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TRANSVERSE BEAM DYNAMICS WITH NOISE

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The linear transverse beam dynamics perturbed by magnet nonlinearities and noise is analysed, using averaging techniques.

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1 INTRODUCTION

In this paper we discuss the linear transverse beam dynamics perturbed by magnet nonlinearities and noise. This is a progress report, there is still much work to do. In Section 2, we formulate the equations of motion in a form for the analysis using averaging techniques. This yields the so-called averaged problem, which is deterministic and autonomous. The unaveraged problem, which is equivalent to the original problem, can be viewed as a perturbation of the averaged problem. Our approach has a perhaps novel feature in that it includes both the resonant and nonresonant cases in the same formalism. In Section 3, we discuss the averaged problem and introduce the idea of a coarse-grain equilibrium. We show that in the one degree-of-freedom (DOF) case the phase space density limits to a coarse-grain equilibrium in large time. Section 4 is a series of remarks on what can be proved about the exact deterministic(no noise) problem with knowledge of the averaged problem using the averaging theorem. Section 5 gives an overview of ideas from stochastic processes which will be useful in what follows. We very briefly discuss stochastic and Markov processes, the Itô stochastic differential equation and Markov diffusion processes, and the idea of weak convergence of a family of stochastic processes to an Itô diffusion. These ideas are used in

Section 6 to discuss a theory of stochastic adiabatic invariance and a 4/3 law for phase randomization. Finally, in Section 7 we apply the ideas of Section 6 to cases of tune, dipole and quadrupole noise with an octupole nonlinearity. This is brief because of space limitations.

2 TRANSVERSE EQUATIONS OF MOTION IN AVERAGING FORM

We begin with the one DOF case and then briefly indicate how to incorporate the second degree of freedom. The transverse equation of motion is written

$$x'' + k(s;\nu)x = \varepsilon g(x,s;\gamma)\,. \tag{2.1}$$

Here ν denotes the tune and we write $k(s;\nu)$, even though this is not strictly correct, in order to make explicit that given a $k(s)$ the tune ν, which is a functional of $k(s)$, is uniquely defined. The γ in the g means that g can be a random function. We will use $(\Gamma, \mathcal{A}, \mathcal{P})$ to denote the probability space, where Γ is the sample space, $\gamma \in \Gamma$ is a sample point, $\mathcal{A}$ is a class of subsets of Γ called events (also called a σ-algebra or a σ-field) and P is the probability measure on $\mathcal{A}$. In the usual convention Ω and ω are used instead of Γ and γ, however here we wish to use ω to denote frequencies. The function $k(s;\nu)$ is periodic with frequency $\omega_1 = 2\pi/C$ where C is the circumference of the machine.

Equation (2.1) can be written in system form as

$$\chi' = K(s;\nu)\chi + \varepsilon G(\chi,s;\gamma) \tag{2.2}$$

where $\chi = \binom{x}{x'}$, $K = \begin{pmatrix} 0 & 1 \\ -k & 0\end{pmatrix}$, $G = \binom{0}{g}$. For $\varepsilon = 0$, (2.2) has fundamental solution matrices $\Psi(s)$ defined by $\Psi' = K(s;\nu)\Psi$, $\det\Psi \neq 0$. We have found it convenient to use $\Psi = \Phi$, where

$$\Phi(s;\nu) := \sqrt{\beta(s)}\begin{pmatrix} 1 & 1 \\ \frac{-\alpha(s)+i}{\beta(s)} & \frac{-\alpha(s)-i}{\beta(s)}\end{pmatrix}\begin{pmatrix} e^{i\psi(s)} & 0 \\ 0 & e^{-i\psi(s)}\end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i\end{pmatrix}, \tag{2.3}$$

where $\alpha(s) := -\frac{1}{2}\beta'(s)$, $\psi(s) := \int_0^s \frac{1}{\beta(t)}dt = \psi_p(s) + 2\pi\nu s/C$, and where the last equality defines the periodic function $\psi_p(s)$ with ν defined by $2\pi\nu := \psi(C)$. We also define $\omega_2 := 2\pi\nu/C = \nu\omega_1$. The fundamental solution matrix has the Floquet decomposition

$$\Phi(s;\nu) = \Phi_p(s)e^{\mathcal{J}\omega_2 s}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.4}$$

where $\Phi_p(s)$ is the periodic matrix, with frequency ω_1, obtained from (2.3) by replacing $\psi(s)$ by $\psi_p(s)$.

We now transform (2.2) into a standard form for the method of averaging. Define the transformation from χ to u by

$$\chi = \Phi(s;\nu_0)u, \tag{2.5}$$

where ν and ν_0 are related by $\omega_2 = \nu\omega_1 = \omega_{20} + \varepsilon a = \nu_0\omega_1 + \varepsilon a$. Note that it is ν_0 in (2.5) and not ν. This trick allows us to treat near resonance and nonresonance simultaneously. If ν is near a resonant tune ν_0 then a measures the distance to resonance; if ν is nonresonant we take $\nu_0 = \nu$ (and thus $a = 0$). The transformed Equation (2.2) becomes

$$u' = \varepsilon\left[a\mathcal{J}u + \Phi^{-1}(s;\nu_0)G(\Phi(s;\nu_0)u, s;\gamma)\right]. \tag{2.6}$$

In the u coordinates, the emittance becomes

$$\gamma(s)x^2 + 2\alpha(s)x\,x' + \beta(s)x'^2 = 4(u_1^2 + u_2^2)\,. \tag{2.7}$$

Perturbations due to magnet nonlinearities, g_m, dipole noise, η_d, and quadrupole noise, η_q, enter (2.1) as follows:

$$x'' + (k(s;\nu) + \varepsilon\eta_q(s;\gamma))x = \varepsilon(g_m(x,s) + \eta_d(s;\gamma))\,.$$

The function g_m could also represent the beam-beam force. To incorporate tune noise explicitly, we note that the solution of $\chi' = K(s;\nu)\chi$ can be written $\chi(s) = \Phi_p(s)e^{\mathcal{J}\omega_2 s}\Phi_p^{-1}(0)\chi(0)$. If $\varepsilon\eta_T(s)$ is the tune noise then $\omega_2 s$ is replaced by $\omega_2 s + \varepsilon\int_0^s \eta_T(t)dt$ which leads to $\chi' = K(s;\nu)\chi + \varepsilon\eta_T(s)J(s)\chi$, where

$$J(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}.$$

Combining all the perturbations gives

$$G(\chi, s; \gamma) = \begin{pmatrix} 0 \\ g_m(\chi_1, s) \end{pmatrix} + \begin{pmatrix} 0 \\ \eta_d(s; \gamma) \end{pmatrix}$$

$$- \begin{pmatrix} 0 \\ \eta_q(s; \gamma)\chi_1 \end{pmatrix} + \eta_T(s; \gamma) J(s) \chi \,. \qquad (2.8)$$

In many examples $g_m(\chi_1, s) = e(s) D'(\chi_1)$, where $e(s)$ has period C, and (2.6) can be written

$$u' = \varepsilon \left\{ a \mathcal{J} u + \frac{1}{4} e(s) \begin{pmatrix} -D(\chi_1)_{u_2} \\ D(\chi_1)_{u_1} \end{pmatrix} + \right.$$

$$\left. \frac{1}{4} \eta_d(s) \begin{pmatrix} -(\chi_1)_{u_2} \\ (\chi_1)_{u_1} \end{pmatrix} - \frac{1}{4} \eta_q(s) \begin{pmatrix} -(\frac{1}{2}\chi_1^2)_{u_2} \\ (\frac{1}{2}\chi_1^2)_{u_1} \end{pmatrix} + \eta_T(s) \mathcal{J} u \right\}, \qquad (2.9)$$

where $\chi_1 = \chi_1(u, s) = \Phi_{11}(s; \nu_0) u_1 + \Phi_{12}(s_i \nu_0) u_2$, and the subscripts u_1 and u_2 denote partial derivatives. Note the Hamiltonian structure of (2.9). We write our basic Equation (2.9), as

$$u' = \varepsilon (a \mathcal{J} u + f(u, s) + F(u, s; \gamma)) \,. \qquad (2.10)$$

Recall that the first term on the right hand side is present if we are analyzing a near resonance case, the deterministic second term corresponds to the second term in (2.9) and the third term collects together the three random terms in (2.9). We assume that the noise terms, η, all have zero mean, thus $EF = 0$ where E denotes expected value. Note that $f(u, s)$ is quasiperiodic with two frequencies ω_1 and ω_2. If we included a quasiperiodic power supply ripple into the g_m term then f would be quasiperiodic with several frequencies. In either case we can write the Fourier expansion of f as $f(u, s) = \sum_{n \in \mathbf{Z}^p} f_n(u) e^{i(\omega \cdot n)s} = \bar{f}(u) + \tilde{f}(u, s)$ where n is a p-dimensional integer vector, $\omega = (\omega_1, \ldots, \omega_p)$ is the vector of the p frequencies, $\bar{f}(u) = \sum_{n \in \mathcal{M}} f_n(u)$ is the average of $f(u, s)$ over s for fixed u, where $\mathcal{M} = \{n \in \mathbf{Z}^p \mid \omega \cdot n = 0\}$ and $\tilde{f}$, which is defined by the second equality, is called the zero mean part of f. $\bar{f}$ is said to be resonant if $\bar{f}(u) \neq f_0(u)$.

We now define the averaged problem, in scaled time, to be

$$v' = a\mathcal{J}v + \bar{f}(v) =: f_A(v), \tag{2.11}$$

and its general solution will be denoted $v(\tau) = \varphi(\tau, v_0), \varphi(0, v_0) = v_0$. Thus the unaveraged problem (2.10) can be viewed as a perturbation of the averaged problem (2.11) with zero mean perturbations $\tilde{f}$ and F. At this point we can characterize averaging techniques, both deterministic and stochastic, as techniques for proving properties of the solutions of (2.10) (and thus (2.2)) in terms of properties of solutions of (2.11).

The generalization to two degrees of freedom is as follows. The equations are $x''+k_h(s;\nu_h)x = \varepsilon g_h(x, y, s; \gamma)$, and $y''+k_v(s;\nu_v)y = \varepsilon g_v(x, y, s; \gamma)$, which in system form becomes $\chi' = K(s;\nu)\chi + \varepsilon G(\chi, s; \gamma)$ where $\chi = (x, x', y, y')^T$, $G = (0, g_h, 0, g_v)^T$, K is block diagonal, and $\nu = (\nu_h, \nu_v)$ is the vector of horizontal and vertical tunes. The 4×4 fundamental solution matrix, $\Phi(s;\nu)$, is block diagonal with the 2×2 blocks on the diagonal of the form of (2.3) for the horizontal and vertical motions. Letting $\nu = \nu_0 + \varepsilon a$ where a and ν_0 are vectors, an equation analogous to (2.6) is obtained.

A development similar to this section can be found in Ref. 1, although the treatment of resonance has been improved here.

3 THE AVERAGED PROBLEM

The averaged problem is given in (2.11). In the case where G is a Hamiltonian perturbation the averaged problem is Hamiltonian, i.e. $f_A(v)$ is a Hamiltonian vector field.

In the one DOF Hamiltonian case, the phase plane portrait for (2.11) is easy to construct and generically the equilibria are either saddle points or centers. Around each center there is a coarse-grain equilibrium if the period of oscillation varies with amplitude. To see this introduce action-angle variables (J, θ) around the center, then $\dot{J} = 0$ and $\dot{\theta} = \omega(J)$. The ensemble density ρ evolves via $\rho(J, \theta, s) = \rho_0(J, \theta - \omega(J)s)$ where $\rho_0(J, \theta) = \rho_{00}(J) + \sum_{n\neq 0} \rho_{0n}(J)e^{in\theta}$ is the initial density. A simple stationary phase argument (using integration by parts) gives

$$\int_A \rho(J, \theta, s)dJd\theta = \int_A \rho_{00}(J)dJd\theta + 0\left(\frac{1}{s}\right)$$

as $s \to \infty$ under suitable regularity conditions, if $\omega'(J)$ is bounded away from zero. Here A can be a union of polar coordinate area elements, thus $\rho_{0\,0}(J)$ can be viewed as a coarse-grain equilibrium (note that the equilibrium density is uniform in the canonical angle). The coarse-grain equilibrium is discussed in Ref. 2 but we only recently devised this simple proof.

In the two DOF Hamiltonian case, f in (2.10) depends on three frequencies ω_1, ω_h and ω_v. The averaged problem may be integrable, as in the nonresonant case or in the case where ω_1 is rationally related to one of ω_h or ω_v, or nonintegrable. In the integrable case the averaged dynamics is easy to understand, whereas the nonintegrable case presumably requires some effort. We have taken a preliminary look at the question of coarse-grain equilibrium, but even in the integrable case, the situation is not yet clear.

4 DETERMINISTIC PERTURBATIONS: BRIEF REMARKS

In this section we consider the case where F=0 in (2.10), and discuss the question, "what can be proven about the solutions of Eq. (2.1) or equivalently the solutions of (2.10),

$$u' = \varepsilon[f_A(u) + \tilde{f}(u, s)], \; u(0) = u_0, \tag{4.1}$$

in terms of the solutions of the averaged problem (2.11)." The main tool for answering this is the avergaging theorem[1] which states that

$$\begin{aligned} \chi(s) &= \Phi(s; \nu_0)u(s) \\ &= \Phi(s; \nu_0)\left\{\varphi(\varepsilon s, u_0) + \varepsilon P(\varphi(\varepsilon s, u_0), s) + 0(\varepsilon^2 s)\right\} \end{aligned} \tag{4.2}$$

uniformly for $0 \le s \le T/\varepsilon$, where the solution of (2.11) exists on $[0, T]$ and $P(v, s) = \int_0^s \tilde{f}(v, t)dt$. Note that $P(v, 0) = 0$ and P is periodic is s. In words, the averaging theorem states that solutions of (4.1) stay close to solutions of (2.11) for times of order $1/\varepsilon$. However, this in itself is not particularly interesting from a beam dynamics point of view for two reasons: (1) It is not orbits that we want to approximate but rather the surfaces on which they evolve. Note however that $u(s)$ stays within $0(\varepsilon)$ of an invariant of the averaged problem at $0(1/\varepsilon)$ times. (2) For realistic beam dynamics perturbations $0(1/\varepsilon)$ may not be large enough. The main interest is in the applications of (4.2) and in extensions to finding approximate invariants at longer times.

To illustrate the applications, we introduce the Poincaré map in the case where $G(\chi, s)$ in (2.2) has period C in s. The flow of (2.2) with $\chi(0) = \zeta$ then defines an N-turn Poincaré map $\mathcal{P}_N$ defined from (4.2) by

$$\begin{aligned}\mathcal{P}_N(\zeta) := \chi(NC, \zeta) &= \Phi(NC; \nu_0)u(NC) \\ &= \Phi(NC; \nu_0)\left\{u_0 + \varepsilon(NCf_A(u_0) + P(u_0, NC)) + 0(\varepsilon^2)\right\}, \quad (4.3)\end{aligned}$$

where $u_0 = \Phi_P^{-1}(0)\zeta$ and we have used $\varphi(\varepsilon s, u_0) = u_0 + \varepsilon s f_A(u_0) + 0(\varepsilon^2)$ for $s = 0(1)$. If $v = v_e$ is an equilibrium solution of (2.11) then it often follows by the implicit function theorem that $\mathcal{P}_N$ has a nearby fixed point and thus (2.2) has a periodic solution. If v_e is a center then it may be possible to prove the existence of invariant tori, and thus quasiperiodic solutions of (2.2), using KAM type results.

As an elementary example consider the octupole nonlinearity with $D(\chi_1) = \frac{1}{4}\delta\chi_1^4$ and $e(s) = \sum_n e_n e^{in\omega_1 s}$, $e_n = e_{-n}$ real and $e_0 = 1$ (see (2.9)). Assume that $e(s)$ is localized near $s = 0$, so that $e(s)\Phi(s; \nu) \simeq \Phi_P(0)e^{\mathcal{J}\omega_2 s}$. If we consider the 1:4 resonance then $\nu_0 = M + \frac{1}{4}$ yields $\overline{e(s)D(\chi_1(s))} = \frac{1}{2}\delta\beta_0^2 R^4(3 + e_{4M+1}\cos 4\theta)$, where $u_1 = R\cos\theta$ and $u_2 = R\sin\theta$. This determines f_A in (2.11) (See (2.9), (2.10) and the discussion preceding (2.11). For $a > 0$ we find 4 saddle points and 4 centers in addition to the center corresponding to $v = 0$. This gives the standard phase plane portrait (see e.g. Ref. 3, p. 221). Since $\Phi(s; \nu_0)$ and $\tilde{f}(u, s)$ are 4C periodic, we have

$$\mathcal{P}_4(\zeta) = \zeta + \varepsilon g(\zeta, \varepsilon)\,; \quad g(\zeta, \varepsilon) = 4\Phi_P(0)f_A(\Phi_P^{-1}(0)\zeta) + 0(\varepsilon)\,. \quad (4.4)$$

Let $\zeta_e = \Phi_P(0)v_e$, then $g(\zeta_e, 0) = 0$ and the implicit function theorem guarantees the existence of a function $\hat{\zeta}(\varepsilon)$ with $\hat{\zeta}(0) = \zeta_e$ such that $g(\hat{\zeta}(\varepsilon), \varepsilon) = 0$ if $f_A'(v_e)$ is invertible and ε is sufficiently small. In this case $\mathcal{P}_4(\hat{\zeta}(\varepsilon)) = \hat{\zeta}(\varepsilon)$ and (2.2) has a periodic solution of period $4C$. To see that $\mathcal{P}_4(\zeta)$ is a twist map near the fixed point, we let $\zeta = \hat{\zeta}(\varepsilon) + \xi$ and define $\hat{\mathcal{P}}(\xi) := \mathcal{P}_4(\zeta) - \hat{\zeta}(\varepsilon)$ then $\hat{\mathcal{P}}(\xi) = \xi + \varepsilon 4\Phi_P(0)f_A(v_e + \Phi_P^{-1}(0)\xi) + 0(\varepsilon^2)$ which is a perturbed twist map with an $0(\varepsilon)$ twist. Since the twist enters at $0(\varepsilon)$, the Moser twist theorem[4] does not apply, however an extension (see p. 41 of Ref. 4) does apply. While we have not seen a proof of the extension in the literature, it does seem likely to be a straight forward extension of the proof in Siegel and Moser.[5] Given the existence of invariant curves in the Poincaré map, we then have the existence of invariant tori for (2.2) with their associated quasiperiodic orbits.

However, even when this can be justified for a problem of interest, one could argue that it may not be so useful because the ε in the proof of the Moser twist theorem may need to be too small. The next step might be to try for a Nekhorshev type theorem which gives results at exponentially long times in ε, for presumably larger ε than in the twist theorem, but still ε is probably too small to be useful. Another approach is to extend the averaging result of (4.2) to longer times. However only in special cases is it possible to follow orbits to longer than $0(1/\varepsilon)$ times. It is however possible to track invariants of the averaged problem to $0(1/\varepsilon^2)$ times and with much less restrictions on ε than in either the twist theorem or Nekhoroshev type results (the Nekhoroshev theorem can be viewed as the ultimate averaging theorem[6]). For example, if $I(v)$ is an invariant of (2.11), it can be shown, under certain conditions, that there exists an ε_0 such that $u(s)$ defined by (4.1) satisfies $I(u(s)) = I(u(0)) + 0(\varepsilon)$ for $0 \leq s \leq T/\varepsilon^2$ and for $0 \leq \varepsilon \leq \varepsilon_0$. This result is a special case of the theory of stochastic adiabatic invariance to be discussed in Section 6. In a specific problem ε_0 can be estimated and if say $\varepsilon_0 \simeq 10^{-3}$ then for perturbations of the order of 10^{-3} in (2.2) the orbits would stay near the invariant of the averaged problem for 10^6 turns. This is rough and needs to be investigated further if it seems likely to be useful.

If (2.11) has a coarse-grain equilibrium as discussed in Section 3, then it is natural to ask if there is a related property for (2.2). A discussion of this from the averaging point of view as well as some numerical results can be found for a one DOF example in Ref. 2. As a final remark, we note that in the case where (2.11) has a separatrix then under perturbation we expect separatrix splitting and the emergence of a stochastic layer. However this layer is probably exponentially small[7] in ε and thus is probably insignificant for beam dynamics applications.

The above example was one DOF, the analysis in two DOF is more difficult and this is under study.

5 STOCHASTIC PROCESSES

A stochastic process, $X(t, \gamma)$, on a probability space $(\Gamma, \mathcal{A}, P)$ is a function such that for each t, $X(t, \cdot)$ is a random vector. It follows that the finite dimensional distributions of the process $P(X(t_1) < x_1, \dots, X(t_n) < x_n)$ are defined and constitute the *probability law* of the process. For ease in exposition we assume X to be a scalar in this section. The two natural spaces for Γ are $C[0, T]$ and $D[0, T]$; the spaces of continuous and piecewise continuous functions. This leads to the technical problem of measures on function spaces and the idea of the convergence of such measures. A good but difficult reference for the ideas of this section is Ref. 8.

Markov processes are an important class of stochastic processes because they are a rich class of relatively well understood processes. The future depends on the past only through the present is a Mantra which roughly defines a Markov process. More precisely, if $t_1 < t_2 < \cdots < t_n$ then

$$P(X(t_n) < x_n \mid X(t_i) = x_i, i = 1, \dots, n-1) = P(X(t_n)$$
$$vps < x_n \mid X(t_{n-1}) = x_{n-1})$$

is satisfied by the conditional probabilities. It follows that the probability law is completely determined by the joint distribution $P(X(t_1) < x_1,\ X(t_2) < x_2)$ or if densities exist by the initial density $p_0(x, t)$ and the transition probability density $p(x, t|y, \tau)$. Under certain regularity conditions the transition probability density evolves by the differential Chapman-Kolmogorov equation[9]

$$\frac{\partial p(x,t|y,\tau)}{\partial t} = -\frac{\partial}{\partial x}A(x,t)p(x,t|y,\tau) + \frac{1}{2}\frac{\partial^2}{\partial x^2}B(x,t)p(x,t|y,\tau)$$
$$+ \int_{-\infty}^{\infty} dz\,[W(x|z,t)p(z,t|y,\tau) - W(z|x,t)p(x,t|y,\tau)]\,. \tag{5.1}$$

This defines a Markov jump process in $D[0, T]$ if $W \neq 0$. If $W = B = 0$, then (5.1) is the Liouville equation. If $W = 0$ it defines a diffusion process in $C[0, T]$, and (5.1) is called the Fokker-Planck equation. Under additional regularity conditions, these Markov diffusion processes are solutions of the Itô stochastic differential equation[10,11]

$$dX = A(X, t)dt + B(X, t)^{\frac{1}{2}} dW(t), \tag{5.2}$$

where $W(t)$ is standard Brownian motion. Equation (5.2) is shorthand for the integral equation $X(t) = X(t_0) + \int_{t_0}^{t} A(X(s), s)ds + \int_{t_0}^{t} B(X(s), s)^{\frac{1}{2}} dW(s)$ where the second integral is to be interpreted as an Itô integral.

Stochastic processes that arise in applications are typically not Markov processes but do often depend on a small parameter, say ε. If the process is denoted by $X(t;\varepsilon)$, then it is natural to ask if the ε-family of processes can be approximated by a Markov process for small ε. A common way to proceed is in terms of the notion of weak convergence of a family of stochastic processes. The family $X(t;\varepsilon)$ is said to converge weakly to a stochastic process $X_0(t)$ as $\varepsilon \to 0$, if (1) the finite dimensional distributions of $X(t;\varepsilon)$ converge to the finite dimensional distributions of $X_0(t)$ at all continuity points of the distributions for $X_0(t)$, (2) all continuous functionals of $X(t,\varepsilon)$ converge in distribution to the corresponding functional of $X_0(t)$. Two examples of such functionals are $\int_a^b X(t,\varepsilon)dt$ and $\sup_{a\le t\le b} X(t,\varepsilon)$. Of course, the hope is that $X_0(t)$ is a relatively simple process. In fact, we will be interested in weak convergence results when $X_0(t)$ is a Markov diffusion defined by its associated Itô stochastic differential equation or Fokker-Planck equation. In this case, weak convergence is the function space analogue of the central limit theorem for sequences of random variables satisfying a mixing condition (see Ref. 12 p. 375ff).

6 STOCHASTIC PERTURBATION THEORY

Equation (2.10) can be written

$$u' = \varepsilon(f_A(u) + \tilde{f}(u,s) + F(u,s;\gamma)), \tag{6.1}$$

where the averaged problem $v' = f_A(v)$ has been discussed in Section 3, $\tilde{f}$ is quasiperiodic in s of zero mean, and F is a random field (with $\gamma \in \Gamma$) with $E\,F(u,s) = 0$ and satisfies a so-called mixing condition. The precise conditions on f_A, $\tilde{f}$ and F will not be stated, they can be found in the references. The mixing condition on F is a condition on the rate at which $F(u,s;\gamma)$ and $F(u,s+t;\gamma)$ become independent as t increases. For example, if $F(u,s;\gamma) = \eta(s;\gamma)\hat{F}(u,s)$ where $\hat{F}$ is deterministic and η is weakly stationary then the mixing condition will require that the covariance $K(t) = E(\eta(s)\eta(s+t))$ be such that $K(t)$ approach zero sufficiently fast as $t \to \infty$.

The first result[13] is that

$$u(s,\varepsilon) \simeq \varphi(\varepsilon s, u_0) + \sqrt{\varepsilon}Y_0(\varepsilon s;\gamma) \tag{6.2}$$

for $0 \leq \varepsilon s \leq T$, where φ is defined after (2.11) and Y_0 is the solution of the Itô stochastic differential equation $dY_0 = f'_A(\varphi(\tau, u_0))Y_0 d\tau + \sigma(\varphi(\tau, u_0))^{\frac{1}{2}} dW(\tau)$; $Y_0(0) = 0$. Here $\sigma(v) = \lim_{\ell\to\infty} \frac{1}{\ell} \int_0^\ell \int_0^\ell E(F(v, s; \gamma) F^T(v, r; \gamma))dsdr$. Thus Y_0 is a Gauss-Markov process, completely determined by $\varphi(\tau, u_0)$. The symbol $\simeq$ in (6.2) and in the following means approximately in the sense of weak convergence; more precisely, let $Y_\varepsilon(\tau; \gamma) := (u(\tau/\varepsilon, \varepsilon) - \varphi(\tau, u_0))/\sqrt{\varepsilon}$ then $Y_\varepsilon(\tau; \gamma)$ converges weakly to $Y_0(\tau; \gamma)$ for $0 \leq \tau \leq T$. In words, (6.2) says that for ε sufficiently small the motion defined by (6.1) follows the motion of the averaged (deterministic) flow with a Gauss-Markov correction on $0(1/\varepsilon)$ time intervals.

The theory of stochastic adiabatic invariants[14] (SAI) extends this to $0(1/\varepsilon^2)$ time intervals but focuses on the evolution of invariants of the averaged problem rather than solutions of (6.1). Suppose $y = I(v)$ is an invariant for the averaged problem, that is $I'(v)f_A(v) = 0$. Let $y(s, \varepsilon) = I(u(s, \varepsilon))$ then

$$y' = \varepsilon[I'(u)\tilde{f}(u, s) + I'(u)F(u, s; \gamma)]. \tag{6.3}$$

In Ref. 14, it is shown, under conditions of regularity, mixing and ergodicity, that $y(\tau/\varepsilon^2, \varepsilon)$ converges weakly to a Markov-diffusion process for $0 \leq \tau \leq T$. That is

$$y(s, \varepsilon) = I(u(s, \varepsilon)) \simeq J(\varepsilon^2 s) \tag{6.4}$$

for $0 \leq \varepsilon^2 s \leq T$, where $J(\tau)$ satisfies the Itô stochastic differential equation $dJ = \bar{\mu}(J)d\tau + \bar{\Sigma}(J)^{\frac{1}{2}} dW(\tau)$ and $\bar{\mu}$ and $\bar{\Sigma}$ are defined below. Thus at times of $0(1/\varepsilon^2)$ the SAI theory implies that $y(s, \varepsilon)$ behaves like a Markov diffusion process for sufficiently small ε.

First the drift and diffusion are obtained as functions of u: The drift term is $\hat{\mu}(u) = \lim_{\ell\to\infty} \frac{1}{\ell} \int_0^\ell \int_0^s \{\tilde{\mu}_d(u, s, r) + \tilde{\mu}_r(u, s, r)\}drds$ where $\tilde{\mu}_d(u, s, r) = [\frac{\partial}{\partial u}(I'(u)\tilde{f}(u, s))]\tilde{f}(u, r)$ and $\tilde{\mu}_r(u, s, r) = E\{[\frac{\partial}{\partial u}(I'(u)F(u, s))]F(u, r))\}$. The diffusion term is $\hat{\Sigma}(u) = I'(u)\sigma(u)I'(u)^T$ where σ is defined after (6.2). The $\bar{\mu}$ and $\bar{\Sigma}$ are obtained as ergodic averages of $\hat{\mu}$ and $\hat{\Sigma}$ as follows. Let $h = h(u)$ then its ergodic average is defined by $\bar{h}(I(u)) = \lim_{\ell\to\infty} \frac{1}{\ell} \int_0^\ell h(\varphi(\tau, u))d\tau$. Condition (A6) of Ref. 14 requires the ergodic averages of $\hat{\mu}$ and $\hat{\Sigma}$ to exist uniformly in u. This typically holds for 1 DOF systems but not in higher dimensions and (A6) is replaced by (A6′)

which relaxes the uniformity condition but adds a condition on the stochastic perturbation which is difficult to verify in practice.

A companion result to the SAI theory, a 4/3 law for phase randomization, is presented in Ref. 15. Suppose there is a third process, in addition to u and y, defined by

$$\theta' = \varepsilon[\nu(y) + \tilde{h}(u, s) + H(u, s; \gamma)] \tag{6.5}$$

with $EH = 0$. Then under certain conditions (A1 - A6 of Ref. 15)

$$y(s, \varepsilon) \simeq y_0 + \varepsilon^{1/3}\bar{\Sigma}(y_0)^{1/2}W(\varepsilon^{4/3}s),$$
$$\theta(s, \varepsilon) \simeq \theta_0 + \varepsilon s \nu(y_0) + \nu'(y_0)\bar{\Sigma}^{1/2}(y_0) \int_0^{\varepsilon^{4/3}s} W(\tau)d\tau\,, \tag{6.6}$$

for $0 \le \varepsilon^{4/3}s \le T$. Thus at times of $0(\varepsilon^{-4/3})$, y behaves like Brownian motion and θ behaves as in the averaged problem with a Gauss-Markov correction. If θ is an angle and if $\nu'(y_0)\bar{\Sigma}(y_0)\nu'^T(y_0)$ is positive definite then θ mod 2π becomes uniform at times $s = 0(1/\varepsilon^{4/3})$ and there is phase randomization at these times. We have proven phase randomization under weaker conditions (A1 - A4, A7 of Ref. 15) and this may help to simplify condition (A6′) of Ref. 14.

The applications in the next section should help clarify these ideas.

7 DIPOLE, QUADRUPOLE AND TUNE NOISE

We discuss the one DOF case with an octupole nonlinearity and make brief remarks on the two DOF case. We are reasonably sure the deterministic part of $\hat{\mu}$, which results from $\tilde{\mu}_d$, is zero and will assume this in the following.

In the one DOF nonresonant case the averaged vector field, f_A, can be derived from the Hamiltonian $-3/2\delta\beta_0^2(v_1^2 + v_2^2)^2$. The phase plane portrait is a one parameter family of circles and thus $I(v) = \frac{1}{2}(v_1^2 + v_2^2)$ is an invariant. Since the period is not constant, there is a coarse-grain equilibrium for the averaged problem. For tune noise, we find that $\hat{\mu}(u) = \hat{\Sigma}(u) = 0$ and the ergodic averaging condition of the SAI theory is trivially satisfied. The 4/3 law does not give phase randomization at $0(\varepsilon^{-4/3})$ times because

$\bar{\Sigma} = 0$, although (6.6) does hold. The SAI theory implies that $I(u(s))$ remains essentially constant at $0(1/\varepsilon^2)$ times and thus from (2.7) so does the emittance. We are studying the extension of this to longer times in the case where g_m represents the beam-beam force.[16] To illustrate the dipole and quadrupole noise cases, we take $\eta(s) = \hat{\eta}(s;\gamma)e(s)$, where $\hat{\eta}(s;\gamma)$ is weakly stationary, and $e(s) = \sum_n e_n e^{i\omega_1 s}$ is localized at $s = 0$ so that $e(s)\beta(s) \simeq e(s)\beta(0)$. Let $E(\hat{\eta}(s)\hat{\eta}(r)) = K(s-r)$ and $e_n = e_{-n}$ be real. After a lengthy calculation we find

$$\hat{\Sigma}_d(u) = \frac{1}{4}\beta_0 I(u) \int_0^\infty K_d(s)\hat{e}(s)\cos\omega_2 s ds$$

and $$\hat{\Sigma}_q(u) = \frac{1}{4}\beta_0 I^2(u) \int_0^\infty K_q(s)\hat{e}(s)\cos 2\omega_2 s ds,$$

where $\hat{e}(s) = \sum_n e_n^2 e^{i\omega_1 ns}$. Because $\hat{\Sigma}(u)$ depends on u only through I the ergodic averaging condition is again trivially satisfied and $\bar{\Sigma}(I(u)) = \hat{\Sigma}(u)$. Furthermore $\bar{\mu}(I) = \frac{1}{2}\bar{\Sigma}'(I)$. Thus the 4/3 law gives phase randomization at $0(\varepsilon^{-4/3})$ times and from the SAI theory $I(u(s))$ behaves like a Markov diffusion at $0(1/\varepsilon^2)$ times. The Fokker-Planck equation

$$\frac{\partial\rho}{\partial\tau} = \frac{1}{2}\frac{\partial}{\partial J}\bar{\Sigma}(J)\frac{\partial}{\partial J}\rho, \quad \rho(\tau, J_b) = 0,$$

in each case, can be solved analytically as discussed in Ref. 16.

In the 1:4 resonance case we choose the parameters such that the averaged vector field is given by the Hamiltonian $I(v) = \frac{1}{2}(v_1^2 + v_2^2) - \frac{1}{4}(v_1^4 + v_2^4)$ and we also choose the Hamiltonian as our invariant. For tune noise $\eta_T(s) = \eta(s)e(s)$, with η weakly stationary, we find $\hat{\Sigma}(u) = u_1^2 u_2^2(u_2^2 - u_1^2)\bar{K}$ where $\bar{K} = \lim_{\ell\to\infty} \frac{1}{\ell}\int_0^\ell \int_0^\ell K(s-r)e(s)e(r)dsdr$. Thus $\bar{\Sigma}(I(v)) = \frac{1}{T(v)}\int_0^{T(v)} \hat{\Sigma}(\varphi(s, v))ds$ off the separatrix and zero on the separatrix, where $T(v)$ is the period associated with the phase point v. There is a similar calculation for $\bar{\mu}(I)$. Off the separatrix, the 4/3 law gives phase randomization and the SAI theory implies that the invariant for the unperturbed problem evolves like a diffusion at $0(1/\varepsilon^2)$ times. The associated Fokker-Planck equation $\frac{\partial \rho}{\partial s} = \frac{\partial}{\partial J}\bar{\mu}(J)\rho + \frac{1}{2}\frac{\partial^2}{\partial J^2}\bar{\Sigma}(J)\rho$ is easily solved numerically (see Ref. 17). The basic problem left to resolve is the dynamics near the separatrix where neither the 4/3 law nor the SAI theory apply.

There is much work to do on the 2-DOF case. Here we just mention some small results in the octupole nonresonant case. In this case the averaged problem essentially uncouples the transverse vertical and horizontal dynamics and the motion is quasiperiodic (on two-tori) with coupling only in the frequencies. We choose $I = (I_h, I_v)$ where $I_h(u) = \frac{1}{2}(u_1^2+u_2^2)$, $I_v(u) = \frac{1}{2}(u_3^2 + u_4^2)$ as the two invariants. The tune noise case is trivial, as in the one DOF case, since $\hat{\mu}(u) = \hat{\Sigma}(u) = 0$; thus there is no phase randomization at $0(\varepsilon^{-4/3})$ times and $I(u(s))$ is essentially constant at $0(\varepsilon^{-2})$ times. We are studying the extension of this to longer times in the beam-beam case.[16] In the dipole stationary noise case, analogous to the one DOF case, we have found that $\hat{\Sigma}(u) = \bar{\Sigma}(I(u))$ so that ergodic averaging is trivial and the 4/3 law and SAI theory apply. However, here we have assumed that the horizontal and vertical components of the noise are uncorrelated, which is presumably unphysical. For more general noise, we expect that $\hat{\Sigma}(u)$ will not be a function of I and this will lead to a problem with the verification of the ergodic averaging condition in the SAI theory because of resonant tori in the unperturbed problem. This in addition to the resonant cases in the octupole dynamics represent future work.

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