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SEMI-ANALYTICAL COMPUTATION OF THE ELECTROMAGNETIC FIELD IN A CIRCULAR WAKE FIELD TRANSFORMER

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When a charged particle beam moves through the pipes and cavities of an accelerator the currents excite electromagnetic fields and waves. These fields are called wake fields and are undesired in conventional accelerators as nowadays synchrotrons. The main idea of a wake field transformer is to induce a high wake field by a beam of high intensity but low energy per particle (called the drive beam) accelerating a second beam with low intensity but high energy gain per particle. A geometry suitable for a wake field accelerator is a beam widened to a ring passing a thin pill-box cavity (that is a cylinder with a circular base) (Voss, Weiland 1982). For an idealized device (the apertures for the passage of the beam are neglected in order to simplify the computation) and for a special charge distribution of the moving ring the field is computed analytically. The field is derived from the potentials. The only non–zero component, the longitudinal one, of the vector potential is computed with the help of the method of Green's function. The scalar potential is computed with help of the Lorentz gauge. The resulting series expansions are summed by computer. Results for fields and wake potentials generated by a narrow ring and by a ring of realistic length are shown in several figures.

KEY WORDS: Electromagnetic field calculations, wakefields

1 INTRODUCTION

When a bunched beam of charged particles moves through metallic beam pipes and cavities of an accelerator, an electromagnetic field, the wake field is induced. Especially when a bunch passes near a metallic edge (at a sudden transition from a narrow pipe to a wide cavity for example) a shock wave emerges. In conventional accelerators as nowadays synchrotrons the wake fields are a nuisance, because they can lead to instabilities of the trajectories of the beam particles by reaction forces and can lower the beam energy.

The main idea of a wake field transformer[1,2] is to use a high wake field produced with a first beam, called the drive beam, to accelerate a second beam passing the device thereafter.

A device suitable for a wake field transformer is a stack of pill-box cavities (see Figure 1) with a ring-shaped drive beam.

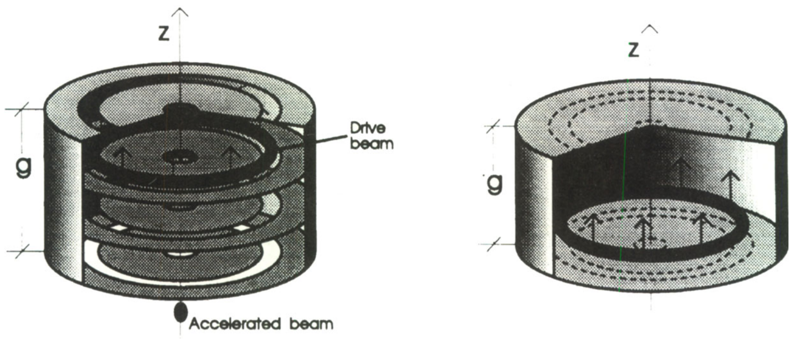

FIGURE 1: Stack of pill-box cavities with a ring-shaped drive beam suitable as Wake Field Transformer, (a) real device, (b) idealized model

The beam to be accelerated passes along the axis. When the drive beam enters and leaves one cell shock waves emerge moving radially outward and inward. With appropriately chosen dimensions the reflection of the shock wave starting outwards at the entrance of the drive beam and the shock wave starting inwards at the exit of the beam superpose.

In this report the electromagnetic field induced by the ring-shaped drive beam in a model of a circular wake field transformer is computed analytically. In this model the beam ports were neglected in order to keep a reasonable degree of complexity of the problem. Of course this main restriction of the study presented disregards some important properties of the structure; such a model omits effects of diffraction at the corners of the beam ports as well as fields reflected or diffracted by spaces beyond the slits. But nevertheless the results should give a guide line to what happens in the real device.

For the computation of wake potentials similar problems have been investigated. In the scope of studies for the Electron Ring Accelerator[7] Morton and Neil[8] have considered also a ring moving through a pill-box cavity. Since they did not compute the electromagnetic field but the energy radiated into the cavity, they could deal with an infinitely thin ring. Weiland and Zotter[9] have considered a bunch with a longitudinal Gaussian distribution as in this report, but there the charge is concentrated on the axis and passes the pill-box cavity at the speed of light. Their representation for the longitudinal electric field can be deduced from the results of this report.

In section 2 it is shown, that this problem can be solved with the help of potentials, whereby only the component parallel to the axis of the vector potential is non-zero. In order to take advantage of the symmetry of the problem appropriate cylindrical coordinates are introduced. In section 3 the Green's function used for the computation of the longitudinal component of the vector potential is derived. In section 4 the charge distribution of the moving ring is assumed to be the product of an unspecified radial and an unspecified longitudinal function. The vector potential is computed with the Green's function and from it the scalar potential is derived using the Lorentz gauge. The field is derived from the potentials and with the help of the longitudinal component of the electric field the wake potential is computed. The results for the potentials, the field components and the wake potential are series expansions involving integrals over the yet unspecified functions. In section 5 the charge distribution is the product of a Gaussian distribution in the longitudinal direction and a rectangular function in the radial direction. In section 6 the resulting double series expansions are summed by computer and the resulting numerical results are shown in several figures.

2 MATHEMATICAL DESCRIPTION OF THE MODEL

In order to take advantage of the problem, appropriate cylindrical coordinates (r,φ,z) are used, whereby the z-axis and the axis of the device coincide. The bottom and the top of the pill-box cavity with radius R and height g are given by $z = 0$, g and the outer cylindrical surface is given by $r = R$. Due to the symmetry of the problem neither the components of the field nor the potentials being introduced below depend on the azimuth φ; furthermore only the components B_φ, E_r and E_z of the field are non-zero. Assuming the metallic surface of the pill-box to be perfectly conducting the parallel component of the electric field on the bounding surface must vanish. Thus one obtains for the boundary conditions

$$E_z(R, z, t) = 0 \quad \text{and} \tag{1}$$

$$E_r(r, z = 0 \text{ or } g, t) = 0 \tag{2}$$

for the electric field. The boundary conditions for the magnetic field (the component normal to the ideally conducting bounding surface must vanish) are automatically fulfilled since B_r and B_z vanish due to the symmetry.

The charge distribution of the ring moving with speed v is assumed to be the product of the total charge Q and of two functions $\xi(r)\,\lambda(z)$ depending on the radius r, the longitudinal coordinate z respectively; where the functions are normalized in the sense that

$$\int_{-\infty}^{\infty} dz\,\lambda(z) = 1 \quad \text{and} \quad \int_0^R dr\;2\pi r\,\xi(r) \;=\; 1.$$

The current distribution due to the moving ring is:

$$j_z(r, z, t) = Q\,v\,\lambda(z - vt)\,\xi(r)\,, \qquad \vec{j}(r, z, t) \;=\; j_z(r, z, t)\,\vec{e}_z. \tag{3}$$

In section 5 the charge distribution of the moving ring is specialized to be the product of a Gaussian in the longitudinal coordinate and a rectangular function in radial direction. Then one obtains for the current distribution:

$$j_z(r, z, t) = \frac{Q\,v}{\pi(b^2 - a^2)\,\sqrt{2\pi}\,\sigma}\;\Theta(r-a)\;\Theta(b-r)\;e^{-\frac{(z-vt)^2}{2\sigma^2}} \tag{4}$$

A standard method for solving Maxwell's equations is to derive the electromagnetic field from the vector potential $\vec{A}$ and the scalar potential Φ:

$$\vec{B}(r,z,t) = rot\ \vec{A}(r,z,t) \tag{5}$$

$$\vec{E}(r,z,t) = -grad\ \Phi(r,z,t) - \frac{\partial}{\partial t}\vec{A}(r,z,t).$$

Introducing the Lorentz gauge $div\ \vec{A} + (1/c^2)(\partial\Phi/\partial t) = 0$ one obtains from Maxwell's equations the wave equation

$$\left\{\Delta\vec{A}(r,z,t) - \frac{1}{c^2}\frac{\partial^2\vec{A}(r,z,t)}{\partial t^2}\right\}_z = \left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] A_z(r,z,t) \tag{6}$$

$$= -\mu_0 j_z(r,z,t) \tag{7}$$

$$\vec{A}(r,z,t) = A_z(r,z,t)\,\vec{e}_z$$

to be fulfilled by the vector potential. The scalar potential is computed with the help of the Lorentz gauge:

$$\Phi(r,z,t) = -\frac{1}{\epsilon_0\mu_0}\int_{-\infty}^{t} dt'\ \mathrm{div}\ \vec{A}(r,z,t') = -\frac{1}{\epsilon_0\mu_0}\int_{-\infty}^{t} dt'\ \frac{\partial A_z(r,z,t')}{\partial z}. \tag{8}$$

One can show, that the boundary conditions (1) and (2) for the field are fulfilled, when one introduces

$$A_z(R,z,t) = 0 \quad \text{and} \quad \left.\frac{\partial A_z(r,z,t)}{\partial z}\right|_{z=0,g} = 0 \tag{9}$$

to be the boundary conditions for the z-component of the vector potential. With this choice also the expression $\partial A_z/\partial z$ vanishes everywhere on the boundary. Thus the scalar potential $\Phi(r,z,t)$ computed with Eq. (8) also vanishes everywhere on the bounding surface. The non-vanishing components of Eqs. (5) become:

$$B_\varphi(r,z,t) = -\frac{\partial A_z(r,z,t)}{\partial r}, \tag{10}$$

$$E_r(r,z,t) = -\frac{\partial\Phi(r,z,t)}{\partial r} \quad \text{and} \tag{11}$$

$$E_z(r,z,t) = \frac{\partial A_z(r,z,t)}{\partial t} - \frac{\partial\Phi(r,z,t)}{\partial z}. \tag{12}$$

One is interested in the accelerating effect of the wake field onto a test charge; i.e. in the work done by the longitudinal electric field acting onto the test charge moving parallel to the longitudinal coordinate z. It is a good approximation to assume, that the test charge moves with constant velocity a distance s behind the (maximum of) the drive beam. Then the test charge passes the point z at time $(s+z)/v$. One defines the longitudinal wake potential to be:

$$W_{\|}(s) := \int_0^g dz\; E_z\left(r, z, \frac{z+s}{v}\right). \tag{13}$$

3 GREEN'S FUNCTION FOR THE Z-COMPONENT OF THE VECTOR POTENTIAL

The Green's function to be used for the computation of the z-component of the vector potential is defined by the differential equation

$$\left[\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(r, z, t; r', z', t') = -\frac{\delta(r-r')}{2\pi r'}\,\delta(z-z')\,\delta(t-t') \tag{14}$$

together with the boundary conditions

$$\left.\frac{\partial G(r, z, t; r', z', t')}{\partial z}\right|_{z=0,\,g} = 0 \quad \text{and}$$

$$G(R, z, t; r', z', t') = 0. \tag{15}$$

corresponding to Eq. (9). In addition causality is required and thus the Green's function must fulfill the initial condition

$$G(r, z, t; r', z', t') = 0 \quad \text{for} \quad t < t'. \tag{16}$$

For the computation of the Green's function the ansatz

$$G(r, z, t; r', z', t') = \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} \tilde{G}_{pn}(t; r', z', t')\,\cos\left(p\frac{\pi}{g}z\right)\,J_0\left(j_n\frac{r}{R}\right) \tag{17}$$

leading to the "Oscillatory Mode representation" (Here the nomenclature given by Felsen[4] is used) is used; i.e. the Green's function is expanded in the radial direction, the longitudinal direction respectively as a Fourier–Bessel series, as an eigenfunction expansion in $\cos(p\pi z/g)$ respectively. j_n denotes the nth zero of the Bessel function J_0. The boundary conditions are automatically fulfilled by this representation. In order to satisfy the causality condition (16) one requires

$$\tilde{G}_{pn}(t; r', z', t') = 0 \quad \text{for} \quad t < t'. \tag{18}$$

The right hand side of the differential equation (14) which is to solve is expanded w.r.t. the same systems of eigenfunction. At first the Dirac distribution $\delta(x - x')$ in the interval [0, 1] is written as a Fourier–Bessel series with the help of Eqs. (1) and (2) in section 18.1 of Watson's book:[6]

$$\delta(x-x') = \sum_{n=1}^{\infty} a_n\, J_0(j_n x) \;=\; 2x' \sum_{n=1}^{\infty} \frac{J_0(j_n x')\, J_0(j_n x)}{J_1^2(j_n)}$$

$$a_n = \frac{2}{J_1^2(j_n)} \int_0^1 t\, \delta(t-x')\, J_0(j_n t)\, dt \;=\; 2x' \frac{J_0(j_n x')}{J_1^2(j_n)}.$$

The above relation is used to find the expansion

$$\frac{\delta(r-r')}{2\pi r'} = \frac{1}{2\pi r' R}\, \delta\left(\frac{r}{R} - \frac{r'}{R}\right) \;=\; \frac{1}{R^2\pi} \sum_{n=1}^{\infty} \frac{1}{J_1^2(j_n)}\, J_0\left(j_n \frac{r'}{R}\right) J_0\left(j_n \frac{r}{R}\right).$$

in the interval $[0, R]$. For the expansion of $\delta(z-z')$ with $z, z' \in [0, g]$ the transformation

$$\delta(z-z') = \delta(z-z') + \delta(z+z') \;=\; \frac{1}{2g} \sum_{p=-\infty}^{\infty} \left[e^{-ip\frac{\pi}{g}z'} + e^{ip\frac{\pi}{g}z'}\right] e^{ip\frac{\pi}{g}z}$$

$$= \frac{1}{g} \sum_{p=-\infty}^{\infty} \cos\left(p\frac{\pi}{g}z'\right) e^{ip\frac{\pi}{g}z} \;=\; \frac{1}{g} \sum_{p=0}^{\infty} (2-\delta_{p0}) \cos\left(p\frac{\pi}{g}z'\right) \cos\left(p\frac{\pi}{g}z\right)$$

is used. When the ansatz for the Green's function and the appropriate expansions for the right hand side are inserted into the differential equation (14) one obtains:

$$\frac{\partial^2 \tilde{G}_{np}(t; r', z', t')}{\partial t^2} + \omega_{np}^2 \tilde{G}_{np}(t; r', z', t') = \frac{c^2}{R^2 \pi g} \frac{(2-\delta_{p0})}{J_1^2(j_n)}\, J_0\left(j_n \frac{r'}{R}\right) \times$$
$$\times \cos\left(p\frac{\pi}{g}z'\right) \delta(t-t')$$

with

$$\omega_{np} := c\sqrt{(j_n/R)^2 + (p\pi/g)^2} \tag{19}$$

The solution of this differential equation for $\tilde{G}_{np}$ satisfying the initial condition (18) is:

$$\tilde{G}_{np}(t; r', z', t') = \frac{c^2 \Theta(t-t')}{\pi g R^2} \frac{2-\delta_{p0}}{J_1^2(j_n)}\, J_0\left(j_n \frac{r'}{R}\right) \cos\left(p\frac{\pi}{g}z'\right) \frac{\sin[\omega_{np}(t-t')]}{\omega_{np}}.$$

Finally inserting the last result into the ansatz (17) gives the Green's function

$$G(r,z,t;r',z',t') = \frac{c^2\Theta(t-t')}{\pi g R^2}\sum_{p=0}^{\infty}\sum_{n=1}^{\infty}\frac{2-\delta_{p0}}{J_1^2(j_n)}\cos\left(p\frac{\pi}{g}z'\right)\cos\left(p\frac{\pi}{g}z\right)\times$$

$$\times J_0\left(j_n\frac{r'}{R}\right)J_0\left(j_n\frac{r}{R}\right)\frac{\sin[\omega_{np}(t-t')]}{\omega_{np}} \tag{20}$$

in the oscillatory mode representation with ω_{np} defined in Eq. (19).

4 BUNCH WITH A CHARGE DISTRIBUTION CONSISTING OF THE PRODUCT OF A RADIAL AND A LONGITUDINAL FUNCTION

4.1 Computation of the Potentials

The longitudinal component A_z of the vector potential satisfying the differential equation (7) with the boundary condition (9) is computed with the help of the Green's function:

$$A_z(r,z,t) = \mu_0\int_0^R dr'\,2\pi r'\int_0^g dz'\int_{-\infty}^{\infty} dt'\;G(r,z,t;r',z',t')\,j_z(r',z',t')$$

Inserting the oscillatory mode representation (20) of the Green's function and the current distribution in Eq. (3) one obtains:

$$A_z(r,z,t) = \frac{2\mu_0 Q v c^2}{R^2 g}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{2-\delta_{p0}}{J_1^2(j_n)}\,\Xi_n J_0\left(j_n\frac{r}{R}\right)\cos\left(p\frac{\pi}{g}z\right)\times$$

$$\times\underbrace{\int_0^g dz'\int_{-\infty}^{t} dt'\;\cos\left(p\frac{\pi}{g}z'\right)\frac{\sin[\omega_{np}(t-t')]}{\omega_{np}}\,\lambda(z'-vt')}_{I} \tag{21}$$

with

$$\Xi_n = \int_0^R dr'\,r'\,J_0\left(j_n\frac{r'}{R}\right)\,\xi(r') \tag{22}$$

In order to reduce the double integral for I to a single integral the following transformation is applied. Since the problems to perform one of the integrations arises from the fact, that the argument of the unknown function λ involves both integration variables z' and t', it is natural to use the substitution $\tau = z' - vt'$ (Then one has to insert $dt' = -\frac{1}{v}d\tau$ and

$t' = \frac{z'-\tau}{v}$) for the t' integration. Then the resulting area of integration in the $\tau - z'$ plane is split into two parts corresponding to the two terms in the formula below:

$$I = \frac{1}{v} \int\limits_{g-vt}^{\infty} d\tau\, \lambda(\tau) \int\limits_{0}^{g} dz' \cos\left(p\frac{\pi}{g}z'\right) \frac{\sin\left[\omega_{np}\left(t - \frac{z'-\tau}{v}\right)\right]}{\omega_{np}} +$$

$$+ \frac{1}{v} \int\limits_{-vt}^{g-vt} d\tau \lambda(\tau) \int\limits_{0}^{\tau+vt} dz' \cos\left(p\frac{\pi}{g}z'\right) \frac{\sin\left[\omega_{np}\left(t - \frac{z'-\tau}{v}\right)\right]}{\omega_{np}}$$

Now the integrations w.r.t. z' can be performed analytically and one obtains with

$$\alpha_{np} = \frac{j_n^2}{R^2} + (1 - \beta^2)\left(p\frac{\pi}{g}\right)^2 \tag{23}$$

$$I = \frac{1}{c^2\alpha_{np}} \left\{ \int\limits_{g-vt}^{\infty} d\tau\, \lambda(\tau) \left[(-)^p \cos\left(\omega_{np}\left(t - \frac{g-\tau}{v}\right)\right) - \cos\left(\omega_{np}\left(t + \frac{\tau}{v}\right)\right) \right] + \right.$$

$$\left. + \int\limits_{-vt}^{g-vt} d\tau\, \lambda(\tau) \left[\cos\left(p\frac{\pi}{g}(\tau + vt)\right) - \cos\left(\omega_{np}\left(t + \frac{\tau}{v}\right)\right) \right] \right\}$$

With some further transformation and with the abbreviations:

$$r_{np}^c(t) = \int\limits_{-vt}^{\infty} d\tau\, [\, (-)^p \lambda(\tau + g) - \lambda(\tau)\,]\, \cos\left(\omega_{np}\left(t + \frac{\tau}{v}\right)\right) \quad \text{and} \tag{24}$$

$$q_p^c(t) = \int\limits_{-vt}^{g-vt} d\tau\, \lambda(\tau) \cos\left(p\frac{\pi}{g}(vt + \tau)\right) \tag{25}$$

one finally obtains:

$$A_z(r, z, t) = \frac{2\mu_0 Q v}{g R^2} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{2 - \delta_{p0}}{J_1^2(j_n)} \left[r_{np}^c(t) + q_p^c(t) \right] \frac{\Xi_n}{\alpha_{np}} \cos\left(p\frac{\pi}{g}z\right) J_0\left(j_n \frac{r}{R}\right) \tag{26}$$

together with (22), (24) and (25).

For the scalar potential it would be possible to start with the appropriate differential equation and to perform the computation analogously to the vector potential with the help of the corresponding Green's function (which differs from the Green's function defined above for the vector potential, because the boundary conditions are different); but as mentioned above we will use Eq. (8) for this task. One obtains:

$$\Phi(r,z,t)=\frac{4Q}{gR^2\epsilon_0}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{1}{J_1^2(j_n)}\left[\frac{vp\pi}{\omega_{np}g}r_{np}^s(t)+q_p^s(t)\right]\frac{\Xi_n}{\alpha_{np}}\cos\left(p\frac{\pi}{g}z\right)J_0\left(j_n\frac{r}{R}\right) \tag{27}$$

with

$$r_{np}^s(t)=\omega_{np}\int_{-\infty}^{t}dt'\,r_{np}^c(t')=\omega_{np}\int_{-\infty}^{t}dt'\int_{-vt'}^{\infty}d\tau[(-)^p\lambda(\tau+g)-\lambda(\tau)]\cos\left(\omega_{np}\left(t'+\frac{\tau}{v}\right)\right)$$

$$q_p^s(t)=p\frac{\pi}{g}v\int_{-\infty}^{t}dt'\,q_p^c(t')=p\frac{\pi}{g}\int_{-\infty}^{t}dt'\int_{-vt'}^{g-vt'}d\tau\,\lambda(\tau)\cos\left(p\frac{\pi}{g}(vt'+\tau)\right)$$

For r_{np}^s the t' integration can easily be performed after the two integrations have been interchanged and one obtains:

$$r_{np}^s(t)=\int_{-vt}^{\infty}d\tau\,[\,(-)^p\lambda(\tau+g)-\lambda(\tau)\,]\,\sin\left(\omega_{np}\left(t+\frac{\tau}{v}\right)\right). \tag{28}$$

The computation of q_p^s is more complicated. As shown in Figure 2, at first a crosshatched

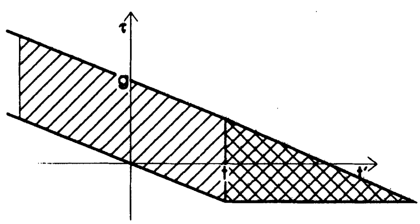

FIGURE 2: Original area (hatched) of integration for the computation of q_p^s and area added (crosshatched) for the interchange of the two integrations

area is added to the original region of integration and the additional term so introduced is subtracted at the end; this is done in order to allow an easy interchange of the integrations. The term due to the sum of the two areas vanishes; one finally obtains:

$$q_p^s(t)=\int_{-vt}^{-vt+g}d\tau\,\lambda(\tau)\,\sin\left(p\frac{\pi}{g}(vt+\tau)\right). \tag{29}$$

4.2 Computation of the Field

The electromagnetic field is derived by inserting the potentials given in Eqs. (26) and (27) into Eqs. (10), (11) and (12). The electromagnetic field components are:

$$B_\varphi(r,z,t) = \frac{2\mu_0 Q v}{gR^2} \sum_{n=1}^{\infty}\sum_{p=0}^{\infty} \frac{2-\delta_{p0}}{J_1^2(j_n)} \left[r_{np}^c(t) + q_p^c(t) \right] \times$$
$$\times \frac{\Xi_n}{\alpha_{np}} \frac{j_n}{R} \cos\left(p\frac{\pi}{g}z\right) J_1\left(j_n \frac{r}{R}\right) \tag{30}$$

$$E_r(r,z,t) = \frac{4Q}{gR^2\epsilon_0} \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{1}{J_1^2(j_n)} \left[\frac{v}{\omega_{np}} p\frac{\pi}{g} r_{np}^s(t) + q_p^s(t) \right] \times$$
$$\times \frac{\Xi_n}{\alpha_{np}} \frac{j_n}{R} \sin\left(p\frac{\pi}{g}z\right) J_1\left(j_n \frac{r}{R}\right) \tag{31}$$

$$E_z(r,z,t) = \frac{2Q}{gR^2\epsilon_0} \sum_{n=1}^{\infty}\sum_{p=0}^{\infty} \frac{2-\delta_{p0}}{J_1^2(j_n)} \left[\frac{v}{\omega_{np}} \frac{j_n^2}{R^2} r_{np}^s(t) - p\frac{\pi}{g}(1-\beta^2) q_p^s(t) \right] \times$$
$$\times \frac{\Xi_n}{\alpha_{np}} \cos\left(p\frac{\pi}{g}z\right) J_0\left(j_n \frac{r}{R}\right) \tag{32}$$

For the computation of the longitudinal component E_z of the electric field the relations

$$\frac{\partial}{\partial t} r_{np}^c(t) = -\omega_{np}\, r_{np}^s(t) + v\,[\,(-)^p \lambda(g - vt) - \lambda(-vt)\,]$$

and

$$\frac{\partial}{\partial t} q_p^c(t) = -p\frac{\pi}{g} v\, q_p^s(t) - v\,[\,(-)^p \lambda(g - vt) - \lambda(-vt)\,]$$

are used. They are derived directly from the definitions of the coefficients r_{np}^c and q_p^c in Eqs. (24) and (25) and from those of r_{np}^s and q_p^s in Eqs. (28) and (29).

4.3 Computation of the Wake Potential

When one inserts the longitudinal electric field from Eq. (32) into the definition of the wake potential, i.e. into Eq. (13) one ends up with two integrals involving the coefficients r_{np}^s and q_p^s defined above. The first integral is:

$$\int_0^g dz\, r_{np}^s\left(\frac{z+s}{v}\right)\cos\left(p\frac{\pi}{g}z\right) =$$

$$= \int_0^g dz \int_{-z-s}^{\infty} du[(-)^p\lambda(g-vt) - \lambda(-vt)]\sin\left(\omega_{np}\frac{z+s+u}{v}\right)\cos\left(p\frac{\pi}{g}z\right) =$$

$$= \frac{\omega_{np}\beta^2}{v\alpha_{np}}\left[r_{np}^c\left(\frac{s}{v}\right) - (-)^p r_{np}^c\left(\frac{s+g}{v}\right) + q_p^c\left(\frac{s}{v}\right) - (-)^p q_p^c\left(\frac{s+g}{v}\right)\right]$$

For the last transformation the definition of r_{np}^c in Eq. (24) is inserted and then the expression is split into two terms belonging to two parts of the initial area of integration as indicated in Figure 3a. This allows an easy interchange of the integrations w.r.t. z and u and performing the integration w.r.t. z yields the result stated.

The second integral is:

$$o_p(s) := \int_0^g dz\, q_p^s\left(\frac{z+s}{v}\right)\cos\left(p\frac{\pi}{g}z\right) =$$

$$= \int_0^g dz \int_{-s-z}^{g-s-z} du\, \lambda(u)\sin\left[p\frac{\pi}{g}(z+s+u)\right]\cos\left(p\frac{\pi}{g}z\right).$$

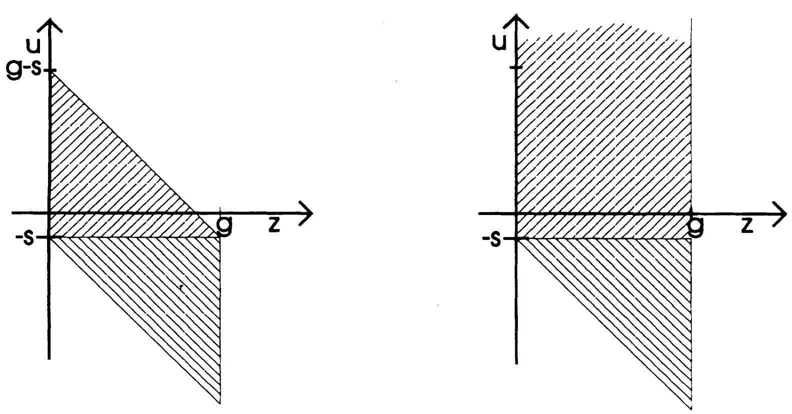

FIGURE 3: Split of the areas of integration in order to allow interchange of the integrations, (a) for integration involving r_{np}^c, (b) for integration involving q_p^c

Also this expressions is split into two parts corresponding to two parts of the area of integration in the z-u-plane as indicated in the Figure 3b. After interchanging integrations w.r.t. z and u and performing integration w.r.t. z one obtains:

$$o_p(s) = \int_{-s-g}^{-s} du\lambda(u)\frac{g+s+u}{2}\sin\left[p\frac{\pi}{g}(s+u)\right] + \int_{-s}^{g-s} du\lambda(u)\frac{g-s-u}{2}\sin\left[p\frac{\pi}{g}(s+u)\right] \tag{33}$$

Finally the wake potential is written down as:

$$W_{\|}(s,r) = \frac{2Q}{gR^2\epsilon_0}\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\frac{2-\delta_{p0}}{J_1^2(j_n)}\left\{\frac{\beta^2 j_n^2}{\alpha_{np}R^2}\left[r_{np}^c\left(\frac{s}{v}\right) - (-)^p r_{np}^c\left(\frac{s+g}{v}\right) + q_p^c\left(\frac{s}{v}\right) - (-)^p q_p^c\left(\frac{s+g}{v}\right)\right] - p\frac{\pi}{g}(1-\beta^2)o_p(s)\right\}\frac{\Xi_n}{\alpha_{np}}J_0\left(\frac{j_n r}{R}\right) \tag{34}$$

5 BUNCH WITH A GAUSSIAN AS LONGITUDINAL DISTRIBUTION AND A RECTANGULAR FUNCTION AS RADIAL DISTRIBUTION

When one specializes the longitudinal distribution λ to be a Gaussian and the radial distribution ξ to be a rectangular function one ends up with the current distribution given in Eq. (4). In this case the integrations defining Ξ and r_{np}^c, q_p^c, r_{np}^s and q_p^s can be performed and be reduced to standard functions. Inserting $\xi(r) = \Theta(r-a)\,\Theta(b-r)/[\pi(b^2-a^2)]$ on obtains for Ξ_n defined in Eq. (22):

$$\Xi_n = \int_a^b dr'\, r'\,\frac{J_0\left(j_n\frac{r'}{R}\right)}{\pi(b^2-a^2)} = \frac{R}{j_n}\frac{1}{\pi(b^2-a^2)}\left[b\,J_1\left(j_n\frac{b}{R}\right) - a\,J_1\left(j_n\frac{a}{R}\right)\right] \tag{35}$$

The coefficients r_{np}^c, q_p^c, r_{np}^s, q_p^s and o_p are expressed with the help of the complex error function

$$w(z) = e^{-z^2}\,erfc(-iz) = e^{-z^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}\,dt\right) \tag{36}$$

as defined in Ref. 5. Inserting $\lambda(\tau) = 1/(\sqrt{2\pi}\,\sigma)\exp(-\tau^2/(2\sigma^2))$ into Eqs. (24), (25), (28), (29) and (33) gives with $w_1 = w[\omega_{np}\sigma/(v\sqrt{2}) - i(vt-g)/(\sigma\sqrt{2})]$, $w_2 = w[\omega_{np}\sigma/(v\sqrt{2}) - ivt/(\sigma\sqrt{2})]$, $w_3 = w[p\pi\sigma/g\sqrt{2} - ivt/(\sigma\sqrt{2})]$, $w_4 = w[p\pi\sigma/(g\sqrt{2}) - i(vt-g)/(\sigma\sqrt{2})]$:

$$r_{np}^{c}(t) = \frac{1}{2} Re \left\{ (-)^{p} \exp\left(-\frac{(g-vt)^{2}}{2\sigma^{2}}\right) w_{1} - \exp\left(-\frac{(vt)^{2}}{2\sigma^{2}}\right) w_{2} \right\} \tag{37}$$

$$q_{p}^{c}(t) = \frac{1}{2} Re \left\{ \exp\left(-\frac{(vt)^{2}}{2\sigma^{2}}\right) w_{3} - (-)^{p} \exp\left(-\frac{(g-vt)^{2}}{2\sigma^{2}}\right) w_{4} \right\} \tag{38}$$

$$r_{np}^{s}(t) = \frac{1}{2} Im \left\{ (-)^{p} \exp\left(-\frac{(g-vt)^{2}}{2\sigma^{2}}\right) w_{1} - \exp\left(-\frac{(vt)^{2}}{2\sigma^{2}}\right) w_{2} \right\} \tag{39}$$

$$q_{p}^{s}(t) = \frac{1}{2} Im \left\{ \exp\left(-\frac{(vt)^{2}}{2\sigma^{2}}\right) w_{3} - (-)^{p} \exp\left(-\frac{(g-vt)^{2}}{2\sigma^{2}}\right) w_{4} \right\} \tag{40}$$

$$\begin{aligned} o_{p}(s) = \frac{\sigma}{2\sqrt{2}} Re \Bigg\{ & (-)^{p} \left(p\frac{\pi\sigma}{\sqrt{2}g} - i\frac{s+g}{\sqrt{2}\sigma}\right) \exp\left(\frac{(s+g)^{2}}{2\sigma^{2}}\right) w\left(p\frac{\pi\sigma}{\sqrt{2}g} - i\frac{s+g}{\sqrt{2}\sigma}\right) - \\ & - 2\left(p\frac{\pi\sigma}{\sqrt{2}g} - i\frac{s}{\sqrt{2}\sigma}\right) \exp\left(\frac{(s)^{2}}{2\sigma^{2}}\right) w\left(p\frac{\pi\sigma}{\sqrt{2}g} - i\frac{s}{\sqrt{2}\sigma}\right) + \\ & + (-)^{p} \left(p\frac{\pi\sigma}{\sqrt{2}g} - i\frac{s-g}{\sqrt{2}\sigma}\right) \exp\left(\frac{(s-g)^{2}}{2\sigma^{2}}\right) w\left(p\frac{\pi\sigma}{\sqrt{2}g} - i\frac{s-g}{\sqrt{2}\sigma}\right) \Bigg\} \end{aligned} \tag{41}$$

For the components of the electromagnetic field and the wake potential one finally obtains

$$\begin{aligned} B_{\varphi}(r,z,t) = & \frac{2\mu_{0} v Q}{g\pi R^{2}(b^{2}-a^{2})} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{2-\delta_{p0}}{J_{1}^{2}(j_{n})} \left[r_{np}^{c}(t) + q_{p}^{c}(t) \right] \times \\ & \times \frac{1}{\alpha_{np}} \cos(p\frac{\pi}{g}z) \left[bJ_{1}(j_{n}\frac{b}{R}) - aJ_{1}(j_{n}\frac{a}{R}) \right] J_{1}\left(j_{n}\frac{r}{R}\right) \end{aligned} \tag{42}$$

$$\begin{aligned} E_{r}(r,z,t) = & \frac{4Q}{g\pi R^{2}\epsilon_{0}(b^{2}-a^{2})} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{1}{J_{1}^{2}(j_{n})} \left[\frac{v}{\omega_{np}} p\frac{\pi}{g} r_{np}^{s}(t) + q_{p}^{s}(t) \right] \times \\ & \times \frac{1}{\alpha_{np}} \sin\left(p\frac{\pi}{g}z\right) \left[bJ_{1}\left(j_{n}\frac{b}{R}\right) - aJ_{1}\left(j_{n}\frac{a}{R}\right) \right] J_{1}\left(j_{n}\frac{r}{R}\right) \end{aligned} \tag{43}$$

$$\begin{aligned} E_{z}(r,z,t) = & \frac{2Q}{g\pi R^{2}\epsilon_{0}(b^{2}-a^{2})} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{2-\delta_{p0}}{J_{1}^{2}(j_{n})} \left[\frac{v}{\omega_{np}} \frac{j_{n}}{R} r_{np}^{s}(t) - p\frac{\pi}{g}(1-\beta^{2}) \times \right. \\ & \left. \times \frac{R}{j_{n}} q_{p}^{s}(t) \right] \frac{1}{\alpha_{np}} \cos\left(p\frac{\pi}{g}z\right) \left[bJ_{1}\left(j_{n}\frac{b}{R}\right) - aJ_{1}\left(j_{n}\frac{a}{R}\right) \right] J_{0}\left(j_{n}\frac{r}{R}\right) \end{aligned} \tag{44}$$

and

$$W_{\|}(s,r) = \frac{2Q}{g\pi R^2\epsilon_0(b^2-a^2)} \sum_{n=1}^{\infty}\sum_{p=0}^{\infty} \frac{2-\delta_{p0}}{J_1^2(j_n)} \left\{ \frac{\beta^2 j_n}{\alpha_{np} R} \left[r_{np}^c\left(\frac{s}{v}\right) - (-)^p r_{np}^c\left(\frac{s+g}{v}\right) + q_p^c\left(\frac{s}{v}\right) - \right.\right.$$

$$\left.\left. -(-)^p q_p^c\left(\frac{s+g}{v}\right)\right] - \frac{\pi}{g}(1-\beta^2)\frac{R}{j_n} o_p(s) \right\} \frac{1}{\alpha_{np}} \left[b J_1\left(j_n \frac{b}{R}\right) - a J_1\left(j_n \frac{a}{R}\right)\right] J_0\left(j_n \frac{r}{R}\right) \tag{45}$$

with the above coefficients. Zotter and Weiland[9] treated the similar problem of a bunch with a longitudinal Gaussian distribution but concentrated on the axis and moving with the speed of light ($v = c$). In order to obtain their result for the longitudinal component E_z of the electric field, one must insert $a = 0$ and $v = c$ and consider the limit $b \to 0$. With the help of

$$\lim_{b\to 0} \frac{1}{b} J_1\left(j_n \frac{b}{R}\right) = \frac{j_n}{R}\frac{1}{2} \tag{46}$$

one obtains agreement with this previous result in Ref. 9.

6 NUMERICAL EVALUATION OF THE ELECTROMAGNETIC FIELD

The electromagnetic field is numerically evaluated summing the series by computer for two cases. On the one hand a "thin" (i. e. σ and $b-a$, but not necessarily a and b are small) ring passing the pill-box cavity was considered; in this case shock waves, starting at the entrance and the exit of the drive beam into and from the device, appear. On the other hand the device described in Ref. 2 was approximated in order to calculate the field for a realistic wake field transformer.

6.1 *Thin Ring in a Pill-Box Cavity*

The following parameters were chosen:
$R = 12.0$ cm, $g = 6.0$ cm,
$\beta = 0.5 \quad \to \quad v = 1.449\ 10^8$ m/s,
$Q = 1\ \mu$C, $a = 7.7$ cm, $b = 8.3$ cm and $\sigma = 0.3$ cm.
The field obtained with these parameters is shown in Figure 4. The field is shown as a function of time t with the radius r as a parameter in four graphics belonging to different values of the longitudinal coordinate z. Especially near the axis of the device (i. e. at $r = 0$) the arrival of the wave fronts can be observed.

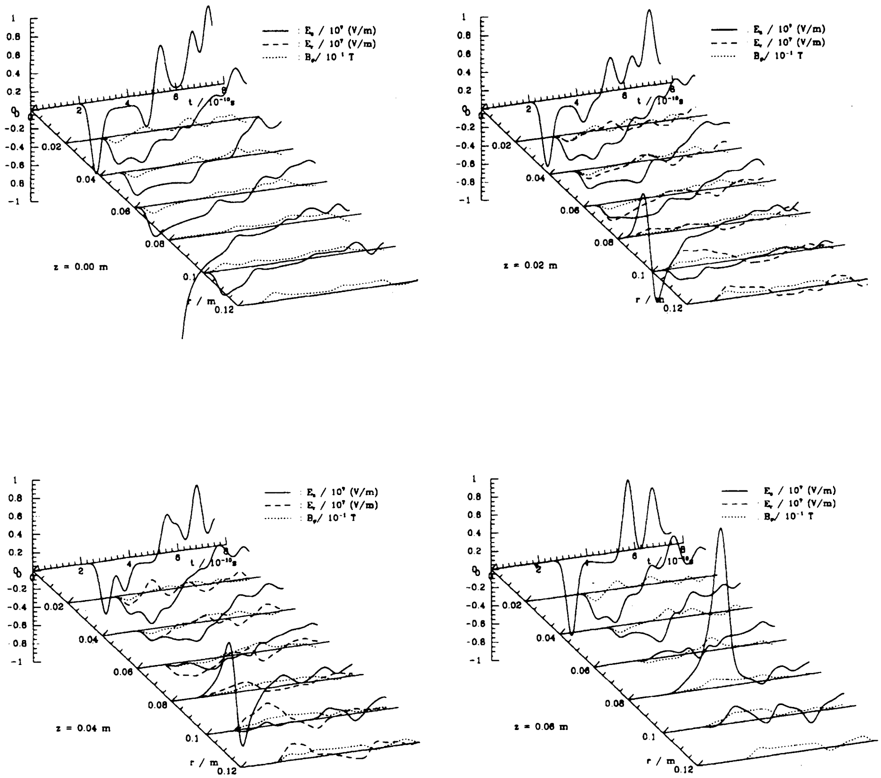

FIGURE 4: Electromagnetic field induced by a thin charged ring passing the pill-box cavity. The parameters of the device are given in section 6.1.

6.2 *Geometry Realistic for a Wake Field Transformer*

The following parameters were chosen:
$R = 55$ mm, $g = 4$ mm,
$\beta = 0.997 \quad \rightarrow \quad v = 2.989\ 10^8$ m/s,
$Q = 1\ \mu$ C, $a = 49$ mm, $b = 51$ mm and $\sigma = 2$ mm.
The field obtained with these parameters is shown in Figure 6. The field is shown as a function of time t with the radius r as a parameter in three graphics belonging to different values of the longitudinal coordinate z. Due to the large widths of the ring (above all in the longitudinal direction) the wave fronts are smeared out and thus are no longer well separated. The longitudinal electric field E_z reaches values very suitable for the acceleration of a second beam near the axis of the device.

For this approximation to the real device also the wake potential $W_{\|}$ was numerically evaluated. The resulting curve is shown in Figure 5. This should be compared to Figure 1.6

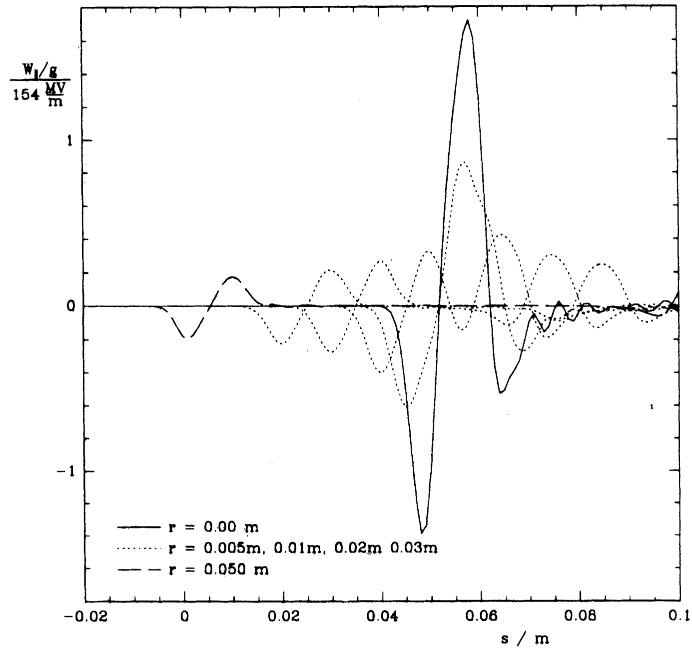

FIGURE 5: Wake potential in the geometry useful as wake field transformer. For the parameters of the device see section 6.2.

of Ref. 2, where the wake potential for a drive beam with no radial extension (i.e. $a = b$) has been computed with purely numerical methods.

More details of the calculations and more figures showing the field distribution may be found in the author's thesis.[10]

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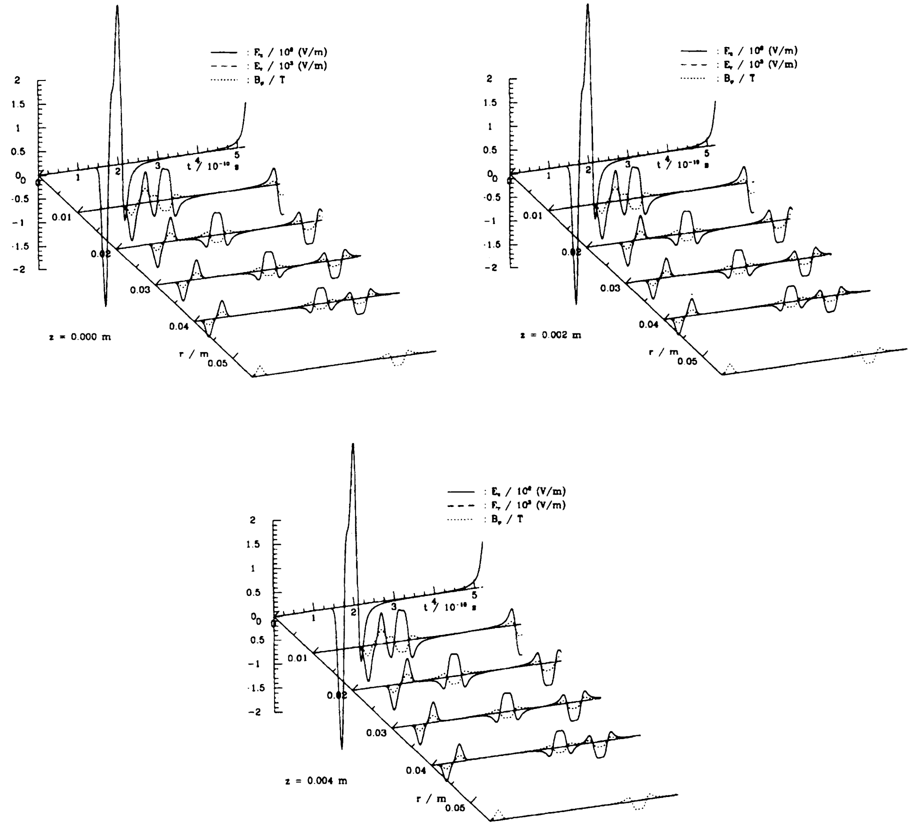

FIGURE 6: Electromagnetic field in a geometry useful as Wake Field Transformer. The parameters of the device are given in section 6.2.

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