XVII. COGNITIVE INFORMATION PROCESSING*

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A. COGNITIVE PROCESSES

1. INTERLINGUAL TRANSFER OF READING SKILL

In previous reports we have shown that a stable order characterizes the ease with which college students can read text that has been transformed geometrically: equal amounts of practice with mathematically equivalent transformations do not yield equivalent levels of performance. Some transformations are considerably more difficult than others.[1] Practice on any transformation, however, facilitates performance on any other; this suggests a generalized habituation to the fact of transformation itself. How generalized that habituation is was studied in the experiment described here.

Ten bilingual subjects, German nationals who had been in the United States for at least nine months, were tested. All were students at the Massachusetts Institute of Technology. Five of these men read 20 pages of English that had been printed in inverted form, and then read 4 pages of German in the same transformation; the other five read 20 pages of German in inverted form, and then 4 pages of English. Also, on the first day, before reading any of the transformed text, and on the fourth day, after all of the transformations had been read, all of the subjects read 1 page of normal English and 1 page of normal German. The time taken by the subjects to read each page was measured with a stop watch. The results are shown in Fig. XVII-1. The speed with which transformed English or German is read increases sharply with practice, from an initial 13 min/page to approximately 4 min/page. Even the latter rate, however, is considerably slower than that for normal text, while normal English (circles) takes a little longer than normal German (triangles). The transfer tests, however, produce asymmetrical results. The subjects trained on 20 pages of inverted English (closed circles) then read four pages of inverted German with no change in the level of performance; but the subjects trained on inverted German (crosses) did not do as well when tested on inverted English.

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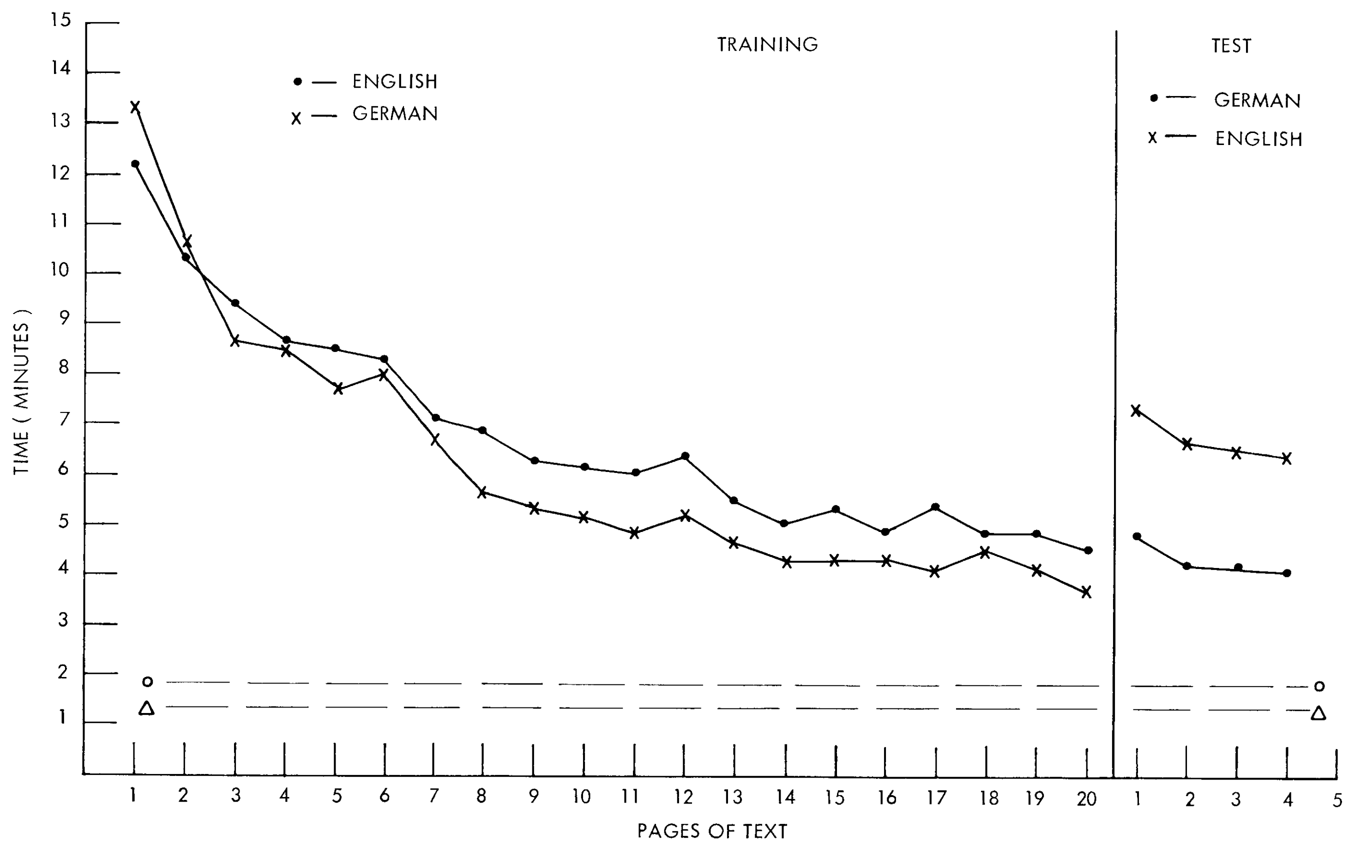

Fig. XVII-1. Results of transfer tests.

This curious asymmetry of transfer has an analog in a number of sensori-motor coordinations, for which the general finding is that practicing the less favored organ permits more transfer to the more favored than the reverse direction does; for example, training a right-handed man's left hand on a complex task enables him thenceforward to perform the task with his right hand, but training his right hand does not usually enable him to perform the task with his left.[2] In the present case we find that training in English enabled native speakers of German to transfer their skill without decrement, but training in German yielded some decrement for performance in English.

The more interesting aspect of these results has to do with what is learned when a subject learns how to decode transformed text. If he were learning only to recognize letters that had been transformed, transfer between the languages would be perfect, for the German and English alphabets are almost identical when Roman type is used, the only difference being the use of the diaresis, which does not affect letter shapes. If he were learning the shapes of words, transfer would be relatively poor, since German and English word shapes are somewhat different. The results indicate that the learning cannot be as simple as either of these alternatives would have it.

P. A. Kolers, Ann C. Boyer

References

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2. R. S. Woodworth, Experimental Psychology (Henry Holt and Company, New York, 1938).

B. PICTURE PROCESSING

1. OPTIMUM SCANNING DIRECTION IN TELEVISION TRANSMISSION

In television transmission, the two-dimensional picture is first scanned to produce a video signal which is then sent through some channel to the receiver. At the receiver, the picture is reconstructed from the video signal by scanning. For any given picture, different scanning methods usually give rise to different video signals and reconstructed pictures. In this report we shall discuss the relative merits of the members of a subclass of scanning methods. We restrict our attention to constant-speed sequential scanning along equidistant parallel lines of slope a (Fig. XVII-2) and try to study the effect of scanning direction on the video signal and the reconstructed picture.

First, we shall find the direction of scanning (that is, the value of a) which yields the minimum-bandwidth video signal, assuming that the two-dimensional Fourier spectrum of the original picture is given. Then we describe some preliminary results

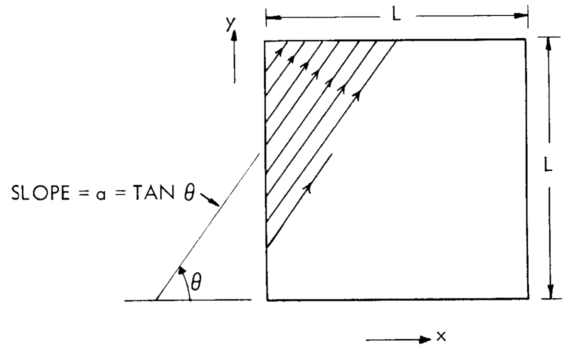


Fig. XVII-2. A subclass of scanning methods.

concerning the subjective effect of scanning direction. Finally, we shall discuss some miscellaneous factors that might affect the choice of a scanning direction.

Minimization of Video Signal Bandwidth

For the sake of simplicity, we shall assume in the following analysis that there is no interlacing. Notice, however, that the addition of interlacing will not change the result of the analysis.

Consider a single picture frame. Let f(x, y) denote the brightness of the picture point (with the average value substracted) as a function of its spatial coordinates (x, y), under the assumption that f(x, y) = 0 if (x, y) lies outside the picture. Let $\phi(\tau_1, \tau_2)$ be the autocorrelation function of f(x, y); and $\Phi(u, v)$, the Fourier transform of $\phi(\tau_1, \tau_2)$, that is, the energy spectrum of f(x, y). Let $\phi_a(\tau)$ and $\Phi_a(\omega)$ be the autocorrelation function and the energy spectrum, respectively, of the video signal $f_a(t)$, derived from f(x, y) by scanning along the direction a. The question is: If $\Phi(u, v)$ is given, what value of a will give the minimum bandwidth $\Phi_a(\omega)$? Without loss of generality, we assume that the scanning speed is 1 unit length/unit time. We also assume that the energy of the picture signal is much larger than that of the synchronous and blanking pulses so that the latter can be neglected. Then, we have

$$\phi_a(\tau) \approx \phi(\tau \cos \theta, \tau \sin \theta), \tag{1}$$

where $\theta = \tan^{-1} a$, assuming that both the distance between sucessive scanning lines and the width of $\phi(\tau \cos \theta, \tau \sin \theta)$ are much smaller than L, the width of the picture. In the case $a \approx 0$ or ∞, $\phi_a(\tau)$ will have peaks at multiples of L, and the right-hand side of Eq. 1 gives only the central peak (at $\tau = 0$); however, the bandwidth of $\Phi_a(\omega)$ is determined mainly by the central peak. In the sequel we shall assume that (1) is an equality.

It follows[1] from Eq. 1 that

$$\Phi_a(\omega) = \frac{1}{2\pi|\cos\theta|} \int_{-\infty}^{\infty} \Phi\left(\frac{\omega}{\cos\theta} - av,\ v\right) dv. \tag{2}$$

Assuming that $0 \leq \theta \leq \frac{\pi}{2}$, so that $\cos\theta \geq 0$, we have

$$\Phi_a(\omega) = \frac{\sqrt{1+a^2}}{2\pi} \int_{-\infty}^{\infty} \Phi\left(\sqrt{1+a^2}\ \omega - av,\ v\right) dv. \tag{3}$$

Let us define the bandwidth of $\Phi_a(\omega)$ as

$$B_a \equiv \frac{\int_{-\infty}^{\infty} \Phi_a(\omega)\, d\omega}{\Phi_a(0)}. \tag{4}$$

This definition is reasonable because $\Phi_a(\omega) \geq 0$ for all ω, and for most pictures, $\Phi_a(\omega)$ have their maxima at $\omega = 0$. Now

$$\int_{-\infty}^{\infty} \Phi_a(\omega)\, d\omega = 2\pi\phi_a(0) = 2\pi\phi(0,0) \tag{5}$$

is independent of a. So in order to minimize B_a, we have to maximize

$$\Phi_a(0) = \frac{\sqrt{1+a^2}}{2\pi} \int_{-\infty}^{\infty} \Phi(-av, v)\, dv. \tag{6}$$

Hence, we want to choose that value of a which will maximize the right-hand side of Eq. 6. Referring to Fig. XVII-3, we have

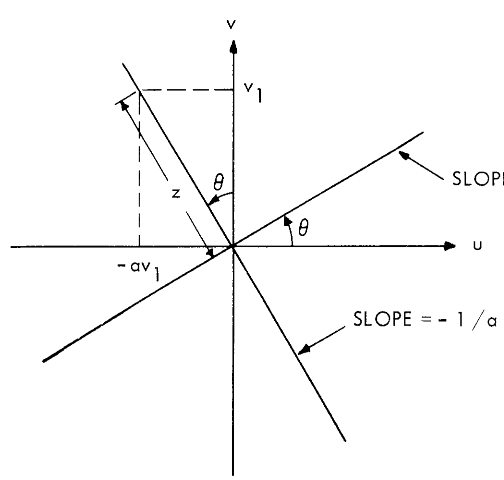


Fig. XVII-3. Pertaining to the interpretation of the right-hand side of Eq. 6.

$$\sqrt{\frac{1+a^2}{2\pi}} \int_{-\infty}^{\infty} \Phi(-av, v)\, dv = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(-z \sin\theta, z\cos\theta)\, dz. \tag{7}$$

Notice that $\Phi(-z \sin\theta, z\cos\theta)$ is the value of $\Phi(u, v)$ along the straight line $v = -\frac{1}{a}u$, which is perpendicular to the straight line $v = au$. To minimize B_a, therefore, we want to maximize the right-hand side of (7), or equivalently, to maximize

$$W_{-\frac{1}{2}} \equiv \frac{\int_{-\infty}^{\infty} \Phi(-z\sin\theta, z\cos\theta)\, dz}{\Phi(0,0)} \tag{8}$$

which is defined as the bandwidth of $\Phi(u, v)$ along the direction $-\frac{1}{a}$.

We conclude, therefore, that in order to obtain the video signal of minimum bandwidth, one should scan the original picture along a direction perpendicular to the direction of maximum bandwidth of $\Phi(u, v)$. This result is perhaps not in accord with one's intuition because, intuitively, one might think that to obtain the minimum-bandwidth video signal, one should scan along the direction of minimum bandwidth of $\Phi(u, v)$; this is not the case according to our analysis.

To verify the result of our analysis, we generated some two-dimensional lowpass Gaussian noise with power density spectra (Fig. XVII-4).

$$\Phi(u, v) = \begin{cases} \dfrac{\text{Constant}}{k_1, k_2}, & \text{for } -k_1 \leq u \leq k_1, \text{ and } -k_2 \leq v \leq k_2 \\ 0, & \text{elsewhere} \end{cases} \tag{9}$$

where k_1 and k_2 are positive real constants. The results of this noise generation (with DC level added) are shown in Fig. XVII-5. According to our analysis, to obtain the minimum-bandwidth video signal, we should scan along the directions $\pm\frac{k_1}{k_2}$. The appearance of the noise does seem to verify our contention.

We note in passing that if the scanning speed and the distance between successive scanning lines (which is assumed to be much smaller than L) are kept constant, then the scanning time per picture frame is independent of the direction of scanning.

Subjective Effect of Scanning Direction

At the ordinary viewing distance (4 or 6 times the picture height), one can see the line structures in the received picture. Do people prefer line structures of a particular orientation to those of other orientations? To try to find an answer to this question, we generated pictures scanned along various directions on a closed-circuit television system. Some of these pictures are shown in Fig. XVII-6.

We showed these pictures to some of our colleagues, and we have listed their

$$\Phi(u,v) = \begin{cases} \dfrac{\text{CONSTANT}}{k_1 k_2}, & \text{IN SHADED REGION} \\ 0, & \text{ELSEWHERE} \end{cases}$$

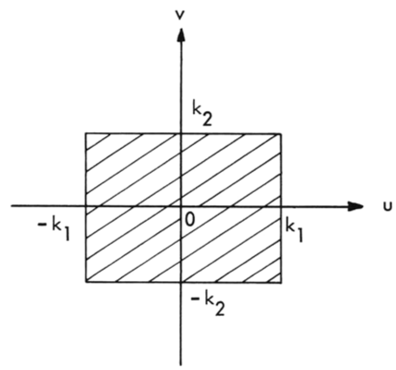
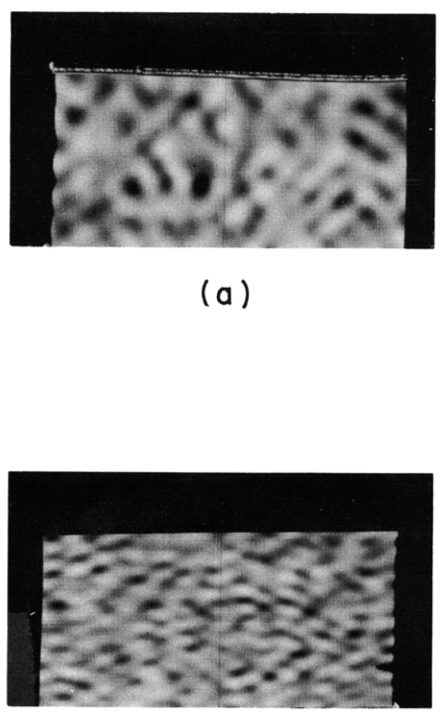


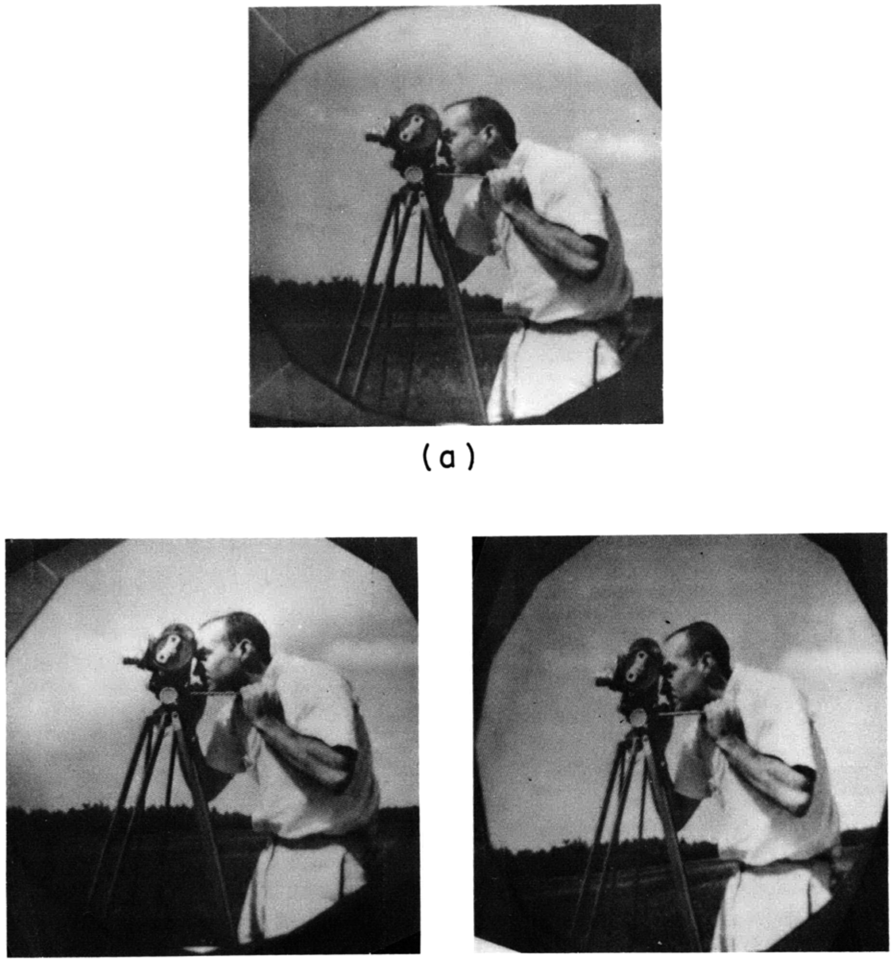
Fig. XVII-4. Spectrum of two-dimensional lowpass Gaussian noise.



(a)

(b)

Fig. XVII-5. Two-dimensional lowpass Gaussian noise. (a) $k_1/k_2 = 1$. (b) $k_1/k_2 = \frac{1}{2}$.



(a)   (b)   (c)

Fig. XVII-6. Picture scanned along various directions.

preferences as follows.

Orders (in the order of decreasing preference):

Subject A: Vertical, horizontal, skew.

Subject B: Horizontal, skew, vertical.

Subject C: Horizontal, vertical, skew.

Subject D: Skew, horizontal, vertical.

The preference, however, was by no means strong.

It is interesting to note that Subject C disliked skew scanning because it seemed to cause anxiety, while Subject D liked skew scanning because it made the picture look "dynamic." Subject B disliked vertical scanning because vertical lines seemed most visible, and Subject D disliked vertical scanning because the picture seemed ready to fall apart.

Jumping to a tentative conclusion, we might say that the preference is not strong but horizontal scanning seems to have a slight lead.

Other Factors

The pictures mentioned in the preceding section are essentially noiseless. In practice, however, the received picture contains additive random noise and ghosts (caused by multipath). How do the effects of random noise and ghosts depend on scanning direction? Also, how is motion affected by scanning direction? These questions are being investigated.

Finally, we wish to remark that there are still other factors that one might consider in choosing a scanning direction. For example,[2] in skew scanning, the lines are not of equal length, therefore the power of the video signal does not have peaks at multiples of line frequency. Hence, when several video signals share the same channel, the use of skew scanning will reduce cross modulation. On the other hand, skew scanning complicates line synchronization.

T. S. Huang

References

1. T. S. Huang, Two-dimensional power density spectrum of television random noise, Quarterly Progress Report No. 69, Research Laboratory of Electronics, M. I. T., April 15, 1963, p. 147.

2. Dr. B. Prasada, Bell Telephone Laboratories, Inc., Private communication, 1964.

2. BOUNDS ON TWO-ELEMENT-KIND IMPEDANCE FUNCTIONS

In a previous report,[1] we discussed some bounds on the impedance functions of R, ±L, ±C, T networks. In this report, we shall present bounds for various types of two-element-kind impedance functions. We first prove a theorem for R, ±C and R, ±L networks.

THEOREM 1. Let $[Z_{ik}(s)]$ be an n^{th}-order R, ±C (or R, ±L) open-circuit impedance matrix. Then $Z_{ik}(jw)$ satisfies

$$\left|Z_{ik}(jw)-\frac{1}{2}(Z_{ik}(0)+Z_{ik}(\infty))\right| \leq \sqrt{\frac{1}{2}(Z_{ii}(0)-Z_{ii}(\infty))\,\frac{1}{2}(Z_{kk}(0)-Z_{kk}(\infty))} \qquad (1)$$

for all real w.

The proof of Theorem 1 follows readily from the two following lemmas.

LEMMA 1. Let $[Z_{ik}(s)]$ be an n^{th}-order R, ±C open-circuit impedance matrix. Then

$$Z_{ik}(s) = h_{ik}^{(\infty)} + h_{ik}^{(o)}/s^2 + \sum_{m=1}^{n} h_{ik}^{(m)}/(s+a_m), \tag{2}$$

where the real numbers a_m are independent of i and k, the $\left[h_{ik}^{(r)}\right]$ $(r=1,2,\ldots,m;0,\infty)$ are real and symmetrical, and

$\left[h_{ik}^{(\infty)}\right]$ is positive semidefinite (psd)

$\left[h_{ik}^{(o)}\right]$ is negative semidefinite (nsd)

$\left[h_{ik}^{(m)}\right]$ is psd if $a_m \geq 0$, and nsd if $a_m < 0$.

LEMMA 2. If $[p_{ik}]$ and $[q_{ik}]$ $(i,k=1,2)$ are real, symmetrical, and psd, then

$$2p_{12}q_{12} \leq p_{11}q_{22} + q_{11}p_{22}. \tag{3}$$

By making appropriate impedance transformations, we deduce from Theorem 1 two theorems about ±R, C and ±R, L networks.

THEOREM 2. Let $[Z_{ik}(s)]$ be an n^{th}-order ±R, C open-circuit impedance matrix. Then $Z_{ik}(jw)$ satisfies

$$\left|Z_{ik}(jw) - \tfrac{1}{2}(Z'_{ik}(0)/jw + Z'_{ik}(\infty)/jw)\right| \leq \sqrt{\tfrac{1}{2}(Z'_{ii}(0)/w - Z'_{ii}(\infty)/w)\tfrac{1}{2}(Z'_{kk}(0)/w - Z'_{kk}(\infty)/w)} \tag{4}$$

for any real w, where $Z'_{ik}(s) = sZ_{ik}(s)$.

THEOREM 3. Let $[Z_{ik}(s)]$ be an n^{th}-order ±R, L open-circuit impedance matrix. Then $Z_{ik}(jw)$ satisfies

$$\left|Z_{ik}(jw) - \tfrac{1}{2}(jwZ''_{ik}(0) + jwZ''_{ik}(\infty))\right| \leq \sqrt{\tfrac{1}{2}(wZ''_{ii}(0) - wZ''_{ii}(\infty))\tfrac{1}{2}(wZ''_{kk}(0) - wZ''_{kk}(\infty))} \tag{5}$$

for any real w, where $Z''_{ik}(s) = Z_{ik}(s)/s$.

Notice that inequalities (1), (4), and (5) are properties of the impedance functions and are independent of the manner in which one realizes these functions. When $i \neq k$, the inequalities give bounds on transfer functions; when $i = k$, they give bounds on driving-point functions.

It is clear that for RC(RL) networks, both Theorem 1 and Theorem 2 (Theorem 3) apply. For any particular realization, N, of an RC n-port, the quantities in (1) and (4) have the following physical interpretations:

$Z_{ik}(0)$ = open-circuit impedance matrix of N, when all capacitances are open-circuited

$Z_{ik}(\infty)$ = open-circuit impedance matrix of N, when all capacitances are short-circuited

$Z'_{ik}(0)/jw$ = open-circuit impedance matrix of N, when all resistances are short-circuited and s=jw

$Z'_{ik}(\infty)/jw$ = open-circuit impedance matrix of N, when all resistances are open-circuited and $s = jw$.

(For any RL n-port realization, we have similar physical interpretations.) The quantities $Z_{ii}(0)$, $Z_{ii}(\infty)$, $Z'_{ii}(0)$, and $Z'_{ii}(\infty)$ are not independent. In fact, we have the following lemmas.

LEMMA 3. $Z_{ii}(0)$ is finite, if and only if $Z'_{ii}(0)$ is zero.

LEMMA 4. $Z_{ii}(\infty)$ is zero, if and only if $Z'_{ii}(\infty)$ is finite.

In order to get useful bounds, one would like the right-hand sides of (1) and (4) to be finite. Hence, one would like to have $Z_{ii}(\infty) = 0 = Z'_{ii}(0)$. One can achieve this by the following procedure. Given an RC open-circuit impedance matrix $[Z_{ik}(s)]$, we form a new RC open-circuit impedance matrix

$$[z_{ik}(s)] = [Z_{ik}(s)] - [Z_{ik}(\infty)] - [Z'_{ik}(0)]/s. \tag{6}$$

Then $z_{ii}(\infty) = 0 = z'_{ii}(0)$, where $z'_{ii}(s) = sz_{ii}(s)$, and we can apply inequalities (1) and (4) to $z_{ik}(s)$. For any nonzero finite w, the right-hand sides of (1) and (4) are finite, and $z_{ik}(jw)$ must lie in the intersection of two nondegenerate closed circular disks.

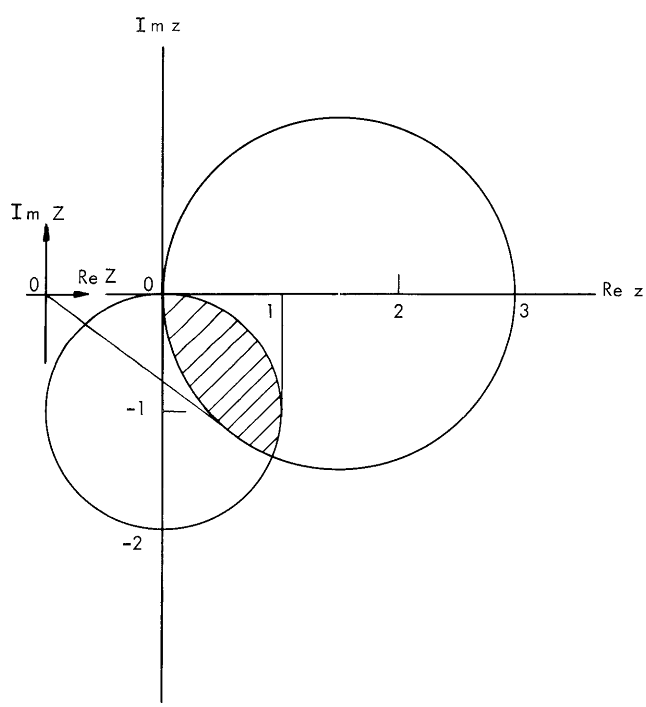


Fig. XVII-7. Example illustrating Theorems 1 and 2.

We conclude with an example. Consider the RC driving-point impedance function $Z(s) = (s+2)(s+6)/(s+1)(s+3)$. Then $Z(0) = 4$ and $Z(\infty) = 1$, and inequality (1) implies $|Z(jw) - 5/2| \leq 3/2$. Let $Z'(s) = sZ(s)$. Then $Z'(0) = 0$ and $Z'(\infty) = \infty$. Therefore, inequality (4), when applied to $Z(s)$ directly, does not give any useful bounds. We can, however, define $z(s) = Z(s) - Z(\infty) - Z'(0)/s = 4(s+9/4)/(s+1)(s+3)$. Then $z(0) = 3$, $z(\infty) = 0$, $z'(0) = 0$, and $z'(\infty) = 4$. Hence for $w = 2$, say, $Z(j2)$ must lie in the shaded region of Fig. XVII-7. In particular, we have

$$1 \leq |Z(j2)| < 2.4, \qquad -37° < [\angle Z(j2)] \leq 0;$$

$$1 < \text{Re } Z(j2) < 2.1, \qquad -1.4 < \text{Im } Z(j2) \leq 0.$$

Putting $s = j2$ in the exact expression for $Z(s)$, we find $Z(j2) = 2.22\angle{-33.8°}$.

In the previous report,[1] we proved that if $[Z_{ik}(s)]$ is the open-circuit impedance matrix of an R, ±L, ±C, T network, then

$$\left| Z_{ik}(jw) - \frac{R_{iko} + R_{iks}}{2} \right| \leq \sqrt{\left(\frac{R_{iio} - R_{iis}}{2}\right)\left(\frac{R_{kko} - R_{kks}}{2}\right)}, \tag{7}$$

where $[R_{iko}]$ is the open-circuit impedance matrix of the network when all reactive elements are open-circuited, and $[R_{iks}]$ is the open-circuit impedance matrix of the network

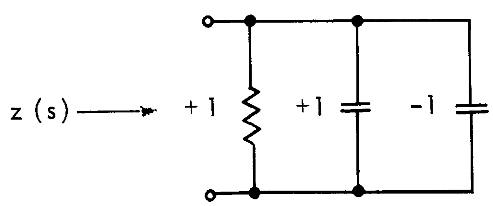


Fig. XVII-8. Example of an R, ±C network.

when all reactive elements are short-circuited.

We remark that Eq. 1 does not follow from Eq. 7, since, in general, for an R, ±C network, $Z_{ik}(0) \neq R_{iko}$ and $Z_{ik}(\infty) \neq R_{iks}$. For example, consider the network of Fig. XVII-8. We have $Z(s) = 1$; therefore, $Z(0) = 1 = Z(\infty)$. But $R_o = 1$ and $R_s = 0$.

T. S. Huang

References

1. T. S. Huang and H. B. Lee, Bounds on impedance functions of R, ±L, ±C, T networks, Quarterly Progress Report No. 75, Research Laboratory of Electronics, M. I. T., October 15, 1964, pp. 214-225.

C. SENSORY AIDS

1. APPROXIMATE FORMULAS FOR THE INFORMATION TRANSMITTED BY A DISCRETE COMMUNICATION CHANNEL

It is often desirable to have an approximate formula for the information transmitted by a discrete communication channel which is simpler than the exact expression.[1] In this report, two approximate expressions are derived. The derivations are instructive, for they show why two systems that operate with the same probability of error can have quite different information transmission capabilities.

Preliminary Theorems

The following theorems will be required. The proofs of Theorem 1 and of the lemma are omitted. Theorem 2 follows directly from Theorem 1, and also from Fano's discussion.[2]

THEOREM 1: Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. If $F(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i, \log x_i$, and if

$$\sum_{i=1}^{n} x_i = p,$$

then

$$F(p/n, p/n, \ldots, p/n) \leq F(x_1, x_2, \ldots, x_n) \leq F(p, 0, 0, \ldots 0).$$

The equality sign on the left applies only if all the x's are equal. The equality sign on the right applies only if all but one of the x's are zero.

THEOREM 2: Define

$p(x_i)$, probability of occurrence of the input x_i to a communication channel,

$p(y_j)$, probability of occurrence of the output y_j from a channel,

Lx, number of inputs having nonzero probability of occurrence, and

Ly, number of outputs having nonzero probability of occurrence.

Let

$$P(e|y_j) = 1 - p(x_j|y_j).$$

Then

$$P(e) = \sum_{j=1}^{Ly} [1 - p(x_j|y_j)]\, p(y_j) = \sum_{j=1}^{Ly} P(e|y_j)\, p(y_j).$$

If

$$H(e|y_j) = -[P(e|y_j) \log P(e|y_j) + [1 - P(e|y_j)] \log [1 - P(e|y_j)]],$$

$$H(e|Y) = \sum_{j=1}^{Ly} H(e|y_j)\, p(y_j),$$

and

$$H(e) = -P(e) \log P(e) - (1 - P(e)) \log (1 - P(e)),$$

then

$$0 \leq H(e|Y) \leq H(e).$$

LEMMA: Let $[p(y|x)]$ be a conditional probability matrix having Lx rows and Ly columns. Consider the matrix $[p(x|y)]$, where

$$p(x_i|y_j) = \frac{p(y_j|x_i)\, p(x_i)}{p(y_j)}.$$

If Q_j denotes the number of nonzero off-diagonal terms in the j^{th} column of the matrix $[p(y|x)]$, then the number of nonzero off-diagonal terms in the j^{th} row of the matrix $[p(x|y)]$ is also Q_j.

Derivation of Upper and Lower Bounds for I(X; Y)

To derive the following bounds on I(X; Y) two different communication channels are considered, each of which is required to transmit information about the same input ensemble. Both channels have the same number of outputs, The two channel matrices have identical elements on the main diagonal. Therefore, $P(e|y_j)$ $(j = 1, 2, \ldots, Ly)$ and P(e) are the same for both channels.

One channel matrix has only one nonzero off-diagonal term in each column. The information transmitted by this channel is a maximum for fixed values of $P(e|y_j)$ $(j = 1, 2, \ldots, Ly)$ and is equal to the upper bound of Eq. 2.

The other channel has a matrix in which all nonzero off-diagonal terms in any one column are equal. The information transmitted by this channel is a minimum for a given number of nonzero off-diagonal terms in each column, and for fixed values of $(Pe|y_j)$ $(j = 1, 2, \ldots, Ly)$. The information transmitted in this case is equal to the lower bound in Eq. 3.

THEOREM 3: Let I(X; Y) be the information transmitted by a discrete channel, and

$$H(X) = -\sum_{i=1}^{Lx} p(x_i) \log p(x_i).$$

Let Q_{max} be the largest of $Q_1, Q_2, \ldots, Q_{Ly}$. Then

$$I(X;Y) \leqslant H(X) - H(e|Y) \leqslant H(X);$$

$$I(X;Y) \geqslant H(X) - H(e|Y) - \sum_{j=1}^{Ly} P(e|y_j)\, p(y_j) \log Q_j \geqslant H(X) - H(e) - P(e) \log Q_{max}.$$

PROOF:

$$I(X;Y) = \sum_{i=1}^{Lx} \sum_{j=1}^{Ly} p(x_i|y_j)\, p(y_j) \log \frac{p(x_i|y_j)}{p(x_i)}$$

$$= H(X) + \sum_{j=1}^{Ly} p(y_j) \sum_{i=1}^{Lx} p(x_i|y_j) \log p(x_i|y_j) \tag{1}$$

$$\sum_{j=1}^{Ly} p(y_j) \sum_{i=1}^{Lx} p(x_i|y_j) \log p(x_i|y_j) = \sum_{j=1}^{Ly} p(y_j) \left[p(x_j|y_j) \log p(x_j|y_j) + \sum_{\substack{i=1 \\ i \neq j}}^{Lx} p(x_i|y_j) \log p(x_i|y_j) \right].$$

If we replace x_i by $p(x_i|y_j)$, p by $\sum_{i=1}^{Lx} p(x_i|y_j)$, and n by Q_j, and if we use

$$1 - p(x_j|y_j) = \sum_{\substack{i=1 \\ i \neq j}}^{Lx} p(x_i|y_j),$$

then the inequalities

$$[1 - p(x_j|y_j)] \log \frac{[1 - p(x_j|y_j)]}{Q_j} \leqslant \sum_{\substack{i=1 \\ i \neq j}}^{Lx} p(x_i|y_j) \log p(x_i|y_j) \leqslant [1 - p(x_j|y_j)] \log [1 - p(x_j|y_j)]$$

follow directly from Theorem 1 and the lemma.

The equality sign on the right applies if, and only if, there is only one nonzero off-diagonal term in the j^{th} row of the p(x|y) matrix. The equality sign on the left applies if, and only if, all the nonzero off-diagonal terms in the j^{th} row of the $[p(x|y)]$ matrix are equal.

Substitution of the inequalities above in Eq. 1 results in

$$I(X;Y) \leq H(X) + \sum_{j=1}^{Ly} p(y_j)\left[p(x_j|y_j) \log p(x_j|y_j) + [1-p(x_j|y_j)] \log [1-p(x_j|y_j)]\right]$$

$$\leq H(X) - H(e|Y). \tag{2}$$

$$I(X;Y) \geq H(X) + \sum_{j=1}^{Ly} p(y_j)\left[p(x_j|y_j) \log p(x_j|y_j) + [1-p(x_j|y_j)] \log \frac{[1-p(x_j|y_j)]}{Q_j}\right]$$

$$\geq H(X) - H(e|Y) - \sum_{j=1}^{Ly} P(e|y_j)\, p(y_j) \log Q_j. \tag{3}$$

Theorem 3 now follows from Theorem 2 and from the fact that

$$\log Q_{max} \geq \log Q_j \qquad (j = 1, 2, \ldots, Ly).$$

Approximate Formula for I(X; Y)

In order to use upper and lower bounds to estimate I(X; Y) in such a way that the expected value of the estimation error is minimized, it is necessary to know the distribution function for I(X; Y). Since the distribution function is not usually available, the estimate for I(X; Y) will be taken as the average of its upper and lower bounds. Such an estimate minimizes the maximum possible estimation error.

It follows that we estimate that

$$I_1(X;Y) = H(X) - H(e|Y) - \frac{1}{2}\sum_{j=1}^{Ly} P(e|y_j)\, p(y_j) \log Q_j \tag{4}$$

$$I_2(X;Y) = H(X) - \frac{1}{2}(H(e) + P(e) \log Q_{max}). \tag{5}$$

The maximum estimation error e is given in each case by

$$e_1 = \frac{1}{2}\sum_{j=1}^{Ly} P(e|y_j)\, p(y_j) \log Q_j \tag{6}$$

$$e_2 = \frac{1}{2}(H(e) + P(e) \log Q_{max}). \tag{7}$$

The maximum error e in per cent, which results because the estimate for I was chosen midway between the upper and lower bound, is

$$e\% = \left(\frac{U-L}{2L}\right) 100\%,$$

where U is the upper bound, and L is the lower bound. Thus

$$e_1\% = \left[\frac{\frac{1}{2}\sum_{j=1}^{Ly} P(e|y_j)\, p(y_j) \log Q_j}{H(X) - H(e|Y) - \sum_{j=1}^{Ly} P(e|y_j)\, p(y_j) \log Q_j}\right] 100\%, \tag{8}$$

$$e_2\% = \frac{\frac{1}{2}(H(e) + P(e) \log Q_{max})}{H(X) - H(e) - P(e) \log Q_{max}}\ 100\%. \tag{9}$$

The use of inequalities

$$H(e|Y) \leq H(e)$$

and

$$\sum_{j=1}^{Ly} P(e|y_j)\, p(y_j) \log Q_j \leq P(e) \log Q_{max}$$

in (6) and (8) results in upper bounds for e_1 and $e_1\%$

$$e_1 \leq \frac{1}{2} P(e) \log Q_{max} \tag{10}$$

$$e_1\% \leq \left(\frac{\frac{1}{2} P(e) \log Q_{max}}{H(X) - H(e) - P(e) \log Q_{max}}\right) 100\%, \tag{11}$$

which are easier to compute than the exact quantities given by Eqs. 6 and 8.

In Figs. XVII-9 and XVII-10 $e_1\%$ and $e_2\%$ are plotted as functions of P(e) for various values of Q_{max} for the cases H(X) = 4 and H(X) = 7. It should be remembered that these graphs represent the maximum errors that can occur as a result of approximating I(X; Y) by $I_1(X;Y)$ and $I_2(X;Y)$. The actual error that results when I(X; Y) is approximated by $I_2(X;Y)$ will equal the maximum error if and only if

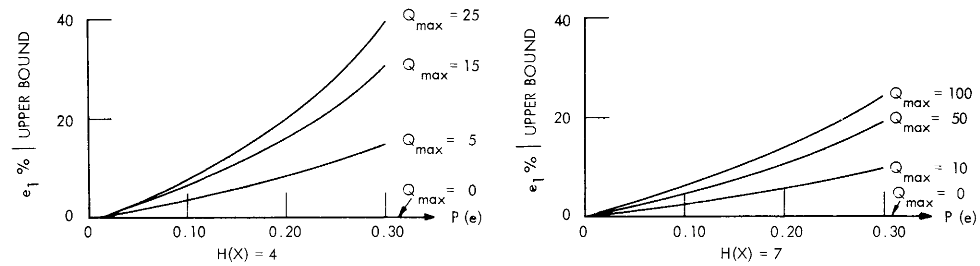


Fig. XVII-9. $e_1\%|$ upper bound vs P(e) and Q_{max}.

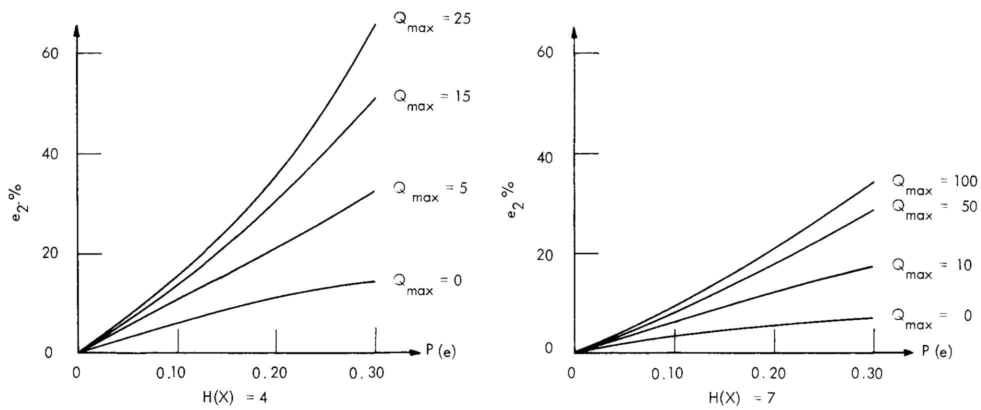


Fig. XVII-10. $e_2\%$ vs P(e) and Q_{max}.

$$I(X;Y) = H(X), \tag{12}$$

or

$$I(X;Y) = H(X) - H(e) - P(e) \log Q_{max}. \tag{13}$$

Equation (12) holds if, and only if, all off-diagonal terms in the channel matrix are zero (a perfect communication system). Equation (13) applies if, and only if, $H(e|y_j) = H(e)$, and $Q_j = Q_{max}$ $(j = 1, 2, \ldots, Ly)$(the same number of off-diagonal terms in each column of the channel matrix, and all these terms equal).

Similarly, the errors that result when I(X; Y) is approximated by $I_1(X;Y)$ are equal to the maximum error only in special cases. If there is only one off-diagonal term in each column of the channel matrix, then

$$I(X;Y) = H(X) - H(e|Y).$$

If all the off-diagonal terms in each column are equal, then

$$I(X;Y) = H(X) - H(e|Y) - \sum_{j=1}^{Ly} P(e|y_j)\, p(y_j) \log Q_j.$$

The estimate $I_1(X;Y)$ is always better than or as good as $I_2(X;Y)$. However, the first estimate requires more computation than the latter. A useful procedure for estimating $I(X;Y)$ is:

1. Evaluate e_2 in per cent. If e_2 is acceptable, evaluate $I_2(X;Y)$ as an approximation to $I(X;Y)$.

2. If e_2 is too large, evaluate the upper bound for e_1 in per cent. If this upper bound is acceptable, evaluate $I_1(X;Y)$ as an approximation to $I(X;Y)$.

3. If the upper bound to e_1 is too large, then compute $I(X;Y)$ from the exact formula (1).

Example

The following channel matrix results when a human subject is required to make one of eight responses to one of eight equiprobable statistically independent stimuli. The information transmitted is to be computed to an accuracy of ±5 per cent of the true value.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
x_1	.95	.05						
x_2	.05	.90	.05					
x_3	.05	.05	.90					
x_4			.10	.80	.10			
x_5					.90	.05		.05
x_6						.95	.05	
x_7						.05	.90	.05
x_8					.05		.05	.90

$= [P(y|x)]$.

Step 1: Computation of e_2 (per cent)

$Q_{max} = 2$

$P(e) = 0.10$

$H(X) = 3$

$e_2 = 11.9\%$.

The maximum error (per cent) resulting from the simpler estimate exceeds the desired ±5 per cent bound.

Step 2: Computation of e_1 (per cent)

$$e_1 \leq 2.2\%.$$

The maximum percentage in error that is caused by using $I_1(X;Y)$ as an estimate for $I(X;Y)$ is within the required limits of accuracy.

$$\frac{1}{2}\sum_{j=1}^{8} P(e|y_j)\ p(y_j)\ \log Q_j = 0.050$$

$$H(e|Y) = 0.383$$

$$I_1(X;Y) = 2.57 \text{ bits/stimulus.}$$

An exact calculation shows that

$$I(X;Y) = 2.59 \text{ bits/stimulus.}$$

Discussion

The amount of computation required for the estimate $I_1(X;Y)$ increases in proportion to the number of messages. The simpler estimate requires little computation and is independent of the number of messages. The maximum error (per cent) for both estimates decreases as H(X) increases, since the influence of H(e) in the denominator of equations (8) and (9) becomes less as H(X) becomes larger. When H(X) is small, the first estimate will usually be required. For larger values of H(X), the second estimate will usually yield acceptable values of e_2 per cent. While it is true that the amount of computation necessary for the evaluation of $I_1(X;Y)$ increases with the number of messages, it is also true that the probability that the simpler estimate will be satisfactory also increases with H(X).

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References

1. R. M. Fano, Transmission of Information (The M. I. T. Press, Cambridge, Mass., and John Wiley and Sons, Inc., New York, 1961).

2. Ibid., pp. 46-47.