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FOCUSING IN DIFFERENT MODELS OF COLD OVERDENSE PLASMA-ELECTRON BUNCH SYSTEM*

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Focusing of relativistic electron (positron) bunches is considered in three different descriptions of cold overdense plasma-rigid electron bunch system. In all three models Coulomb component of field exists but for large values of the bunch Lorentz-factor it is negligible in comparison with the wake field component. Total charge and current densities in general are not compensated. For narrow bunches they are nearly proportional to each other. The resulting focusing force is a complex combination of magnetic and electric forces, whose relative strength depends on bunch parameters. The obtained results in case of narrow bunches are practically independent from the considered models. The general formulae for focusing force are obtained, which can be used for estimates in the planned experiments. Particular cases of narrow, short and long bunches are discussed and focusing gradients are calculated for the experiments performed at ANL, Tokyo University-KEK and UCLA.

KEY WORDS: Next linear colliders, plasma lens, cold neutral overdense plasma, linear approximation, focusing gradient, field, plasma potential flow

1 INTRODUCTION

In the last decade some of the most important discoveries and systematic studies in elementary particle physics have been made at electron-positron colliders. Future (or next) linear colliders will have the c.m. energy 0, 5–1, 0 TeV. At such energies the cross-section for producing μ-pairs is 87 fb and luminocity $L = 10^{33}\text{cm}^{-2}\text{s}^{-1}$ would give 7, 5 μ-pairs per day. So, for producing pointlike particles at such energies the luminocity in the $10^{33} \div 10^{34}\text{cm}^{-2}\text{s}^{-1}$ range is needed. This demanding requirement dominates high energy linear collider design. In particular, the beam vertical size must be of order 1–10 nm.[1,2] Beam height in Final Focus Test Beam (FFTB) project at SLAC energy 50 GeV is 60 nm and this goal is achieved using the system of ordinary quadrupoles and sextupoles.[3] The

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alternative proposals are based on the use of strong transverse fields in plasma, generated by ultrarelativistic electron (positron) bunches.[4–14]

In present work the focusing force is obtained in three different models of overdense cold plasma-rigid relativistic electron (positron) bunch system:

a) model with the generalized vorticity equal to zero;[13,14]

b) model with strong external longitudinal magnetic field, applied along the direction of beam velocity;[12]

c) more general model, considered in[20] and later on in[12].

In cases (a) and (b) the cylindrical electron bunch with a Gaussian distribution of charge in longitudinal direction and parabolic distribution in transverse direction is considered. In the case (c) the flat bunch with the uniform distribution of charge is discussed. In linear approximation the general formulae for focusing force are obtained and the results of different models compared to each other, especially in the most interesting case of narrow beams. Obtained results may be used for the optimization of the conditions of the future experiments.[3,11]

It is possible to use linear approximation in considered overdense plasma regime. The experiments carried out up to now[16–19] also are devoted to this regime. Theoretical treatment of underdense regime needs completely different approach, due to nonlinear character of the phenomena.

All the calculations are based on the rigid electron bunch model. In reality emerging transverse and longitudinal forces changes the bunch charge distribution and adopted model is valid for the finite time intervals, when these changes are negligible. The complete description needs selfconsistent treatment of the plasma-nonrigid electron bunch system, which is performed analytically for one dimensional case.[21] As a result the length of the plasma column l must be limited by $l < v_0\Gamma^{-1} = \frac{2v_0}{3^{1/2}}(\frac{\gamma}{\omega_p})(\frac{2n_0}{n_b})^{1/3}$ in order to escape the development of the instabilities in the beam-plasma system (v_0 – velocity of the beam; ω_p – plasma frequency, n_0, n_b densities of the plasma and bunch electrons respectively, γ-Lorentz factor of the bunch). The length l of the plasma column in experiments did not exist a few meters and this condition practically fulfilled for large values of Lorentz factor γ.

Stability of driving bunch also considered in [23] by 2D simulation. Although the beam is subject of self-focusing (for narrow bunches), filamentation and two-stream instability (for wide bunches) authors of [22–23] find that bunch can be stabilized by introducing on axial B_0 field. This was also one of the reasons for considering the model (b) in present investigation.

2 MODEL WITH A POTENTIAL FLOW OF THE PLASMA

We consider the cylindrical electron (positron) bunch with the Gaussian distribution of charge density in horizontal direction and parabolic distribution in vertical direction

$$n_b(\tilde{z}, r) = \begin{cases} n_{b0}\left(1 - \dfrac{r^2}{a^2}\right)\exp(-\tilde{z}^2/2\sigma^2), & r \le a; \\ 0 \quad , & r > a. \end{cases} \tag{1}$$

where a is the radius of the bunch, $2(2^{1/2})\sigma = d$ – the horizontal dimension of the bunch, n_{b0} – electron density at the middle of the bunch ($\tilde{z} = 0, r = 0$) and we consider the steady state $\tilde{z} = z - v_0 t$ of the electron bunch-plasma system. The electron bunch is moving in cold plasma (with the fixed velocity $\mathbf{v}_0(0, 0, v_0)$), ions are immobile and plasma is neutral in equilibrium. Using the system of the Maxwell equations for electromagnetic fields, generated by bunch and plasma electrons and plasma ions, and hydrodynamic equations for cold neutral plasma electrons, assuming that generalized vorticity for plasma electrons is zero.[13,14]

$$\text{rot}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) = 0, \quad \text{rot rot}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) = 0, \tag{2}$$

where $\mathbf{p}$ is the momentum of the plasma electrons and $\mathbf{A}$ – vector potential of the electromagnetic field, we derive the following nonlinear equation for dimensionless momenta $\vec{\rho} = \vec{p}/mc$ of the plasma electrons:

$$\begin{aligned}\left(\vec{\nabla}\vec{\nabla} - \nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{\rho} + \frac{\vec{\rho}}{(1+\rho^2)^{1/2}}\left[\beta^2 k_p^2 (1 - \frac{n_b}{n_0}) + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\vec{\rho} + \right. \\ \left. + \nabla^2(1+\rho^2)^{1/2}\right] + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}(1+\rho^2)^{1/2} = -\beta^2\frac{\vec{v}_o}{c}k_p^2\frac{n_b}{n_0},\end{aligned} \tag{3}$$

where $k_p = \omega_p/v_0 = 2\pi/\lambda_p$, $\quad \omega_p = (4\pi e^2 n_0/m)^{1/2}$ – is the plasma frequency, $\beta = v_0/c$. In linear approximation, which we shall explore later on, the condition (2) is fulfilled due to Faraday's law:

$$\frac{\partial \mathbf{p}}{\partial t} = -e\mathbf{E}, \quad \frac{\partial}{\partial t} rot\, \mathbf{p} = -e\, rot\, \mathbf{E} = \frac{e}{c}\frac{\partial \mathbf{B}}{\partial t} = \frac{e}{c}\frac{\partial}{\partial t} rot\, \mathbf{A}, \quad \frac{\partial}{\partial t}\left[rot\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)\right] = 0 \tag{2$'$}$$

For $z \to \infty$, $\mathbf{p}$ and $\mathbf{A}$ are equal to zero, so (2) is fulfilled for arbitrary instant of time.

We shall consider the excitation of the axial symmetric E-wave with the field components E_z, E_r, B_θ and components of velocity of the plasma electrons v_z, v_r differ from zero.

In cylindrical coordinates the linearized system of equations for the momenta of the plasma electrons is

$$\begin{aligned}&\frac{\partial^2 \rho_r}{\partial \tilde{z}\partial r} + \frac{1}{r}\frac{\partial \rho_r}{\partial \tilde{z}} - \frac{\partial^2 \rho_z}{\partial r^2} - \frac{1}{r}\frac{\partial \rho_z}{\partial r} + \beta^2\frac{\partial^2 \rho_z}{\partial \tilde{z}^2} + \beta^2 k_p^2 \rho_z = -\beta^2 k_p^2 \frac{n_b}{n_0}, \\ &\frac{\partial^2 \rho_z}{\partial \tilde{z}\partial r} - \frac{1}{\gamma^2}\frac{\partial^2 \rho_r}{\partial \tilde{z}^2} + \beta^2 k_p^2 \rho_r = 0,\end{aligned} \tag{4}$$

where $\gamma = (1 - \beta^2)^{-1/2}$ and $n_b \ll n_0$.

Performing Fourier transformation on $\tilde{z}$, assuming that $\rho_{z,r}$ and $\partial\rho_{z,r}/\partial\tilde{z}$ tend to zero when $\tilde{z} \to +\infty$, and excluding the Fourier transformant $\rho_r(\lambda, r)$ from the obtained system of equations, we came to the equation for $\rho_z(\lambda, r) = \frac{-ie}{mc^2\beta}\lambda E_z(r, \lambda)$, which coincides with the subsequent Eq. (9) from the work.[22] Using the equation of motion and Faraday's law

$$E_r = \frac{mcv_0}{e}\frac{\partial \rho_r}{\partial \tilde{z}}, \quad B_\theta = \frac{mc^2}{e}\left(\frac{\partial \rho_r}{\partial \tilde{z}} - \frac{\partial \rho_z}{\partial r}\right) \tag{5}$$

we come to the following expression for the radial force acting on the bunch electrons

$$\begin{aligned} f_r &= -e(E_r - \beta B_\theta) = -mc^2\beta\frac{\partial \rho_z}{\partial r} = \\ &= 2^{1/2}\pi^{-1/2} m k_p^2 v_0^2 \sigma \frac{n_{b0}}{n_0} \int\limits_{-\infty}^{\infty} \frac{\exp(-(\lambda\sigma)^2/2 + i\lambda\tilde{z})}{\lambda^2 - k_p^2} \times \\ &\times \left[\frac{r}{a^2} - \kappa I_1(\kappa r) K_2(\kappa a)\right] d\lambda, \end{aligned} \tag{6}$$

where I_1, K_2 are the modified Bessel functions, $\kappa = k_p(\beta^2 + \lambda^2/(\gamma k_p)^2)^{1/2}$, $\mathrm{Im}\lambda > 0$. The obtained expression for f_r differs from the subsequent expression from the work,[4] due to some additional approximations explored in [4] (v_0 is set equal to c at the beginning, Gaussian distribution on $\tilde{z}$ approximated by parabolic one). One should note that the radial force tends to zero when $a \to \infty$. Later on we consider the most interesting case of narrow cylindrical bunch when $k_p a \ll 1$.

In this case it is possible to divide the integral contained the product $I_1(\kappa r)K_2(\kappa r)$ in two integrals in limits $|\lambda| \leq \lambda_1$ and $|\lambda| \geq \lambda_1$, and using in the first integral the expansion of $I_1(\kappa r)K_2(\kappa a)$ for small values of the argument and noticing that the second integral is exponentially small, we come to the following expression for radial force $f_r(\tilde{z}, r)$, acting on electrons of the narrow bunch

$$\begin{aligned} f_r(\tilde{z}, r) &= 2mv_0\omega_p\frac{n_{b0}}{n_0}\left\{\left(\frac{\pi}{2}\right)^{1/2}\frac{\sigma r}{a^2}\exp(-(k_p\sigma)^2/2)\times\right. \\ &\times\left[1 - \delta(1 - \frac{(k_p a)^2}{4}(1 - \frac{r^2}{2a^2}))\right]\times \\ &\times\left[\sin(k_p|\tilde{z}|)\left(1 - \mathrm{Re\ erf}(\frac{|\tilde{z}|}{2^{1/2}\sigma} - i\frac{k_p\sigma}{2^{1/2}})\right) + \right. \\ &\left. + \cos(k_p|\tilde{z}|)\mathrm{Im\ erf}(\frac{|\tilde{z}|}{2^{1/2}\sigma} - i\frac{k_p\sigma}{2^{1/2}})\right] + \\ &\left. + \frac{k_p r}{4\gamma^2}\delta\left(1 - \frac{r^2}{2a^2}\right)\exp(-\tilde{z}^2/2\sigma^2) - f^w(\tilde{z}, r)\right\}, \end{aligned} \tag{7}$$

where

$$f^w(\tilde{z},r) = (2\pi)^{1/2}\frac{\sigma r}{a^2}\exp(-(k_p\sigma)^2/2)\,\sin(k_p|\tilde{z}|)\times$$
$$\times\left\{1-\mathrm{erf}\left(\frac{k_p\sigma x_1}{2^{1/2}}\right)\left(1-\frac{(k_pa)^2}{4}(1-\frac{r^2}{2a^2})\right)\right\}\theta(\tilde{z}), \tag{8}$$

$$\theta(\tilde{z}) = \begin{cases} 1, & \tilde{z}<0,\\ 0, & \tilde{z}>0,\end{cases}$$

and δ is the real part of the probability integral

$$\delta = \mathrm{Re}\,\mathrm{erf}\left(\frac{k_p\sigma x_1}{2^{1/2}} - i\frac{|\tilde{z}|}{2^{1/2}\sigma}\right), \tag{9}$$

where $x_1 = \lambda_1/k_p$ and $x_1 \ll \gamma/k_pa$. Using the known expression for the probability integral of the complex variable[15], one can obtain from (6) compact expression for the radial force of the long and short bunches. In the case of the long bunch $k_p\sigma \gg 1$, $a \ll v_0/\omega_p \ll d/8^{1/2}$ when $\tilde{z} \gg 2^{1/2}\sigma$, $\tilde{z} > 0$ we have exponentialy small defocusing force

$$f_r \simeq m\omega_p v_0\frac{n_{b0}}{n_0}\frac{k_pr}{2\gamma^2}\left(1-\frac{r^2}{2a^2}\right)\exp(-\tilde{z}^2/2\sigma^2). \tag{10}$$

In the far rare part of the bunch we have an oscillating force due to the wake-field generated by the bunch

$$f_r \simeq -2mv_0\omega_p\frac{n_{b0}}{n_0}f^w(\tilde{z},r). \tag{11}$$

In the middle of the long bunch $|\tilde{z}| \ll 2^{1/2}\sigma$ the force has three components — defocusing, focusing and oscillating:

$$f_r \simeq mv_0\omega_p\frac{n_{b0}}{n_0}\left[\frac{k_pr}{2}\left(1-\frac{r^2}{2a^2}\right)\left(\frac{1}{\gamma^2}-\frac{k_p\sigma}{(2\pi)^{1/2}}\times\right.\right.$$
$$\times\sum_{n=1}^{\infty}\frac{1}{n}\exp(-(k_p\sigma/2^{1/2}-n/2)^2-n^2/4)- \tag{12}$$
$$\left.\left.-\frac{(k_p\sigma)^2}{2\pi^{1/2}}\exp(-(k_p\sigma)^2/2)\frac{\sin(k_p|\tilde{z}|)}{(k_p|\tilde{z}|)}\right)\right].$$

In the short bunch case $k_p\sigma \ll 1$, $a \ll d/8^{1/2} \ll v_0/\omega_p$, when $\tilde{z} \gg 2^{1/2}\sigma$, the force tends to zero and when $\tilde{z} < 0$, $|\tilde{z}| \gg 2^{1/2}\sigma$ the main contribution to force gives the wake-field component:

$$f_r \approx -2(2\pi)^{1/2}mv_0\omega_p\frac{n_{b0}}{n_0}\frac{\sigma r}{a^2}\sin(k_p|\tilde{z}|). \tag{13}$$

For $k_p\sigma \ll 1,\ |\tilde{z}| \leq 2^{1/2}\sigma$ we have

$$f_r = mv_0\omega_p \frac{n_{b0}}{n_0}(k_pr)(k_p\sigma)\left[\frac{1}{(2\pi)^{1/2}\gamma^2}\left(1-\frac{r^2}{2a^2}\right)-\frac{2.5}{\pi^{1/2}}\frac{\sigma}{k_pa^2}\right] \qquad (14)$$

The value of the forces given by the expressions (13) and (14) is mainly determined by the factor $n_{b0}^{1/2}(n_{b0}/n_0)^{1/2}$ which could be large.

Now we shall give some numerical estimates of the gradient of the focusing force using the data from experiments at ANL,[8] KEK-Tokyo University,[10,16] UCLA[17] and also from the proposal for FFTB experiment at SLAC[11] for overdense case.

For the long bunches $k_p\sigma \gg 1$, when $|\tilde{z}| \ll 2^{1/2}\sigma$ (the middle of the bunch) focusing force is given by (12), and in this case for the parameters of ANL and UCLA experiments the magnitude of focusing gradient $G = |f_r/er|$ is equal to several hundred G/cm (10 kG/m). Focusing length f for thin lens ($f \gg l, l$ — is the length of lens) is (see e.g. [7]) $f \approx \frac{1}{kl} = \frac{\gamma mc^2}{eGl}$ and for UCLA experiment $\gamma = 7.5, l \approx 7.5$ cm and $G \approx 100G/\text{cm}$, $f \approx 18$ cm which practically coincides with obtained experimental value.[17] For the FFTB experiment we get $G \sim 0$ due to the factors $1/\gamma^2$ and $\exp(-(k_p\sigma)^2/2)$ entered in the expression (12). The short bunches $k_p\sigma \ll 1$ were used in the KEK-Tokyo University experiment and the field gradient G from (13) is equal $G \approx 2.4$ kG/cm for conditions of this experiment. In the case of FFTB experiment it is possible to increase essentially the field gradient G by decreasing the bunch length. For example, if the parameters of the FFTB experiment in the overdense case may be chosen as $n_0 = 2\cdot 10^{17}\text{cm}^{-3},\ n_{b0} = 5.3\cdot 10^{16}\text{cm}^{-3},\ \sigma = a = 4.7\cdot 10^{-4}$ cm field gradient from (13) and (14) is $G \sim 4$ GG/cm (the length of the bunch is smaller than the length of wave $\lambda_p = 2\pi/k_p \approx 7.8\cdot 10^{-3}$ cm).

3 MODEL WITH STRONG EXTERNAL LONGITUDINAL MAGNETIC FIELD

Now let us consider the model with the strong external magnetic field $\mathbf{B}(0, 0, B_0)$ applied along the bunch motion. The strength of this constant magnetic field should satisfy the condition that Larmor radius of the plasma electrons must be smaller than plasma wave length and/or bunch transverse dimensions. In this case the plasma electrons have only the longitudinal component of the velocity which is different from zero.

One should note also that in the experiments carried out at ANL[18] and KEK[16,19] for wake field generation, plasma chambers have a constant solenoidal magnetic field in order to confine the plasma column.

Starting, as in the previous section, with the Maxwell and hydrodynamics equations, introducing scalar φ and vector $\mathbf{A}(0, 0, A_z)$ potentials, obeying the Lorentz condition, we came to the following linearized equations for the potential $\varphi(\tilde{z}, r)$[12]

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{\gamma^2}\frac{\partial^2\varphi}{\partial\tilde{z}^2}+\frac{k_p^2}{\gamma^2}\varphi = \frac{m}{e}\frac{n_{b0}}{n_0}k_p^2v_0^2\exp(-\tilde{z}^2/2\sigma^2)f(r), \qquad (15)$$

where $f(r) = 1-r^2/a^2$ and $n_{b0}/n_0 \ll 1$ (linear approximation). The boundary conditions for potential φ are $\varphi \to 0,\ \partial\varphi/\partial\tilde{z} \to 0$ when $\tilde{z} \to +\infty$.

Using the Fourier transformation on $\tilde{z}$ for potential φ, the solution of the equation (15) can be written as the following:

$$\varphi(r,\tilde{z}) = \frac{2^{3/2} m\pi^{-1/2}}{e}\frac{n_{b0}}{n_0}k_p^2 v_0^2 \sigma\gamma^2 \left\{ \int_0^{k_p} \frac{1}{k_p^2-\lambda^2}\left[\frac{\pi}{2}N_2\left(\frac{a}{\gamma}(k_p^2-\lambda^2)^{1/2}\right)\times\right.\right.$$

$$\left.\times J_0\left(\frac{r}{\gamma}(k_p^2-\lambda^2)^{1/2}\right)+\frac{1}{2}(1-\frac{r^2}{a^2})+\frac{2\gamma^2}{a^2(k_p^2-\lambda^2)}\right]\exp(-(\lambda\sigma)^2/2)\cos(\lambda\tilde{z})d\lambda-$$

$$-\int_{k_p}^{\infty}\frac{1}{\lambda^2-k_p^2}\left[K_2\left(\frac{a}{\gamma}(\lambda^2-k_p^2)^{1/2}\right)I_0\left(\frac{r}{\gamma}(\lambda^2-k_p^2)^{1/2}\right)+\frac{1}{2}\left(1-\frac{r^2}{a^2}\right)-\right.$$

$$\left.\left.-\frac{2\gamma^2}{a^2(\lambda^2-k_p^2)}\right]\exp(-(\lambda\sigma)^2/2)\cos(\lambda\tilde{z})d\lambda\right\}, \tag{16}$$

where $J_0,\ N_2$ are the Bessel functions of the first and second kind, and $I_0,\ K_2$ are modified Bessel functions, $\operatorname{Im}\lambda > 0$. When $a \to \infty$ the (16) coincides with the potential of one-dimensional problem.

The field components are:

$$E_z = -\frac{1}{\gamma^2}\frac{\partial\varphi}{\partial\tilde{z}}, \quad E_r = -\frac{\partial\varphi}{\partial r}, \quad B_\theta = \beta E_r$$

and radial force f_r , acting on bunch electrons is

$$f_r = -eE_r + e\beta B_\theta = -\frac{eE_r}{\gamma^2} = \frac{e}{\gamma^2}\frac{\partial\varphi}{\partial r} =$$

$$= 2^{3/2}m\pi^{-1/2}\frac{n_{b0}}{n_0}k_p^2v_0^2\sigma\left\{\frac{r}{2a^2}\int_{-\infty}^{\infty}\frac{\exp(-(\lambda\sigma)^2/2+i\lambda\tilde{z})}{\lambda^2-k_p^2}d\lambda-\right.$$

$$-\frac{\pi}{2\gamma}\int_0^{k_p}N_2\left(\frac{a}{\gamma}(k_p^2-\lambda^2)^{1/2}\right)J_1\left(\frac{r}{\gamma}(k_p^2-\lambda^2)^{1/2}\right)\times \tag{17}$$

$$\times(k_p^2-\lambda^2)^{-1/2}\exp(-(\lambda\sigma)^2/2)\cos(\lambda\tilde{z})d\lambda-$$

$$-\frac{1}{\gamma}\int_{k_p}^{\infty}K_2\left(\frac{a}{\gamma}(\lambda^2-k_p^2)^{1/2}\right)I_1\left(\frac{r}{\gamma}(\lambda^2-k_p^2)^{1/2}\right)\times$$

$$\left.\times(\lambda^2-k_p^2)^{-1/2}\exp(-(\lambda\sigma)^2/2)\cos(\lambda\tilde{z})d\lambda\right\}.$$

The comparison of the expression (17) with the consequent expression (6) for f_r in the model described in previous point shows evident differences.

In particular, the product $I_1 K_2$ in (17) has a different argument and integral is taken in different limits of integration; expression (17) has also an additional term with the product $J_1 N_2$. The value of the focusing force (17) depends on Lorentz factor.

Integrals with the products of the Bessel function, which enter in (17), are impossible in general to calculate analytically. But it is possible to carry out the approximate calculations for the case of small and large values of the arguments of the Bessel function.

Let us start with the case when $k_p r/\gamma \leq k_p a/\gamma \ll 1$. In this case in the first integral it is possible to expand the product $N_2 J_1$ on this small parameter in all range of integration $(0, k_p)$; in the second integral we expand the product $K_2 I_1$ in interval $(k_p, k_p x_1)$, where x_1 is in the region $1 < x_1 \ll \gamma/(k_p a)$ and carry out the integration. It is possible to show that the main contribution in the second integral comes from the vicinity of lower limit k_p and remaining part of the second integral is negligible. Obtained by this way resulting expression for radial force coincides with the expression (7).

This coincidence is connected with the similarity of the relations between the charge and current densities in both models $j_z = v_0 q$, which in considered case (model b) is exact. In the case considered in previous section 2 (model a) from continuity equation

$$\frac{\partial}{\partial \tilde{z}}(j_z - v_0 q) = -\frac{1}{r}\frac{\partial}{\partial r}(r j_r), \tag{18}$$

where $j_r = n_e v_r$ and from the condition of narrow bunches $k_p a \sim v_r/v_0 \ll 1$, we have $j_z \approx v_0 q$. So in both cases the same relation between the z-component of current density and charge density take place.

Let us consider now the cases of long and short "narrow" bunches.

In the case of the long bunch $k_p \sigma \gg 1$, $a \ll v_0\gamma/\omega_p \ll d\gamma/2^{3/2}$ and when we have exponentially small defocusing force (see (10)). In the far rare part of the bunch we have oscillating force due to the wake field generated by the bunch (see (11)). In the middle of the long bunch the force has three components — defocusing, oscillating and focusing, which coincides with expression (12) of model (a).

In the case of the short bunch, when $k_p \sigma \ll 1$, but $k_p \sigma x_1 \gg 1$ we have $a \ll d\gamma/2^{3/2} \ll v_0\gamma/\omega_p$, $2^{3/2} v_0/\omega_p d \ll x_1 \ll v_0\gamma/\omega_p a$ and the results of the previous consideration when $k_p \sigma \gg 1$ are valid if we change the factor $exp(-(k_p\sigma)^2/2)$ on 1.

In the case of the short wide bunch when, $k_p \sigma \ll 1$, $k_p \sigma x_1 \ll 1$ then $d\gamma/2^{3/2} \ll a \lll v_0\gamma/\omega_p$, $1 < x_1 \ll v_0\gamma/\omega_p a$ and the force in the middle of the bunch $|\tilde{z}| \ll 2^{1/2}\sigma$ is defined on formulae (14) of the model a). When $|\tilde{z}| \gg 2^{1/2}\sigma$ exponential decreasing of parts of force take place and on the rare part of the bunch only the force due to the wake field remains finite (see (13)).

When the condition $k_p a/\gamma \gg 1$ is fulfilled then the integral in (17) containing product $N_2 J_1$ of two Bessel functions is small due to the rapid oscillation of Bessel functions, and integral containing the product $K_2 I_1$ is also small due to the exponential decrease of Bessel function. Resulting expression for radial force is

$$f_r \simeq (2\pi)^{1/2} m\omega_p v_0 \frac{n_{b0}}{n_0}\frac{\sigma r}{a^2}\exp(-(k_p\sigma)^2/2)\left\{\left(1-\mathrm{erf}\frac{|\tilde z|}{2^{1/2}\sigma}-\right.\right.$$

$$\left. -\frac{\sigma\exp(-\tilde z^2/2\sigma^2)}{2^{1/2}\pi|\tilde z|}-\frac{2^{3/2}|\tilde z|}{\pi\sigma}\exp(-\tilde z^2/2\sigma^2)\sum_{n=1}^{\infty}\frac{\exp(-n^2/2)}{n^2+2\tilde z^2/\sigma^2}\right)\times \qquad (19)$$

$$\left.\times\sin(k_p|\tilde z|)-\frac{2}{\pi}\exp(-\tilde z^2/2\sigma^2)\sum_{n=1}^{\infty}\frac{\exp(-n^2/2)}{n^2+2\tilde z^2/\sigma^2}n\,\mathrm{sh}\frac{nk_p\sigma)}{2^{1/2}}\right\},$$

which tends to zero when $a\to\infty$.

4 MORE GENERAL MODEL

Following the ideas of the work[20] with some modification performed in [12] the problem of the focusing of the flat beam with the uniform fixed distribution of the charge is considered. As in the preceding section, electron bunch is moving with the velocity $v_0(0,0,v_0)$ in a cold neutral plasma with immobile ions. The horizontal dimension of the bunch $2a$ is much longer than vertical dimension $2b$ and longitudinal dimension is $2d$. Plasma electrons have the velocity $\mathbf{v}_e(0, v_{ey}, v_{ez})$, electric field is $\mathbf{E}(0, E_y, E_z)$ and magnetic field is $\mathbf{B}(B,0,0)$. A steady state case when all physical quantities are functions of y and $\tilde z = z - v_0 t$, is considered. Using relativistic equation of motion for plasma electrons and Maxwell equation, introducing the dimensionless variables

$$t'=\omega_p t,\quad y'=k_p y,\quad z'=k_p z,\quad \mathbf{p}'=mc\mathbf{p},$$
$$\mathbf{v}'=\frac{\mathbf{v}}{c},\quad n'=\frac{n}{n_0},\quad \mathbf{j}'=\frac{\mathbf{j}}{cen_0}, \qquad (20)$$
$$\mathbf{E}=\frac{mc\omega_p}{e}\mathbf{E}',\quad \mathbf{B}=\frac{mc\omega_p}{e}B',\quad \omega_p^2=\frac{4\pi e^2 n_0}{m},\quad k_p=\frac{\omega_p}{c},$$

we obtain the system of the nonlinear equations, presented in [12].

Introducing a new variables by a modified Breizman-Tajima-Fisher-Chebotaev transformation:

$$V_z=\frac{v_z'}{\beta-v_z'},\quad V_y=\frac{v_y'}{\beta-v_z'},\quad N=n'(\beta-v_z'),\quad \beta=\frac{v_0}{c} \qquad (21)$$

we linearize obtained nonlinear system of equation more consistently, than it was done in [12,20].

By decomposition:

$$N=1+\epsilon N_1+\epsilon^2 N_2+\dots,\qquad E_z=\epsilon E_{z1}+\epsilon^2 E_{z2}+\dots,$$
$$V_y=\epsilon V_{y1}+\epsilon^2 V_{y2}+\dots,\qquad E_y=\epsilon E_{y1}+\epsilon^2 E_{y2}+\dots, \qquad (22)$$
$$V_z=\epsilon V_{z1}+\epsilon^2 V_{z2}+\dots,\qquad B=\epsilon B_1+\epsilon^2 B_2+\dots$$

(we omit the superscripts "prime" in what follows; $\epsilon = n_b/n_0$, where n_b is the bunch charge density; we include in what follows the quantity ϵ in N_1, V_{y1}, etc) we came at the first (linear) approximation to the following system of equations

$$
\begin{aligned}
&1.\ \frac{\partial N_1}{\partial \tilde{z}} = \frac{\partial V_{y1}}{\partial y} \\
&2.\ \frac{\partial V_{y1}}{\partial \tilde{z}} = \frac{1}{\beta^2}E_{y1}; \qquad 3.\ \frac{\partial V_{z1}}{\partial \tilde{z}} = \frac{1}{\beta^2}E_{z1}; \\
&4.\ \frac{\partial B_1}{\partial y} = \beta V_{z1} + \beta\frac{n_b}{n_0} + \beta\frac{\partial E_{z1}}{\partial \tilde{z}}; \qquad 5.\ \frac{\partial}{\partial \tilde{z}}(B_1 + \beta E_{y1}) = -\beta V_{y1}; \\
&6.\ \frac{\partial}{\partial \tilde{z}}(\beta B_1 + E_{y1}) = \frac{\partial E_{z1}}{\partial y}; \qquad 7.\ \frac{\partial E_{z1}}{\partial \tilde{z}} + \frac{\partial E_{y1}}{\partial y} = \frac{n_b}{n_0} - (N_1 + V_{z1}).
\end{aligned}
\tag{23}
$$

The continuity equation (23.1) follows from Maxwell equations, but for the convenience we use it explicitly.

Introducing the "potentials" Φ_1 and Ψ_1 by expressions:

$$E_{z1} = -\frac{\partial \Phi_1}{\partial \tilde{z}}, \quad N_1 = \frac{1}{\beta^2}(\Phi_1 - \Psi_1) \tag{24}$$

and omitting the subscript "1" from equations (23.1, 3–6) we have

$$\frac{\partial^2 \Phi}{\partial \tilde{z}^2} + \gamma^2\frac{\partial^2 \Phi}{\partial y^2} + \frac{1}{\beta^2}\Phi = \frac{n_b}{n_0} + \gamma^2 N_1, \tag{25}$$

or

$$\frac{\partial^2 \Phi}{\partial \tilde{z}^2} + \gamma^2\frac{\partial^2 \Phi}{\partial y^2} - \gamma^2\Phi = \frac{n_b}{n_0} - \frac{\gamma^2}{\beta^2}\Psi. \tag{26}$$

From equations (1–3, 7) in (23) we have

$$\frac{\partial^2 \Psi}{\partial \tilde{z}^2} + \frac{1}{\beta^2}\Psi = \frac{n_b}{n_0} \tag{27}$$

Solving equation (27) and putting the solution in (26), we shall have inhomogeneous equation for Φ, which we shall solve using Fourier transformation on y and the technique described previously in [12]. Here we emphasize only that we use the solutions of equations (24, 25), which are continuous on the boundaries of the bunch at $\tilde{z} = \pm d$ and tend to zero when $\tilde{z} \to +\infty$. The wake-field behind the bunch has a form of the trace, concentrated in the region $|y| < b$, $|\tilde{z}| < d$ and charge density has a discontinuity at $y = \pm b$. The magnetic field, associated with the wake is zero, as a consequence of the absence of the energy flow along the wake. Some of the obtained results are the following.

The Coulomb (nonperiodic) component exists. Ahead of the bunch and behind it at large distances from the bunch Coulomb component is proportional to

$$\exp(-\gamma k_p|\tilde{z}| - k_p|y|) \quad |\tilde{z}| \gg d, \quad |y| \gg b,$$

i.e. the screening of the Coulomb field exists, but it differs in $\tilde{z}$ and y directions by factor γ. The range of the screening is not a Debye length (we consider the cold plasma in hydrodynamic description), but a Langmuir wave length in y-direction and $\lambda_p/(2\pi\gamma)$ in $\tilde{z}$-direction.

Inside the beam, the obtained expression for Coulomb component of the potential Φ is more complicated, and it is inverse proportional to γ^2. Considering the large values of Lorentz-factor, it is possible to neglect the Coulomb component Φ^c, leaving wake-field component Φ^w, and obtain the compact expression for focusing force inside the bunch

$$f_y = -E_y - \beta B_1 = \frac{\partial \Phi}{\partial y} \approx \frac{\partial \Phi^w}{\partial y}, \quad \Phi = \Phi^c + \Phi^w. \tag{28}$$

(We use equation (23.6) and the definition of the potential Φ).

For narrow bunches $k_p b < 1$ we use the decomposition of the functions entering in expression for Φ^w and get (in Gauss units):

$$f_y = -2\pi e^2 n_b y \left(1 - \frac{4}{3\pi}(k_p b)\right)\left(1 - \cos(k_p(\tilde{z} - d))\right), \tag{29}$$

$$-d \leq \tilde{z} \leq d, \quad -b \leq y \leq b,$$

and for the field gradient:

$$G = \frac{|f_y|}{e|y|} = 2\pi n_b e \left(1 - \frac{4}{3\pi}(k_p b)\right)\left(1 - \cos(k_p(\tilde{z} - d))\right). \tag{30}$$

Notice that at the initial part of the bunch ($\tilde{z} \approx d$) the focusing force (29) is small, so the Coulomb component, which is defocusing, is essential.

The obtained value by the order of magnitude is equal to that, which is possible to obtain, using simple estimate, based on the picture of completely neutralized charge and non-neutralized current[7] for the flat bunch

$$G = 4\pi n_b e \quad . \quad (\beta = \frac{v_0}{c} \approx 1);$$

For focusing strength for overdense plasma lens, considered in present paper, we have $K = \frac{eG}{\gamma mc^2} = \frac{2\pi n_b e^2}{\gamma mc^2}\left[1 - \frac{4}{3\pi}(k_p b)\right]$, which somewhat differs from expressions (2.2), (2.3) used in [11] for FFTB parameter studies.

For example, for plasma lens experiment at SLAC FFTB[11] $\gamma = 10^5$ and it is possible to neglect nonperiodic component of the field. For the overdense plasma lens assuming the uniform distribution of the bunch charge, when $k_p > 1.8 \cdot 10^2 \div 10^3 \quad \mathrm{cm}^{-1}$, $n_0 > n_b$, $n_b = 10^{16} \div 10^{18}\,\mathrm{cm}^{-3}$, $b = 10^{-4} \div 10^{-5}\,\mathrm{cm}$ and if $k_p b \ll 1$, it is possible to use (30) with the maximum value

$$G = 4\pi n_b e \approx (6 \cdot 10^7 \div 6 \cdot 10^9)\frac{\mathrm{G}}{\mathrm{cm}}.$$

We have for the total charge and current densities inside the narrow bunch:

$$
\begin{aligned}
q &= en_0\left(1-\frac{n_b}{n_0}-\frac{n_e}{n_0}\right) = -e\left(n_b-\frac{n_0}{\beta^2}\Psi\right) = -en_b\cos(k_p(d-\tilde{z})),\\
j_z &= -\beta\left(\frac{n_b}{n_0}+\frac{1}{\beta^2}\Phi\right) = -en_bc\left[1+\frac{2(k_pb)}{\pi}\left(1-\cos(k_p(d-\tilde{z}))\right)\right],\\
&\quad \beta\simeq 1, \quad -d\le\tilde{z}\le d, \quad |y|\le b.
\end{aligned}
\tag{31}
$$

So the current and charge in general are not compensated and the resulting focusing force is the complex combination of magnetic and electric forces. For short bunches (or in the initial part of long bunches), when $k_p(d-\tilde{z}) \ll 1,\ j_z \approx -en_bc \approx qc$ i.e. we have the same relation between charge and current densities as in the previous two models.

Expressions for the focusing force (28) depend essentially on the shape and charge distribution in the bunch. Notice also that linear focusing force (on y) is possible to obtain only in case of narrow beam $k_pb \ll 1$; in cases of medium and wide beams the obtained expressions for the force are nonlinear on y.

5 CONCLUSIONS

1. In linear approximation the general formulae are obtained for the focusing force for three different discriptions of the overdense cold plasma-rigid electron bunch system (see Introduction). The results obtained in models (a) and (b) for the narrow bunch coincide.

2. In all three models the Coulomb component of the field exists. In case (c) it was shown that Coulomb field is screened on the distances equal to Langmuir length in transverse direction and the same quantity divided by γ-factor in longitudinal direction. In all three models for moderate values of the Lorentz-factor the Coulomb component is essential and its effect is always defocusing. Defocusing Coulomb component is essential also in that regions of the bunch, where focusing wake field component is small, e.g. near the bunch front. For the large values of Lorentz-factor in all three models it is possible to neglect Coulomb component, leaving only wake field component in the parts of the bunch, where wake component is essential for focusing.

3. In all three models the charge and current densities are in general not compensated. For narrow (and short) bunches charge and current densities approximately related by simple expression $j_z \approx v_0q$. Focusing force is a complex combination of electric and magnetic forces, which relative strength depends on the shape and parameters of the bunch.

4. The distinct feature of the model (b) is a dependence of the results and applicability conditions on Lorentz-factor of the beam. Maybe it will be possible to use this feature for measuring the Lorentz-factor of the beam.

5. The focusing force is linear on transverse variable for narrow bunch case in all models. In case of uniform charge distribution it is periodic on longitudinal variable, with the period equal to Langmuir length. In cases (a) and (b), when Gaussian distribution of bunch charge is adopted the z-dependence of the focusing force is complicated. The last property may be nonessential for focusing of the driven bunch in wake-field of considered driving bunch, when the length of driven bunch is much smaller than Langmuir length of the plasma.

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REFERENCES

1. G. Loew, *Comparison of Parameters, Techniques and Costs of Different Proposals*, Proc. of ECFA Workshop on e^+e^- – Linear Colliders, ed. R. Settles, MPI-PhE/93–14, ECFA-93–15, v. 2, 481.
2. R.H. Siemann, *Linear Colliders:The Last Ten Years and Next Ten Years*, SLAC-PUB-6417, Jan. 1994; presented at the Sessler Symposium on Beam Physics, LBL, Berkeley, CA, Dec. 6, 1993.
3. D. Burke, *The Final Focus Test Beam Project*, Proc. of ECFA Workshop on e^+e^- – Linear Colliders, ed. R. Settles, MPI-PhE/93–14, ECFA-93–15, v.1, 285.
4. P. Chen, *Part. Acc.*, **20**, 71, (1985).
5. P. Chen, J.J. Su, T. Katsouleas, S. Wilks, J.D. Dawson, IEEE Trans. Plasma Sc., **PS-15**, 218, (1987).
6. P. Chen, SLAC-PUB-5186, Sept. 1990.
7. J.J. Su, T. Katsouleas, J.D. Dawson, R. Fedele, *Phys. Rev.*, **A41**, 3321, (1990).
8. J.B. Rosenzweig *et al.*, *Phys. Fluids*, **B2**, 1376, (1990).
9. D.H. Whithum, A.M. Sessler, J.J. Stewart, J.J. Su, Preprint LBL-25759-Rev(2), June 30, 1989.
10. H. Nakanishi *et al.*, *Phys. Rev. Lett.*, **6B**, 1870, (1991).
11. B. Barletta *et al.*, Proc of PAC-93, Washington, D.C., 1993.
12. A.Ts. Amatuni, S.S. Elbakian, E.V. Sekhpossian, Preprint LBL-34836, UC-144, Nov. 1993; Sov. J. of Contemp. Phys., Allerton Press, N.Y., **25**, N6, 1, (1990).
13. W.B. Mori, T. Katsouleas, Proc. of EPAC-90, Nice-Paris, v.1, 603, 1990.
14. L.D. Landau, E.M. Livshits, *Hydrodynamics*, Nauka, 1986.
15. M. Abramovitz, I.A. Stegun, *Handbook of Mathematical Functions*, ed., NBS USA, (1984).
16. A. Ogata, KEK-preprint 92–77, 1992.
17. G. Hairapetian *et al.*, *Phys. Rev. Lett.*, **72**, 2403, (1994).
18. J.B. Simpson, NIM B40/41, 908, (1989).
19. H. Nakanishi *et al.*, *Part. Acc.*, **32**, 203, (1990).
20. B.N. Breizman, T. Tajima, P.L. Fisher, P.Z. Chebotaev, INS-preprint, Univ. of Texas at Austin, 1993.
21. A.A. Rukhadze, L.S. Bogdankevich, S.E. Rosinskiy, G. Rukhlin *Physics of high current relativistic electron bunches*, §15, (in Russian), Atomizdat, M. 1980.
22. R. Keinings and M.E. Jones, *Phys. Fluids*, **30**, 252, (1987).
23. J.J. Su, T. Katsouleas, J.M. Dawson, P. Chen, M. Jones and R. Keinings, IEEE Trans. on Plasma Sc., PS-15, N2, 192, (1987).