from there the process becomes heavily computational — extensive orbit following (tracking) calculations are made to determine the limits of the phase space region (dynamic aperture) in which the trajectories remain confined.

3 HAMILTONIAN DYNAMICS AND THE ONE-TURN MAP OF AN ACCELERATOR

The motion of particles around an accelerator is governed locally by a Hamiltonian. However, for understanding the long-term dynamics it is sufficient to analyze the return map. In this section we review the relationship between the Hamiltonian and the return map.

We consider the motion defined by a Hamiltonian having N degrees of freedom plus temporal variation, $H(q_1, \ldots, q_N, p_1, \ldots, p_N, s)$. In keeping with the practice in accelerator physics tracking, the temporal variable s in an accelerator is a variable that parameterizes the path around the accelerator ring, while the energy and time form the third canonical pair (Ref. 17, Section 1.2). The Hamiltonian is then the function that gives the rate of change of the other variables with respect to s as the accelerator is circumnavigated:

$$\frac{dq_i}{ds} = \frac{\partial H}{\partial p_i} \tag{1a}$$

and

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial q_i}. \tag{1b}$$

In fact, for single-particle dynamics in accelerators, N is at most three, and the coordinates are the transverse coordinates and momenta, the energy, and the time (which is needed to calculate the temporally varying accelerating kick).

This Hamiltonian theory neglects the phase-space contracting effect of synchrotron radiation and the diffusion caused by quantum fluctuations. These effects are usually neglected in proton machines. In addition they can be neglected during the first several turns in electron machines, when it is nevertheless important to know the dynamic aperture (the region of confined trajectories in phase space), as that controls the success of the injection process.

For most analyses of the single-particle dynamics in accelerators, the energy-time canonical pair is considered separately. The characteristic (synchrotron) frequency of the energy-time oscillations is much smaller than the characteristic (betatron) frequencies of the transverse oscillations, and so can be treated later by adiabatic theory. When this decoupling applies, the picture of the motion in phase space can be pieced together by understanding the constant-energy dynamics for the range of energies implied by the synchrotron oscillations.

For the systems we consider, the temporal variable is periodic. By this we mean not only that the Hamiltonian is periodic in this variable,

$$H(s+S) = H(s) \tag{2}$$

with some period S, but also that phase space is toroidal in the variable s so that $(q_1, \dots, q_N, p_1, \dots, p_N, s)$ and $(q_1, \dots, q_N, p_1, \dots, p_N, s+S)$ are the same point. This obviously holds for the case of the accelerator as described above. This requirement implies that single-valuedness and periodicity as in Equation (2) are the same condition.

The integration of the trajectories through one period of the Hamiltonian gives a map,

$$\bar{z} = M(z) \tag{3}$$

of the phase space of initial conditions at $s = 0$ onto the phase space at $s = S$. Here, is the points obtained by starting at z and integrating the equations of motion for one period S of the temporal variable. This map is symplectic (Ref. 18, Section 7-1) because the equations of motion come from a Hamiltonian. Furthermore, this map is the same for each period, as the Hamiltonian is periodic. This implies that one can find the trajectory after going through n periods of the Hamiltonian by applying the map n times.

A Hamiltonian system of N degrees of freedom is integrable if it has N single-valued invariants $\mathbf{J} = (J_1, \dots, J_N)$ in *involution* (See Ref. 19, Section 147), known as *actions*. In *involution* means that the Poisson brackets of the actions with themselves vanish. Assuming, as we will, that the surfaces of constant action are compact, the surfaces must be tori in phase-space-time. The variables canonical to the actions are the *angles* $(\theta_1, \ldots, \theta_N)$, which, with time, specify a point on a given torus. Because the actions are constants of the motion, the Hamiltonian does not depend on the angles. Further, it can be shown that the time-dependence of the Hamiltonian can be eliminated by a subsequent transformation. Hence, the Hamiltonian for an integrable system has the form, $H(\mathbf{J})$. Because the Hamiltonian has this form, the angle variables increase linearly in time,

$$\frac{d\theta_i}{ds} = \frac{\partial H}{\partial J_i} \equiv \omega_i(\mathbf{J}). \tag{4}$$

Any observable O has time dependence only through the evolution of the angles,

$$O = f(\theta_1, \ldots, \theta_N, s) \tag{5}$$

Furthermore, the dependence on these angles is periodic. This, together with the fact that the angle variables increase linearly in time, implies that the time-dependence of any observable is *quasiperiodic*,

$$O = f(\theta_1 = \theta_{10} + \omega_1 s, \ldots, \theta_N = \theta_{N0} + \omega_N s, s), \tag{6}$$

which means, in essence, that the evolution has the form of Equation (6) — that of a function periodic in $N+1$ variables with the evolution of each of these variables being linear. Such evolution has $N + 1$ fundamental frequencies, one (ω_i) for each of the degrees of freedom and one,

$$\Omega \equiv 2\pi/S, \tag{7}$$

associated with the periodicity of the temporal variable.

In fact, the fundamental temporal frequency (7) provides a scale for the other frequencies. The scaled frequencies,

$$\nu_i \equiv \omega_i/\Omega, \tag{8}$$

are known as the winding numbers in most of the nonlinear dynamics literature, as these give the number of times a trajectory circulates around the torus "the i^{th} way" (i.e., increasing θ_i by 2π) per each circulation around the torus "the s^{th} way", that is, per increase of the temporal variable by one fundamental period S. In accelerator physics the quantities (8) are known as the *tunes*, as they give the number of oscillations of the various degrees of freedom per circulation around the accelerator.

For practical applications integrable nonlinear systems have many advantages. The simplicity of the motion makes it easily calculated. Moreover, integrable systems have linear, not exponential, divergence of trajectories, and so calculations are not accompanied by the loss of predictability associated with chaotic dynamics. In addition, the regions of trajectory loss can be known precisely as the region inside the separatrix. In nonintegrable systems, the boundaries of invariant regions tend to be complex with structure at all scales (Ref. 17, Ch. 3).

Finally, nonlinearity can have a stabilizing effect in at least two ways. The effect of perturbations on nonlinear systems is guaranteed to be small; nonlinear stabilization guarantees that the associated resonant regions have finite size. In addition, the spread in tunes can make such systems less susceptible to collective instabilities,[20] which arise when the dynamical frequencies of the particles match a natural frequency of a cavity in an accelerator, and so are strongest when all particles have the same frequency and resonate together.

4 CHAOTIC MOTION AND RESONANCES

When an integrable system, such as that described in the last section, is perturbed, resonances (or islands) form. The perturbation generally contains fourier harmonics (in the unperturbed angles) that do not change for a particle moving along an unperturbed trajectory. These resonant perturbations cause large oscillations in the actions.

It is possible to calculate the effect of a single resonance in isolation.[21] Such a calculation shows that the oscillations in the action are bounded. In this calculation the maximum excursion for a particle started at some particular resonance [the place in action space where $l_1\nu_1(\mathbf{J}) + l_2\nu_2(\mathbf{J}) = m$ for integer l_1, l_2, and m] is called the island width. Hence, a resonance can be illustrated in action space by a curve of varying width. A sketch of a collection of resonances and their widths is shown in Figure 1.

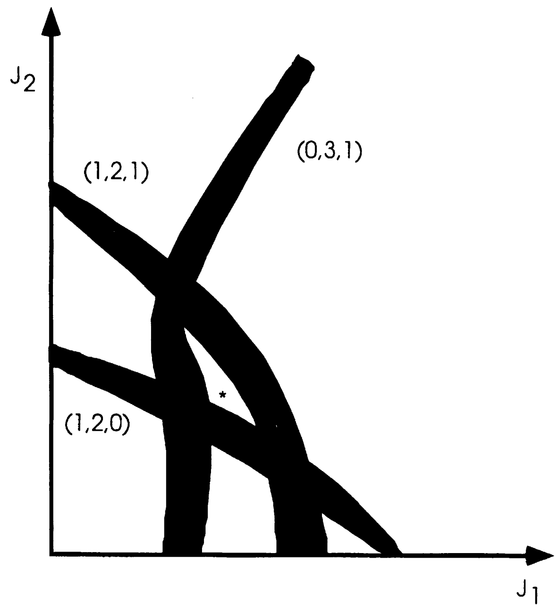

FIGURE 1 Resonances of varying width in action space. The asterisk indicates where a KAM surface might be.

The resonance width calculation predicts integrability if the resonances are separated, as they would be at the asterisk in Figure 1. Chirikov[21] proposed that once the perturbation is large enough that the above calculation would predict the overlap of two neighboring islands, the system is chaotic. Hence, there is chaotic motion at the resonance crossings, the corners of the *resonance triangle* of Figure 1.

5 MOTION NEAR FIXED POINTS

Analyzing the resonances for a nonlinear system is problematic, as typically the underlying integrable system is not known. Instead, we consider the fixed points. At each of the resonance crossings (cf. Figure 1), there are 2^N fixed points. By studying the fixed points, we will be able to deduce the strength of the resonances at those points in the action plane.

The linear stability of fixed points of four dimensional symplectic maps has been studied extensively by Howard and MacKay.[22] The linearized motion near an L^{th} order fixed point is governed by the tangent map, the derivative of the L-times composed map. The tangent map $\mathbf{M}$ is represented by a symplectic matrix. The linear stability is determined by the eigenvalues of this matrix. For symplectic matrices, if λ is an eigenvalue, then so are $1/\lambda$, λ^*, and $1/\lambda^*$. Thus, eigenvalues come in complex conjugate pairs on the unit circle ($\lambda = 1/\lambda^*$), inverse pairs on the real line ($\lambda = \lambda^*$), or complex quadruplets in other parts of the complex plane.

The eigenvalues can be found by first defining the stability index,[23]

$$\rho = \lambda + 1/\lambda,$$

for each inverse pair. Given the stability indices, which can be complex, one can solve for the inverse pair of eigenvalues. From the characteristic equation for the polynomial, it follows that the stability indices are the roots of a polynomial,

$$Q(\rho) \equiv \rho^2 - A\rho + B - 2 = 0, \tag{9}$$

where

$$A \equiv Tr(M), \tag{10}$$

and

$$B \equiv \{[Tr(M)]^2 - Tr(M^2)\}/2. \tag{11}$$

Our interest is in obtaining maps that are nearly integrable and, furthermore, have invariant surfaces that are simply nested tori. (We do not want island structures.) Integrable maps preserve vectors of the the form $(\delta\theta)$: two orbits on the same torus separated by some angles initially are always separated by those angles as the angles increase linearly in time. This implies that each canonical pair has one eigenvalue that is unity. As the generalized eigenvalues of each canonical pair are inverses, this implies that all of the solutions of the characteristic equation for the linearized map of an integrable system with simply nested tori are unity. This implies that the stability index must be 2, and so Equation (9) must have a degenerate pair of roots of value 2. This implies that the values of A and B for integrable systems are

$$A = 4, \tag{12a}$$

and

$$B = 6. \tag{12b}$$

This discussion implies that we will be able to reduce the resonance widths at the crossings by requiring the conditions (12).

6 METHOD FOR FINDING FOUR DIMENSIONAL SYMPLECTIC MAPS WITH REDUCED CHAOS

Our analysis leads us to a method for reducing the chaotic region of phase space. The fixed points corresponding to the relevant resonance pairs are found. The parameters of the Hamiltonian are varied so as to solve for the values at which the fixed-point parameters satisfy Equations (12). We now turn to a discussion of how this might be carried out.

The simplest accelerator lattices are made up entirely of dipole, quadrupole, and sextupole magnets. The dipole magnets bend the trajectories so they will close. The quadrupoles are chosen so that the central tunes are far from low order resonances. This provides a stable region of phase space for orbits having the design energy. The sextupoles are chosen so that the derivative of the central tune with energy vanishes. This prevents nearby off-energy particles from being in some low-order resonance, and so makes a sufficiently large confined region in six dimensional phase space.

If it is no longer required that the central tune avoid low-order resonances to eliminate chaotic trajectories, the standard design principles no longer apply. One might choose the central tune to have any value. In this case, the tune is chosen arbitrarily, and any chromaticity would work provided the tunes remain within the stable region. Hence, the locations and strengths of the sextupoles are varied freely within the limits of linear stability, and one attempts to solve for parameter values for which Equations (12) are satisfied for the design energy, i.e.,

$$A_j(\delta \equiv \Delta E/E_0 = 0) - 4 = 0 \tag{13a}$$

and

$$B_j(\delta \equiv \Delta E/E_0 = 0) - 6 = 0 \tag{13b}$$

are satisfied for fixed-point j. In this case we say that our *parameters* are the sextupole strengths and locations, while the *residuals*, the quantities we want to vanish, are the left sides of Equations (13).

We have no rigorous algorithm for choosing the fixed points to use in the optimization. In previous work[10,13] we have found that it is sufficient to use a few of the lowest order fixed points that are closest to resonance. Our experience in two dimensional maps is that in the minimization of the residues of the low-order fixed points, the higher-order fixed point residues do not typically increase, but instead decrease also. The explanation may be that the higher-order resonances are primarily due to nonlinear beats of the lower-order resonances.

It is straightforward to show from a perturbation analysis, that in the case where the perturbation of a given resonance is dominated by two harmonics, the conditions (12) imply that resonance amplitudes vanish. Hence, this procedure results in a diminishing of the resonance widths at the point of crossing. One could then imagine that the resonances remain large between the crossings, and so there will be chaotic trajectories near the separatrix, but that the global transport is eliminated because the chaotic trajectories may no longer move along a resonance past the crossing point. A sketch of the action plane in this case is shown in Figure 2. In fact, we expect the reduction of the resonance width at two points along a resonance to result in a reduction between, just a dragging a rope by its ends causes the middle to follow also.

Actually, many resonances cross at a given intersection. That is, many resonance will pass through the corners of the central "resonance triangle" in Figures 1–2. The analysis of the resonance structure, the relations between the A and B coefficients and the resonance amplitudes, and the dynamics in this complicated region is an area where much future research is needed.

Our procedure for chaos reduction will need to be augmented to provide for a stable range of energies, because the above procedure may lead to lattices for which slightly off-energy particles see large islands and chaotic regions. To prevent this, one could add to the list of the residuals, the quantities,

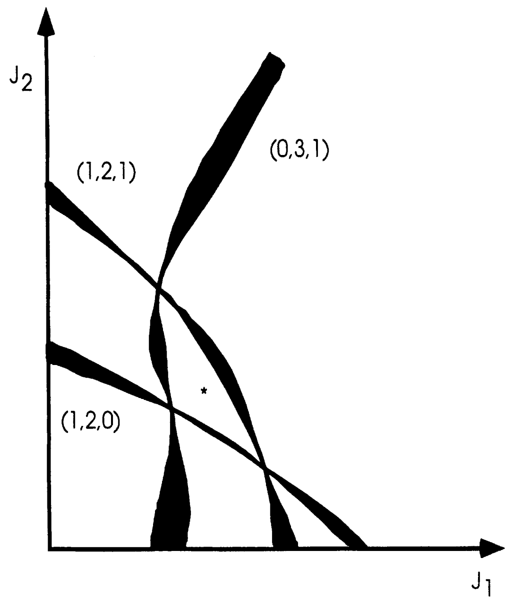

FIGURE 2 Resonances in action space when the condition (12) holds.

$$R_{Ajn} \equiv A_j(\delta_n) - 4 \tag{14a}$$

and

$$R_{Bjn} \equiv B_j(\delta_n) - 6, \tag{14b}$$

where the offsets δ_n are chosen in the range of desired stable energies. Alternatively, if only a small stable energy interval is needed, one could, in the spirit of conventional lattice design, require the vanishing of the A-chromaticity,

$$\chi_{Ajn} \equiv \frac{dA_j}{d\delta}, \tag{15a}$$

and the B-chromaticity,

$$\chi_{Bjn} \equiv \frac{dB_j}{d\delta}. \tag{15b}$$

(These chromaticities are not the usual chromaticities, but they can be related to them.)

Alternatively, one might ask whether this method can be used to eliminate troublesome resonances in existing designs. In this case the quadrupoles would be chosen to keep the same values of central tune, and the sextupoles would be chosen to make the central chromaticity vanish. This would still leave considerable freedom, in that the locations of sextupoles within a cell would be free, and the relative strengths, if there is more than one, could also be varied. In addition, one could, as was done in Ref. 13, add multipoles. This would provide two new parameters (strength and location) for each.

In the work of Ref. 13 it was found that one additional residual had to be added to the list, that of the location of the fixed points. Without such a restriction, the variation of parameters can result in the movement of the fixed point into the origin, where nonlinearity is small. (The nonlinear tune shift increases.) This can actually result in a decrease in dynamic aperture, as the more distant fixed points, which are now determining the dynamic aperture, have not been optimized.

In the implementation of this method it will be necessary to be able to find high-order fixed points of four dimensional mappings. This can be difficult, but the difficulty is reduced substantially when one is analyzing maps having inversion symmetry,[24] so that one knows that the odd-order fixed points must lie on a plane. The method could also fail if the sets of fixed points and parameters become too large. However, as mentioned before, experience in the systems of one and a half degrees of freedom has shown that the chaos is often cured by considering only a few low-order fixed points.

Regardless, should this method work, the computational requirements for accelerator design will be greatly reduced. Rather than analyzing lattice after lattice, each time with extensive tracking, it will be possible to solve for good lattices simply by following the stability properties of a few low-order fixed points. Of course, the final lattice will then need to be checked by extensive tracking analysis.

7 CONCLUSION AND FUTURE DIRECTIONS

At this stage it is difficult to draw conclusions. A method for finding four dimensional symplectic maps with reduced chaos has been proposed, but given the heuristic nature of this method, one will know whether it works only once it has been implemented. We are now in the process of implementing this method, and we hope to be able to report on the results next year.

If the method proves to be successful, it promises to be beneficial to accelerator physics in several ways. Having better control over dynamic aperture would allow one to increase the dynamic aperture when this is needed to capture (at injection) more particles or to provide a greater volume of stable orbits so that scattering is less likely to cause particle loss. Alternatively one could decrease dynamic aperture by increasing the residuals in order to trim a beam or extract particles.

Regardless, in the process of implementation we should learn more about transport in Hamiltonian systems of two and a half degrees of freedom. If we are able to diminish transport by imposing the requirements (12) for a few low-order fixed points, then it will lend credence to the picture that transport is caused by motion along resonances, and that the transport to the dynamic aperture is the result of movement along the resonance web.

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