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AND

ENGINEERING

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A. A SPECIAL CLASS OF QUANTIZER-INPUT SIGNALS

In a previous study,[1] the quantizer structure has been restricted in one case to have constrained transition values, and to have constrained representation values. (For nomenclature refer to Fig. XIII-1.) In each of these two cases sufficient conditions have been found on the error-weighting function that the quantization error surface had a

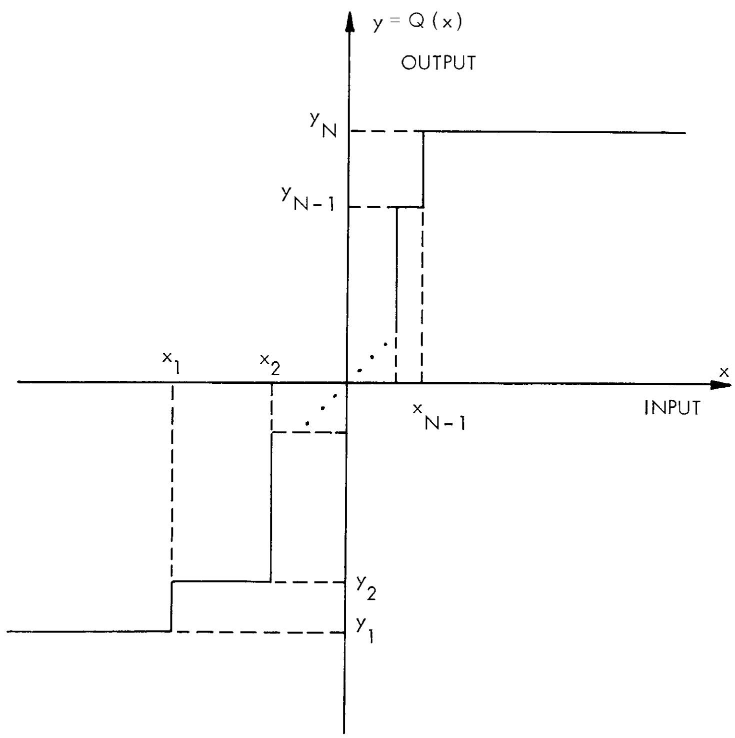

Fig. XIII-1. Input-output characteristic of the N-level quantizer. The x_i are the transition values and the y_i are the representation values.

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single relative extremum, a relative minimum. In a recent paper,[2] P. E. Fleischer has constrained the error-weighting function to be $g(e) = e^2$. He then derived a sufficient condition on the amplitude probability density of the quantizer-input signal so that the associated error surface will also have one relative extremum, a relative minimum. Fleischer's sufficient condition is expressed by the inequality

$$\frac{\partial^2}{\partial \xi^2}\{\ln [p_x(\xi)]\} < 0. \tag{1}$$

where $p_x(\xi)$, the amplitude probability density of the quantizer-input signal, is required to be continuous. In order to derive this condition, he determines a sufficient condition on the matrix of second partial derivatives of the quantization error so that all relative extrema will be relative minima. This condition is equivalent to requiring that the matrix of second partials be positive definite. In determining the condition for which this matrix is positive definite Fleischer used the row-sum condition.[3] The condition that is obtained guarantees that the error surface will have only one relative extremum and that it will be a relative minimum.

In a typical practical problem involving experimental data $p_x(\xi)$ will be specified numerically, rather than by an algebraic expression. Since numerical differentiation cannot be accurately accomplished, this particular form of Fleischer's condition cannot be used to determine whether or not $p_x(\xi)$ has a single relative extremum. Our primary purpose here is to transform (1) into an equivalent condition that can be used to determine whether or not a specific numerically specified amplitude density satisfies Fleischer's condition.

We begin by recalling the definition of a strictly convex function.

A function f(x) is strictly convex if and only if

$$f[\alpha a+(1-\alpha)b] < \alpha f(a) + (1-\alpha)f(b), \tag{2}$$

for all $b > a$ and all α such that $0 < \alpha < 1$. It can be determined, for example, by graphical consideration of the implications of Eq. 2, that strictly convex functions also satisfy the inequality

$$\frac{\partial^2}{\partial x^2}[f(x)] > 0. \tag{3}$$

Therefore, comparing Eqs. 1 and 3, we see that Fleischer's condition is equivalent to requiring that the function

$$\psi(\xi) = -\ln [p_x(\xi)] \tag{4}$$

be strictly convex. Observing that the strictly convex criteria (2), may be alternatively written

$$e^{-\psi[\alpha a+(1-\alpha)b]} > e^{-[\alpha\psi(a)+(1-\alpha)\psi(b)]} \tag{5}$$

for all b > a and for all α so that $0 < \alpha < 1$, we can write for (4)

$$p_x(\xi) = e^{-\psi(\xi)} \tag{6}$$

and, by direct substitution of (6) into (5), have

$$p_x[\alpha a+(1-\alpha)b] > [p_x(a)]^{\alpha}[p_x(b)]^{(1-\alpha)}. \tag{7}$$

If this inequality is satisfied for all b > a and all α so that $0 < \alpha < 1$, then Fleischer's condition is satisfied.

An examination of Eq. 7 indicates several properties of the amplitude probability densities which satisfy this condition. First, if we consider the case for which $p_x(a) = p_x(b)$, we find that (7) can be written equivalently as

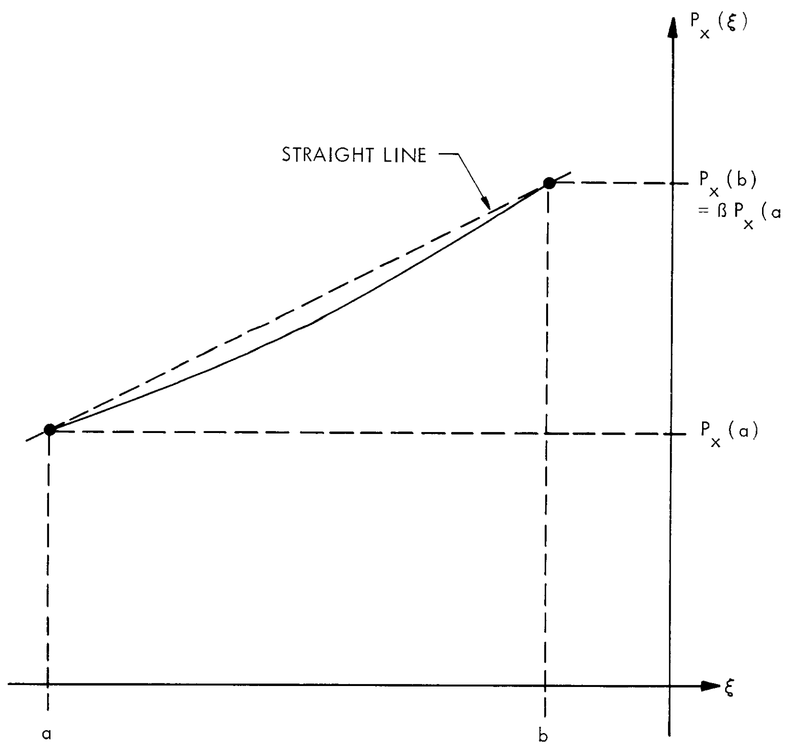

Fig. XIII-2. Illustration of Eq. 7 for $p_x(b) = \beta p_x(a)$. (The figure is drawn for $\beta = 2$.)

$$p_x[\alpha a+(1-\alpha)b] > p_x(a) = p_x(b)$$

or

$$p_x(\xi) > p_x(a) = p_x(b) \tag{8}$$

for $a < \xi < b$. This implies that the $p_x(\xi)$ satisfying this condition must have only one relative extremum and that this relative extremum is a relative maximum. Second, if we consider the case in which

$$p_x(b) = \beta p_x(a),$$

Eq. 7 becomes

$$p_x(\xi) > \beta^{(1-\alpha)} p_x(a) \tag{9}$$

for $a < \xi < b$. From a graphical examination of this condition (Fig. XIII-2), we see that the $p_x(\xi)$ that satisfy Fleischer's condition possess a type of mild convexity property.

In conclusion, we note that Eq. 7, the condition equivalent to Fleischer's condition, is one that can easily be utilized, for example, by means of a digital computer search program, to determine whether or not a given numerically obtained amplitude probability density satisfies Eq. 1.

J. D. Bruce

References

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2. P. E. Fleischer, Sufficient Conditions for Achieving Minimum Distortion in a Quantizer, 1964 IEEE International Convention Record, Part 1, pp. 104-111.

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B. OPTIMUM HOMOMORPHIC FILTERS

1. Introduction

In this report optimum homomorphic filters will be discussed. The problem can be stated as follows: two signals $s_1(t)$ and $s_2(t)$ are combined according to some rule, denoted o. We wish to determine the optimum homomorphic system from the class having o as both the input and output operations such that the error between the output and the desired output is minimized. Hence, we wish to select a nonlinear system from a specified class of nonlinear systems which is optimum according to some error criterion.

The error criterion that will be used depends on the class of homomorphic systems under consideration. Specifically, the error criterion to be associated with any particular filtering problem will be restricted to be the norm of the error vector in the vector space of system outputs. This will guarantee that the error criterion satisfies the properties that we associate with error criteria such as mean-square error or integral-square error. Since the norm that we associate with a particular vector space is not unique, this restriction still permits flexibility in the specific error criterion that we select and still affords analytical convenience.

Our approach is to consider the meaning of the error criterion on the output space in the light of the cascade representation for homomorphic systems. We shall show that the characteristic system to associate with the class can be chosen in such a way that minimization of the error is equivalent to minimization of mean-square error or integral-square error (the choice depending on whether the system inputs are continuing or aperiodic) at the output of the linear portion of the cascade representation. When the characteristic systems have been selected, the choice of the optimum system reduces to a choice of the linear system in the cascade representation.

2. Linear Filtering Problem

The linear filtering problem, considered from the vector-space point of view, consists in choosing a linear operator on the space, which will separate a given vector from a linear combination of vectors in the space. When the vectors represent time functions, with vector addition as the algebraic addition of the functions, we usually impose additional constraints such as realizability and time invariance of the system represented by the linear operator. The primary aspect of the vector-space interpretation of filtering is that it implies the determination of a linear transformation for which the range is a subspace of the domain. In the general case, for which the linear transformation corresponds to a homomorphic system, the output space is a subspace of the input space. On this basis, it is clear that the classes of homomorphic systems which are of interest in this discussion are those for which the input and output operations are the same.

Consider a vector space V. Let $v_1, v_2, \ldots, v_n$ be n vectors in V, and let v be a linear combination of these n vectors, that is,

$$v = \sum_{k=1}^{n} a_k v_k.$$

A necessary and sufficient condition so that there exists a linear transformation T_j on V such that

$$T_j(v) = a_j v_j$$

for all j, is that the set of vectors $v_1, v_2, \ldots, v_n$ be linearly independent.[1] When restated in terms of linear systems, this means that there is always a linear system that will filter an input from a linear combination of inputs, provided only that the inputs that have been combined are linearly independent. This can also be interpreted to state that if the vectors $v_1, v_2, \ldots, v_k$ are chosen from independent subspaces of V, then each of the vectors in the linear combination can be separated from the vector v. The implications of this with respect to linear multiplexing have been discussed by Zadeh and Miller.[2] The existence of the linear transformation T_j does not, of course, guarantee realizability or time invariance of the associated linear system.

When signal separation cannot be performed exactly because the vectors do not lie in independent subspaces or there are additional constraints to impose on the transformations, then an error criterion must be selected on the basis of which the optimum transformation can be chosen. When an inner product is defined on the vector space, this error criterion is generally selected to be the length, or norm, of the error vector. This choice is based on two considerations: (i) it is meaningful as an error criterion; and (ii) it is analytically convenient.

The error vector is defined as the vector sum of the desired output and the inverse of the actual output. That is,

$$v_e = v_d + (-v_o),$$

where v_e is the error vector, v_d is the vector representing the desired output, and v_o is the vector representing the actual output. The vector $(-v_o)$ represents the inverse of v_o. For example, if $s_d(t)$ is the desired output, $s_o(t)$ is the actual output and vector addition is the product of the time functions, then the time function corresponding to the error vector will be

$$e(t) = s_d(t)/s_o(t).$$

When an inner product is associated with the vector space, the norm of a vector is taken as the positive square root of the inner product of the vector with itself. The usefulness of the norm of the error vector as an error criterion arises directly from the algebraic properties of the norm. Specifically, let (v_1, v_2) denote the inner product of a vector v_1 with a vector v_2. Then for any vectors v_1, v_2, and v_3 in V and for any scalar c in the field associated with V, the properties of an inner product require that

(1) $(v_1, v_2) = \overline{(v_2, v_1)}$.

Here, the bar denotes conjugation.

(2) $(v_1+v_2, v_3) = (v_1, v_3) + (v_2, v_3)$

(3) $(cv_1, v_2) = c(v_1, v_2)$

(4) $(v_1, v_1) > 0$ if and only if v_1 is not the zero vector.

The norm of a vector v denoted $\|v\|$ in an inner product space is given by

$$\|v\|^2 = (v, v).$$

It follows from properties (1), (2), (3), and (4) that

(1') $\|v\| > 0$ if and only if v is not the zero vector

(2') $\|v_1+v_2\| \leq \|v_1\| + \|v_2\|$

(3') $\|cv_1\| = |c|\ \|v_1\|$.

From property (1') it is clear that any norm for the error vector satisfies the fundamental restriction that we would like to impose on an error criterion. Specifically, it guarantees that positive and negative errors cannot cancel, since $\|v_e\| = 0$ implies that the actual output and desired output are equal.

3. Optimum Homomorphic Filters

The cascade representation for the class of homomorphic systems having the operation o as both input and output operations is shown in Fig. XIII-3. The inputs are restricted to constitute a Hilbert space with an orthonormal basis, under some inner product, and the set of outputs to constitute a subspace of the vector space of inputs. The cascade representation is derived by mapping the input space linearly in a one-to-one manner onto a space in which vector addition corresponds to algebraic addition. A linear operation is then performed on this space and the resulting outputs are mapped onto the output space, which is a subspace of the inputs, by means of the inverse of the system a_o. I wish to show first that if L is chosen on the basis of any error criterion that is a norm on its output space, then the over-all system will be optimum under some error criterion that is a norm on the output space of the over-all system. This conclusion is based on Theorems I and II which follow.

THEOREM 1: Let V and W denote two inner product spaces, and T denote an invertible linear transformation between them. The inner product of two vectors v_i and v_j in V will be written as (v_i, v_j), and the inner product of two vectors w_i and w_j in W as $[w_i, w_j]$. Then the inner product $[w_i, w_j]$ in W can be taken as

$$[w_i, w_j] = \left(T^{-1}(w_i), T^{-1}(w_j)\right)$$

or equivalently,

$$[w_i, w_j] = (v_i, v_j), \tag{1}$$

where

$$v_i = T^{-1}(w_i)$$

and

$$v_j = T^{-1}(w_j).$$

PROOF: We need only show that the inner product on W satisfies properties (1)-(4), as required of an inner product.

Property 1: $[w_1, w_2] = (v_1, v_2)$

$$[w_2, w_1] = (v_2, v_1).$$

But (v_1, v_2) is an inner product of V; hence

$$(v_1, v_2) = \overline{(v_2, v_1)}.$$

Therefore

$$[w_1, w_2] = \overline{[w_2, w_1]}.$$

Property 2: $[w_1+w_2, w_3] = (v_1+v_2, v_3)$,

since T^{-1} is linear, and therefore

$$T^{-1}(w_1+w_2) = T^{-1}(w_1) + T^{-1}(w_2).$$

But $(v_1+v_2, v_3) = (v_1, v_3) + (v_2, v_3)$.

Also, $[w_1, w_3] + [w_2, w_3] = (v_1, v_3) + (v_2, v_3)$; hence

$$[w_1+w_2, w_3] = [w_1, w_3] + [w_2, w_3].$$

Property 3: $[cw_1, w_2] = \left(T^{-1}(cw_1), T^{-1}(w_2)\right)$

$$= \left(cT^{-1}(w_1), T^{-1}(w_2)\right)$$

$$= (cv_1, v_2)$$

$$= c(v_1, v_2).$$

Therefore

$$[cw_1, w_2] = c(v_1, v_2)$$

or

$$[cw_1, w_2] = c[w_1, w_2].$$

Property 4: Since T is invertible and linear,

$$T^{-1}(w) = 0 \quad \text{if and only if } w = 0.$$

By definition of the inner product in W,

$$[w, w] = (T^{-1}(w), T^{-1}(w)).$$

But $(T^{-1}(w), T^{-1}(w)) = 0$ if and only if $T^{-1}(w) = 0$, since the inner product in V satisfies the required properties. Consequently,

$$[w, w] = 0 \quad \text{if and only if } T^{-1}(w) = 0$$

which in turn requires that

$$[w, w] = 0 \quad \text{if and only if } w = 0.$$

THEOREM 2: Let V and W denote two Hilbert spaces with orthonormal bases. Let (v_a, v_b) denote the inner product of any two vectors v_a and v_b in V, and $[w_c, w_d]$ denote the inner product of any two vectors w_c and w_d in W. Then, an invertible linear transformation T can always be defined which maps V to W in such a way that

$$\begin{aligned}[w_a, w_b] &= \left(T^{-1}(w_a), T^{-1}(w_b)\right) \\ &= (v_a, v_b)\end{aligned}$$

for any vectors w_a and w_b in W.

PROOF: Let v_1, v_2, ... denote an orthonormal basis in V, and w_1, w_2, ... denote an orthonormal basis in W. Consider any vector v in V which, in terms of the basis in V, is

$$v = \sum_{k=1}^{\infty} (v, v_k)\, v_k.$$

Then the transformation T will be defined as

$$T(v) = \sum_{k=1}^{\infty} (v, v_k)\, w_k.$$

It has been shown that this transformation is linear and invertible.[3] The proof is omitted here to avoid a discussion of the convergence of the summation in Eq. 2. Now, consider any two vectors v_a and v_b in V, and let $T(v_a) = w_a$ and $T(v_b) = w_b$. The vector v_a is expressible as

$$v_a = \sum_{k=1}^{\infty} (v_a, v_k)\, v_k$$

and the vector v_b is expressible as

$$v_b = \sum_{k=1}^{\infty} (v_b, v_k)\, v_k.$$

Hence, $T(v_a) = w_a = \sum_{k=1}^{\infty} (v_a, v_k)\, w_k$ and $T(v_b) = w_b = \sum_{k=1}^{\infty} (v_b, v_k)\, w_k$. The inner product of w_a and w_b is, then,

$$[w_a, w_b] = \left(\sum_{k=1}^{\infty} (v_a, v_k)\, w_k, \sum_{r=1}^{\infty} (v_b, v_r)\, w_r \right)$$

$$= \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} (v_a, v_k)(v_r, v_b)[w_k, w_r].$$

But the set $w_1, w_2, \ldots$ is orthonormal in W; and hence

$$[w_k, w_r] = \begin{cases} 1 & k = r \\ 0 & k \neq r \end{cases}$$

Therefore

$$[w_a, w_b] = \sum_{k=1}^{\infty} (v_a, v_k)(v_k, v_b).$$

Similarly, the inner product of v_a and v_b is

$$(v_a, v_b) = \left(\sum_{k=1}^{\infty} (v_a, v_k)\, v_k, \sum_{r=1}^{\infty} (v_b, v_r)\, v_r \right)$$

$$= \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} (v_a, v_k)(v_r, v_b)(v_k, v_r)$$

$$= \sum_{k=1}^{\infty} (v_a, v_k)(v_k, v_b)$$

since the set $v_1, v_2, \ldots$ is orthonormal in V. Thus

$$[w_a, w_b] = (v_a, v_b) = \sum_{k=1}^{\infty} (v_a, v_k)(v_k, v_b).$$

Theorem 1 tells us that given a linear invertible transformation between two vector spaces, a norm can be defined on the output space in such a way that the norm of a vector is invariant under the transformation. Alternatively, if the norm has already been specified on both spaces, then from Theorem 2 we can find an invertible linear transformation between these spaces under which the norm of a vector is invariant. It remains to interpret these theorems within the framework of homomorphic systems.

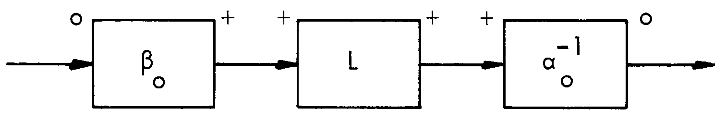

Fig. XIII-3. Cascade representation for the class of homomorphic systems having o as both the input and output operations.

The system represented in Fig. XIII-3 is a linear transformation mapping a vector space into itself. The output of the linear portion is a vector space V under addition (that is, with vector addition as the algebraic sum of the time functions) and the output of the over-all system is a vector space W under the operation o. The system α_o^{-1} is an invertible continuous linear transformation between these spaces. Let the norm of a vector v in V be rewritten as $\|v\|_V$, where restrictions are not placed on the norm other than those required by the definition of a norm. Similarly, let the norm of a vector w in W be written as $\|w\|_W$ defined as

$$\|w\|_W = \|\alpha_o(w)\|_V \tag{3}$$

or equivalently,

$$\|v\|_V = \|\alpha_o^{-1}(v)\|_W.$$

Suppose, now, that we wish to choose L such that the norm of the error vector in W is minimum. Let $f_d(t)$ be the desired output and $f_m(t)$ be the output for the optimum system, that is,

$$\|f_d \circ f^{-1}\|_W > \|f_d \circ f_m^{-1}\|_W \quad \text{for any f in W not equal to } f_m.$$

But $\|f_d \circ f^{-1}\|_W = \|\alpha_o^{-1}(f_d) \circ \alpha_o^{-1}(f^{-1})\|_V$. Hence

$$\|\alpha_o^{-1}(f_d) \circ \alpha_o^{-1}(f^{-1})\|_V > \|\alpha_o^{-1}(f_d) \circ \alpha_o^{-1}\left(f_m^{-1}\right)\|_V$$

This can be rewritten in a more convenient form by using the fact that a linear transformation between vector spaces maps inverses to inverses, which requires that

$$\| \alpha_o^{-1}(f_d) - \alpha_o^{-1}(f) \|_V > \| \alpha_o^{-1}(f_d) - \alpha_o^{-1}(f_m) \|_V$$

for any f in W that is not equal to f_m. Hence, if the over-all system is optimized with respect to the norm defined by Eq. 3, then the linear portion of the cascade representation must be optimum, according to the norm defined in V, when the desired output of the linear portion is taken to be $\alpha_o(f_d)$. Hence the problem of optimizing a homomorphic system within a specified class reduces to the optimization of the linear portion of the cascade representation relative to the desired output transformed by α_o. If the error criterion in W is specified, then the optimization of the linear system L must be carried out according to the error criterion specified by Eq. 3. If the error criterion in V is specified, then when the linear system L is optimum, the over-all system is optimum according to the error criterion dictated by Eq. 3.

As a result of Theorem 2, an alternative is offered. If an error criterion is specified in W, and we wish to optimize L according to a given error criterion in V, then the system α_o^{-1} can be constructed according to Theorem 2. Thus, if the error criterion is unspecified in W, we may make any convenient choice for the system α_o^{-1} and optimize L according to any convenient error criterion. If the error criterion is specified in W but unspecified in V, then a choice of α_o^{-1} will constrain the error criterion in V. Finally, if the error criterion is specified in both V and W, then when L is optimized the choice for α_o^{-1} is constrained if the over-all system is to be optimum. Thus far, we have been concerned with the choice of L and α_o^{-1} in the cascade representation. The question naturally arises as to how the optimization is affected by the system β_o. On the basis of the following theorem, it will be concluded that any choice for β_o can be made. The optimization of L will effectively compensate for the choice of β_o.

THEOREM 3: Let V and W be two vector spaces, and let β_a and β_b represent any two invertible homomorphic transformations from V onto W. Then there always exists a unique linear transformation L on W such that

$$\beta_a = L\beta_b,$$

where $L\beta_b$ represents the cascade of the transformations β_b and L. In other words, for every vector v in V, there exists L such that

$$\beta_a(v) = L[\beta_b(v)]. \tag{4}$$

PROOF: Consider the transformation L on W defined as

$$L(w) = \beta_a\left[\beta_b^{-1}(w)\right]. \tag{5}$$

The transformations β_a and β_b^{-1} are homomorphic, that is,

$$\beta_b^{-1}(w_1+w_2) = \beta_b^{-1}(w_1) + \beta_b^{-1}(w_2),$$

$$\beta_b^{-1}(cw) = c\beta_b^{-1}(w),$$

$$\beta_a(v_1+v_2) = \beta_a(v_1) + \beta_a(v_2)$$

and

$$\beta_a(cv) = c\beta_a(v).$$

We must show, first, that L is linear, that is,

$$L(w_1+w_2) = L(w_1) + L(w_2)$$

and

$$L(cw) = cL(w).$$

But

$$\begin{aligned} L(w_1+w_2) &= \beta_a\left[\beta_b^{-1}(w_1+w_2)\right] \\ &= \beta_a\left[\beta_b^{-1}(w_1)+\beta_b^{-1}(w_2)\right] \\ &= \beta_a\left[\beta_b^{-1}(w_1)\right] + \beta_a\left[\beta_b^{-1}(w_2)\right] \end{aligned}$$

and

$$\beta_a\left[\beta_b^{-1}(w_1)\right] = L(w_1)$$

$$\beta_a\left[\beta_b^{-1}(w_2)\right] = L(w_2).$$

Therefore $L(w_1+w_2) = L(w_1) + L(w_2)$. Similarly,

$$\begin{aligned} L(cw) &= \beta_a\left[\beta_b^{-1}(cw)\right] \\ &= \beta_a\left[c\beta_b^{-1}(w)\right] \\ &= c\beta_a\left[\beta_b^{-1}(w)\right]. \end{aligned}$$

Hence $L(cw) = cL(w)$, and consequently L is linear. But L, defined in Eq. 5, provides the condition required by Eq. 4, for substituting Eq. 4 in Eq. 5, we obtain

$$\beta_a(v) = \beta_a\left[\beta_b^{-1}(\beta_b(v))\right]$$
$$= \beta_a(v).$$

It remains only to show that L as defined by Eq. 4 is unique. This will be done by showing that Eq. 4 implies Eq. 5.

Consider any transformation L having the property required by Eq. 4. If w is any vector in W, then there exists a vector v in V such that

$$v = \beta_b^{-1}(w),$$

since β_b is invertible. Substituting in Eq. 4, we have

$$\beta_a\left[\beta_b^{-1}(w)\right] = L\left[\beta_b\left(\beta_b^{-1}(w)\right)\right]$$
$$= L(w)$$

which is the transformation defined by Eq. 5.

Now, suppose that in the cascade representation of Fig. XIII-3, we have made an arbitrary choice for the system β_o. The error criterion used to optimize L, and the error criterion used to optimize the over-all system have been shown to determine the system α_o. Let L_{opt} denote the optimum choice for L. By virtue of Theorem 3, if the error could be further reduced by varying the system β_o, then the error could be reduced instead by cascading L_{opt} with an invertible linear system, which violates the assumption that L_{opt} is the optimum choice for L.

4. Example

In order to relate the preceding discussion to a specific example, consider the optimization of the systems in the class having multiplication as both input and output operations. This corresponds to the determination of an optimum filter for separating signals that have been multiplied, as is the case, for example, in transmission over a time-variant channel. It should be emphasized at the outset that for this example, the formalism that we have just completed does not seem justified by the intuitively reasonable result. The formalism indicates, however, that the approach is generally applicable to any class of homomorphic systems.

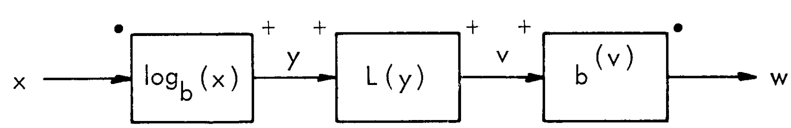

Fig. XIII-4. Cascade representation for the class of homomorphic systems having multiplication as both the input and output operations.

In the cascade representation for the class under consideration, the system α_o can be chosen as a logarithmic amplifier relative to any base b as shown in Fig. XIII-4, if the system inputs and outputs are restricted to be positive. Let us restrict the system L to be a linear, time-invariant, realizable system. If the input time functions for the over-all system are random, then a mean-square-error criterion is an analytically convenient choice for optimization of the linear system. Hence, if $f_d(t)$ is the desired output of the over-all system, then on the basis of the preceding discussion the system L is to be so chosen that the error E given by

$$E = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\log_b(f_d)-\log_b(f_o)]^2\, dt \tag{6}$$

is minimum. If the input to the homomorphic system is $f_i(t)$, then

$$\log_{\beta_1}(f_o) = L[\log_{\beta_1}(f_i)].$$

The solution for the optimum system L is specified by the Weiner-Hopf equation.

The error E is also the error at the output of the exponential amplifier. If we change the error criterion in W but wish to maintain a mean-square-error criterion in V, then the characteristic systems α_o and β_o must be changed.

5. Conclusion

Homomorphic systems can be used for the separation of signals that have been non-additively combined in much the same way as linear systems are used for the separation of signals that have been linearly combined. The manner in which the signals are combined determines the class of homomorphic systems from which the optimum system is selected.

In determining optimum systems, the error criterion is generally based on consideration of its validity as a measure of error, and the convenience that it offers in carrying out the optimization. Since homomorphic systems are represented by linear transformations on vector spaces, the error criterion is restricted in this study to be describable as the norm of the error vector. This restriction does not completely specify the error criterion.

On the basis of the cascade representation for homomorphic systems, we have shown that the optimization within a class of homomorphic systems can be reformulated in terms of the optimization of the linear portion of the cascade representation. If the linear system is optimized according to some error criterion, then the over-all system will also be optimum according to some error criterion. In this case the error criterion under which the system is optimum will depend on the choice of the characteristic system used in the cascade representation to map the outputs of the linear system to the outputs of the over-all system. In particular, a mean-square-error criterion (or integral-square error if the inputs to the linear system are aperiodic) can be used to determine the linear portion of the optimum filter. Alternatively, the error criteria for both the linear system and the over-all system can be selected independently. When this is done, the choice for the characteristic system is restricted.

If a mean-square or integral-square error criterion is used to optimize the linear system and this system is restricted to be time-invariant, then the optimization can be carried out by the methods developed by Wiener in his theory of optimum linear filtering. This does not guarantee that the total system will be time-invariant. To insure this, the characteristic systems must also be chosen to be time-invariant.

The ideas discussed in this paper may be potentially useful in the study of time-variant communication channels, for which the transmitted signal has effectively been multiplied by a noise waveform, and in the study of multipath channels in which signal and noise have been convolved. It is hoped that through this study further work and interest in this approach will be stimulated.

A. V. Oppenheim

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C. A SINGULAR MAXIMIZATION PROBLEM WITH APPLICATIONS TO FILTERING

In this report we generalize a previous result[1] on a problem of extrema under constraints and discuss its application to filtering and estimation.

The mathematical problem is the following: We would like to find the probability density p(x) that maximizes or minimizes the functional

$$L = \int L(x)\, p(x)\, dx \quad \text{under M constraints}$$

$$G = \int G(x)\, p(x)\, dx$$

$$F = \int F(x)\, p(x)\, dx$$

$$\vdots$$

$$1 = \int p(x)\, dx.$$

We require the range of the random variable x to be bounded to some finite interval [A, B].

We shall show that if there exists a p(x) satisfying all of the constraints, then the probability density that maximizes or minimizes L under M constraints is made up of M impulses, at most.

PROOF: Assume that there exists $p_o(x)$ such that all M constraints are satisfied and which results in some value L_o of the functional L. In Fig. XIII-5 we define the regions $R_1, R_2 \dots R_{M+1}$ by arbitrary cuts of the base [A, B].

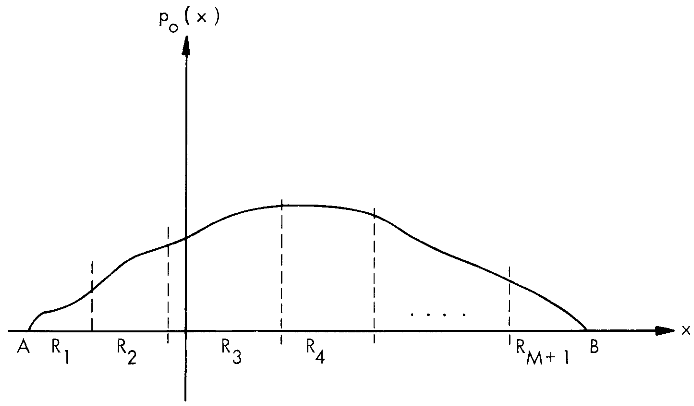

Fig. XIII-5. Probability density $p_o(x)$ that satisfies all constraints.

We shall show that there exists a probability density $p_1(x)$ that is nonzero on only M of the M+1 regions, which satisfies the M constraints and gives $L_1 \leq L_o$. Similarly, $p_2(x)$, which is nonzero on M regions at most, satisfies the M constraints, and yields $L_2 \geq L_o$. It then becomes clear that by successive applications of this result, the base of the probability density p(x) is reduced, at most, to M arbitrary small regions. Therefore the probability density p(x) that maximizes or minimizes L under M constraints will consist, at most, of M impulses.

To show that $p_1(x)$ and $p_2(x)$ with the properties claimed do exist, we define

$$m_1 = \int_{R_1} p_o(x)\,dx$$

$$F_1 = \frac{1}{m_1}\int_{R_1} F(x)\,p_o(x)\,dx$$

$$G_1 = \frac{1}{m_1}\int_{R_1} G(x)\,p_o(x)\,dx$$

$$\vdots$$

$$L_{o1} = \frac{1}{m_1} \int_{R_1} L(x)\, p_o(x)\, dx$$

$$m_2 = \int_{R_2} p_o(x)\, dx$$

$$F_2 = \frac{1}{m_2} \int_{R_2} F(x)\, p_o(x)\, dx$$

$$\vdots$$

$$L_{o2} = \frac{1}{m_2} \int_{R_2} L(x)\, p_o(x)\, dx$$

etc.

We have, therefore, the new set of equations

$$m_1 L_{o1} + m_2 L_{o2} + \dots + m_{M+1} L_{oM+1} = L_o$$

$$m_1 G_1 + \dots + m_{M+1} G_{M+1} = G$$

$$\vdots$$

$$m_1 + \dots + m_{M+1} = 1$$

and since $p(x) \geq 0$, we have also

$$m_1,\ m_2 \dots m_{M+1} \geq 0.$$

Now consider these equations in the M+1 dimensional space of the weights $\underline{m}$. The M+1 equations define a point in this space, and any M equations define a line. Consider the last M equations. They define a line C_1 which intersects the subspace $m_1 L_{o1} + m_2 L_{o2} + \dots + m_{M+1} L_{oM+1} = L_o$ at one point. The situation is represented in three dimensions in Fig. XIII-6.

On the constraint line C_1 all points not in the subspace $L = L_o$ correspond to $L > L_o$ of $L < L_o$. The line C_1 will leave the subspace of all $m_i > 0$, $(i=1\dots M+1)$ at two points, say, $m_k = 0$ $m_e = 0$. These two points are necessarily on opposite sides of the subspace $L = L_o$. Therefore, we have $L > L_o$ at one point, and $L < L_o$ at the other. Therefore, in the space $\underline{m}$, these two points satisfy all constraints and give smaller and larger values to the functional L. In terms of p(x), these two points correspond to a reassignment of the weights in the M+1 regions, making one of the weights zero, which is our result.

It is of interest to discuss briefly the reason why M impulses are needed for the number of constraints used. Assume that we subdivide p(x) in M regions instead of M+1; we have, then, M+1 equations specifying one point in M-dimensional space. The M

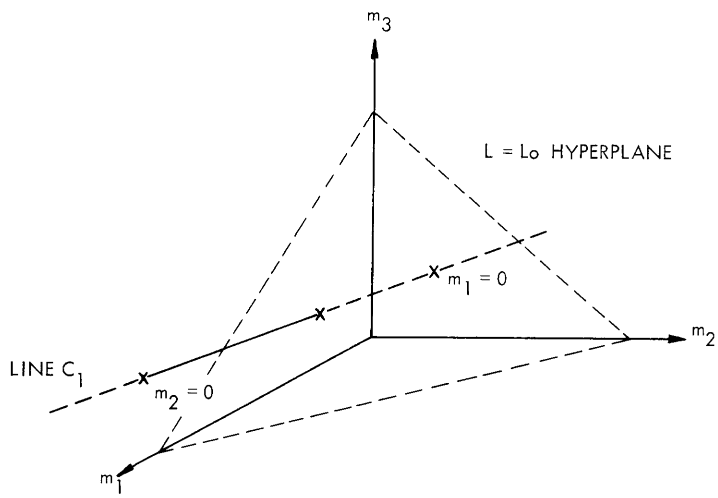

Fig. XIII-6. Discussion of the weight vector m in three dimensions.

constraints alone define one point in this M-dimensional space, the point we started from, and we cannot reassign weights in the M regions to increase or decrease L. Similarly, if we started with M+2 regions, then in the (M+2)-dimensional space, we satisfy M constraints by constraining $\underline{m}$ to a plane, and this allows us to make two of the weights equal to zero. The present result has been established previously[1] for M=2 by a more involved method, which did not lead simply to an extension.

GENERALIZATION: From the method of proof, it is clear that the result is not limited to a one-dimensional probability density p(x) and will hold for probability densities with any finite number of dimensions, as long as the range of $\underline{x} = (x_1, \ldots x_R)$ is bounded for all the variables. Using a vector notation, we have immediately the fact that is $p(\underline{x})$ satisfies M constraints of the form

$$\int G(\underline{x})\ p(\underline{x})\ \underline{dx} = G,$$

then the functional L, $L = \int L(\underline{x})\ p(\underline{x})\ \underline{dx}$ will be maximized or minimized for a probability density $p(\underline{x})$ which consists of M impulses at most.

Applications

1. Extrema of the Average Weighted Error for a Given Nonlinear Filter

Consider an input signal x(t), x(t) = m(t) + n(t) in which m(t) and n(t) are statistically independent processes. We desire to estimate the message m(t) at time $t = \{t_1 \ldots t_k\}$ from knowledge of the input with the same time constant. The noise statistics are known. Let

$$\underline{\hat{m}} = \{\hat{m}_1, \hat{m}_2, \ldots \hat{m}_k\}$$

be the estimates of $m(t_1)$, $m(t_2)$ ..., and assume that the filter characteristic is known. We have

$$\underline{\hat{m}} = \underline{g}(\underline{x})$$

in which $\underline{g}(\underline{x})$ represents a set of k functions of the k input amplitudes. Let the error-weighting function be a scalar function of k variable $W(\underline{e})$ such as $W(\underline{e}) = e_1^2 + e_2^2 + \ldots + e_k^2$. Then the average weighted error can be written

$$\overline{W(\underline{e})} = \iint W[\underline{\lambda} - \underline{g}(\underline{\xi})]\, p_{\underline{m}, \underline{x}}(\underline{\lambda}, \underline{\xi})\, d\underline{\lambda}, d\underline{\xi}.$$

But

$$p_{\underline{m}, \underline{x}}(\underline{\lambda}, \underline{\xi}) = p_{\underline{m}}(\underline{\lambda})\, p_{\underline{n}}(\underline{\xi} - \underline{\lambda}),$$

so that

$$\overline{W(\underline{e})} = \iint W[\underline{\lambda} - \underline{g}(\underline{\xi})]\, p_{\underline{n}}(\underline{\xi} - \underline{\lambda})\, p_{\underline{m}}(\underline{\lambda})\, d\underline{\lambda} d\underline{\xi}.$$

If we let

$$L(\underline{\lambda}) \triangleq \int W[\underline{\lambda} - \underline{g}(\underline{\xi})]\, p_{\underline{n}}(\underline{\xi} - \underline{\lambda})\, d\underline{\xi}$$

in which $L(\underline{\lambda})$ is a known function of $\underline{\lambda}$, then

$$\overline{W(\underline{e})} = \int L(\underline{\lambda})\, p_{\underline{m}}(\underline{\lambda})\, d(\underline{\lambda}). \tag{1}$$

The problem considered is that of finding the specific message density, $p_{\underline{m}}(\underline{\lambda})$, that will maximize or minimize expression (1) under a set of message constraints. This is the mathematical problem discussed earlier; and if the message probability density $p(\underline{x})$ satisfies M constraints of the form $\int G(\underline{\lambda})\, p_{\underline{m}}(\underline{\lambda})\, d\underline{\lambda} = G$, then $\overline{W(\underline{e})}$ will be maximizes or minimized if $p_{\underline{m}}(\underline{\lambda})$ consists of M impulses at most.

2. Minimum of the Average Weighted Error for the Optimum Nonlinear Filter

Since for any given nonlinear filter we have found that the minimum average weighted error corresponds to an impulsive message probability density, it is a simple matter to show that the same result holds if minimization is carried out among the filters, as well as among the message probability densities. Assume that $p_{\underline{m}o}(\underline{\lambda})$ and $\underline{g}_o(\underline{x})$ are the message probability density and the corresponding optimum nonlinear filter, and that the minimum average weighted error is $\overline{W_o(\underline{e})}$. For the filter $\underline{g}_o(\underline{x})$ considered as given there exitst an impulsive message probability density $p_{\underline{m}1}(\underline{\lambda})$ such that $\overline{W_1(\underline{e})} \leq \overline{W_o(\underline{e})}$. Also the optimum nonlinear filter $g_1(\underline{x})$ associated with $p_{\underline{m}1}(\underline{\lambda})$ will give $\overline{W_2(\underline{e})}$ such that

$$\overline{W_2(\underline{e})} \leq \overline{W_1(\underline{e})} \leq \overline{W_o(\underline{e})}. \tag{2}$$

Since, by hypothesis, $\overline{W_o(\underline{e})}$ is the lowest value achievable, we have necessarily equality in Eq. 2; therefore, the lowest average weighted error, $\overline{W_{min}(\underline{e})}$ is given by an impulsive message probability density $p_{\underline{m}1}(\underline{\lambda})$.

Some examples for M=2 have been presented previously.[1] A more comprehensive discussion of these results will appear elsewhere.

V. R. Algazi

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D. ON THE RELATION BETWEEN INTEGRAL AND DIFFERENTIAL CHARACTERIZATION OF NONLINEAR SYSTEMS

Consider the nonlinear system shown in Fig. XIII-7; N_1, N_2, and N_3 are linear systems and N_4 is a multiplier. The behavior of the system can be characterized by the set

$$\frac{dz(t)}{dt} + az(t) = x(t) \tag{1}$$

$$\frac{dw(t)}{dt} + bw(t) = x(t) \tag{2}$$

$$\frac{d^2y(t)}{dt^2} + d\,\frac{dy(t)}{dt} + ey(t) = cr(t) + \frac{dr(t)}{dt} \tag{3}$$

$$r(t) = w(t)\,z(t), \tag{4}$$

where x(t) is the input, y(t) is the output, and w(t), z(t), and r(t) are the inputs and output of the multiplier, as shown in Fig. XIII-7. Equations 1-4 describe the behavior of N_1 through N_4. We assume that all initial conditions are zero.

We would like to find a differential equation relating y(t) and x(t); that is, we would like to eliminate w(t), z(t), and r(t) in Eqs. 1-4. In order to do so, we shall extend the domain of definition from a line to a plane, and look along the 45° line in the plane.

Define $\hat{r}(t_1, t_2) = w(t_1)\,z(t_2)$, and $\hat{y}(t_1, t_2)$ so that y(t) is $\hat{y}(t_1, t_2)\Big|_{t_1=t_2=t}$. Substitute t_1 for t in (2) and t_2 for t in (1). Then multiplication of (1) and (2) and use of the definition of $\hat{r}(t_1, t_2)$ yields

$$\frac{\partial^2 \hat{r}(t_1, t_2)}{\partial t_2 \partial t_1} + a \frac{\partial \hat{r}(t_1, t_2)}{\partial t_2} + b \frac{\partial \hat{r}(t_1, t_2)}{\partial t_1} + ab\hat{r}(t_1, t_2) = x(t_1)\, x(t_2) \tag{5}$$

In order to express (3) in terms of $y(t_1, t_2)$ we must find an expression for $\frac{dy(t)}{dt}$ in terms of $\hat{y}(t_1, t_2)$. Since $\hat{y}(t, t) = y(t)$, the desired derivative will be the directional derivative

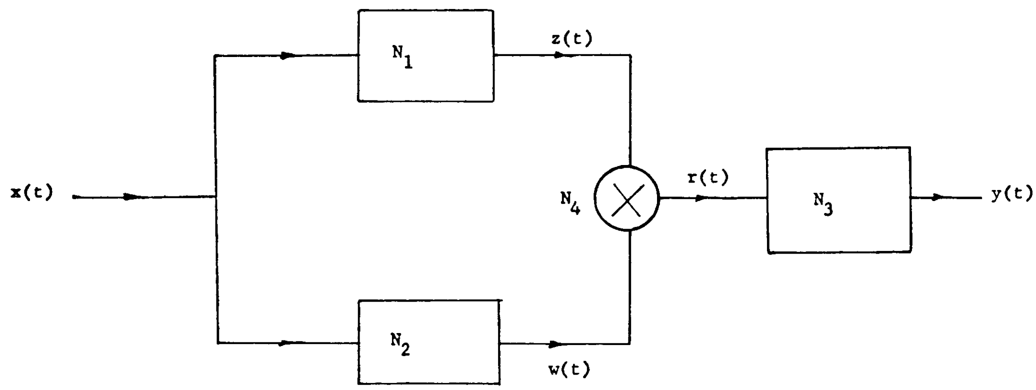

Fig. XIII-7. A simple nonlinear system.

of $\hat{y}(t_1, t_2)$ along the line $t_1 = t_2$, scaled by the factor $\sqrt{2}$ to obtain the proper rate of change. The directional derivative is given by the dot product of the gradient of $\hat{y}(t_1, t_2)$ with the unit vector in the direction of the 45° line. Hence we have the correspondence

$$\frac{dy(t)}{dt} \longleftrightarrow \sqrt{2} \left[\nabla \hat{y}(t_1, t_2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right] = \frac{\partial \hat{y}(t_1, t_2)}{\partial t_1} + \frac{\partial \hat{y}(t_1, t_2)}{\partial t_2}. \tag{6}$$

Repeating this operation, we find the correspondence for the second derivative.

$$\frac{d^2 y(t)}{dt^2} \longleftrightarrow \frac{\partial^2 \hat{y}(t_1, t_2)}{\partial t_1^2} + 2 \frac{\partial^2 \hat{y}(t_1, t_2)}{\partial t_2 \partial t_1} + \frac{\partial^2 \hat{y}(t_1, t_2)}{\partial t_2^2}. \tag{7}$$

By using these results, (3) can be extended to

$$\frac{\partial^2 \hat{y}}{\partial t_1^2} + 2 \frac{\partial^2 \hat{y}}{\partial t_2 \partial t_1} + \frac{\partial^2 \hat{y}}{\partial t_2^2} + d \frac{\partial \hat{y}}{\partial t_1} + d \frac{\partial \hat{y}}{\partial t_2} + e\hat{y} = c\hat{r} + \frac{\partial \hat{r}}{\partial t_1} + \frac{\partial \hat{r}}{\partial t_2}. \tag{8}$$

We must now combine (8) and (5) to eliminate $\hat{r}(t_1, t_2)$. This may be accomplished by taking the partial of both sides of (5) with respect to t_1 to obtain (9), and with respect to t_2 to obtain (10).

$$\frac{\partial^3\hat{r}}{\partial t_2 \partial^2 t_1} + a\frac{\partial^2\hat{r}}{\partial t_2 \partial t_1} + b\frac{\partial^2\hat{r}}{\partial t_1^2} + ab\frac{\partial\hat{r}}{\partial t_1} = \frac{dx(t_1)}{dt_1} x(t_2) \tag{9}$$

$$\frac{\partial^3\hat{r}}{\partial t_2^2 \partial t_1} + a\frac{\partial^2\hat{r}}{\partial t_2^2} + b\frac{\partial^2\hat{r}}{\partial t_2 \partial t_1} + ab\frac{\partial\hat{r}}{\partial t_2} = x(t_1)\frac{dx(t_2)}{dt_2}. \tag{10}$$

Also, we take the partial of (8) with respect to t_1 to obtain (11), with respect to t_2 to obtain (12), and with respect to t_1 and t_2 to obtain (13).

$$\frac{\partial^3\hat{y}}{\partial t_1^3} + 2\frac{\partial^3\hat{y}}{\partial t_2 \partial t_1^2} + \frac{\partial^3\hat{y}}{\partial t_2^2 \partial t_1} + d\frac{\partial^2\hat{y}}{\partial t_1^2} + d\frac{\partial^2\hat{y}}{\partial t_2 \partial t_1} + e\frac{\partial\hat{y}}{\partial t_1} = c\frac{\partial\hat{r}}{\partial t_1} + \frac{\partial^2\hat{r}}{\partial t_1} + \frac{\partial^2\hat{r}}{\partial t_2 \partial t_1} \tag{11}$$

$$\frac{\partial^3\hat{y}}{\partial t_2 \partial t_1^2} + 2\frac{\partial^3\hat{y}}{\partial t_2^2 \partial t_1} + \frac{\partial^3\hat{y}}{\partial t_2^3} + d\frac{\partial^2\hat{y}}{\partial t_2 \partial t_1} + d\frac{\partial^2\hat{y}}{\partial t_2^2} + e\frac{\partial\hat{y}}{\partial t_2} = c\frac{\partial\hat{r}}{\partial t_2} + \frac{\partial^2\hat{r}}{\partial t_2 \partial t_1} + \frac{\partial^2\hat{r}}{\partial t_2^2} \tag{12}$$

$$\frac{\partial^4\hat{y}}{\partial t_2 \partial t_1^3} + 2\frac{\partial^4\hat{y}}{\partial t_2^2 \partial t_1^2} + \frac{\partial^4\hat{y}}{\partial t_2^3 \partial t_1} + d\frac{\partial^3\hat{y}}{\partial t_2 \partial t_1^2} + d\frac{\partial^3\hat{y}}{\partial t_2^2 \partial t_1} + e\frac{\partial^2\hat{y}}{\partial t_2 \partial t_1} = c\frac{\partial^2\hat{r}}{\partial t_2 \partial t_1} + \frac{\partial^3\hat{r}}{\partial t_2 \partial t_1^2} + \frac{\partial^3\hat{r}}{\partial t_2^2 \partial t_1}. \tag{13}$$

Next, multiply through (5) by c, and add the resulting equation to (9) plus (10). Multiply through (12) by a, (11) by b, and (8) by ab, and add the sum of these new equations to (13). We can then write

$$cx(t_1)\, x(t_2) + x(t_1)\frac{dx(t_2)}{dt_2} + \frac{dx(t_1)}{dt_1} x(t_2) =$$

$$\frac{\partial^3\hat{r}}{\partial t_2^2 \partial t_1} + \frac{\partial^3\hat{r}}{\partial t_2 \partial t_1^2} + (a+b+c)\frac{\partial^2\hat{r}}{\partial t_2 \partial t_1} + b\frac{\partial^2\hat{r}}{\partial t_1^2} + a\frac{\partial^2\hat{r}}{\partial t_2^2} + (ab+bc)\frac{\partial\hat{r}}{\partial t_1} + (ab+ac)\frac{\partial\hat{r}}{\partial t_2} + abc\hat{r} =$$

$$\frac{\partial^4\hat{y}}{\partial t_2 \partial t_1^3} + 2\frac{\partial^4\hat{y}}{\partial t_2^2 \partial t_1^2} + \frac{\partial^4\hat{y}}{\partial t_2^3 \partial t_1} + (a+2b+d)\frac{\partial^3\hat{y}}{\partial t_2 \partial t_1^2} + (2a+b+d)\frac{\partial^3\hat{y}}{\partial t_2^2 \partial t_1} + b\frac{\partial^3\hat{y}}{\partial t_1^3}$$

$$+ a\frac{\partial^3\hat{y}}{\partial t_2^3} + (ad+2ab+bd+e)\frac{\partial^2\hat{y}}{\partial t_2 \partial t_1} + (ab+bd)\frac{\partial^2\hat{y}}{\partial t_1^2} + (ab+ad)\frac{\partial^2\hat{y}}{\partial t_2^2}$$

$$+ (abd+be)\frac{\partial\hat{y}}{\partial t_1} + (abd+ae)\frac{\partial\hat{y}}{\partial t_2} + abe\hat{y},$$

or

$$\frac{\partial^4\hat{y}}{\partial t_2 \partial t_1^3} + 2\frac{\partial^4\hat{y}}{\partial t_2^2 \partial t_1^2} + \frac{\partial^4\hat{y}}{\partial t_2^3 \partial t_1} + (a+2b+d)\frac{\partial^3\hat{y}}{\partial t_2 \partial t_1^2} + (2a+b+d)\frac{\partial^3\hat{y}}{\partial t_2^2 \partial t_1} + b\frac{\partial^3\hat{y}}{\partial t_1^3}$$

$$+ a\frac{\partial^3\hat{y}}{\partial t_2^3} + (ad+2ab+bd+e)\frac{\partial^2\hat{y}}{\partial t_2 \partial t_1} + (ab+bd)\frac{\partial^2\hat{y}}{\partial t_1^2} + (ab+ad)\frac{\partial^2\hat{y}}{\partial t_2^2}$$

$$+ (abd+be)\frac{\partial\hat{y}}{\partial t_1} + (abd+ae)\frac{\partial\hat{y}}{\partial t_2} + abe\hat{y} =$$

$$cx(t_1)\, x(t_2) + x(t_1)\frac{dx(t_2)}{dt_2} + \frac{dx(t_1)}{dt_1}\, x(t_2). \tag{14}$$

We have thus obtained a single differential equation relating $y(t)$ and $x(t)$. The equation is a linear partial differential equation with constant coefficients. This linear partial differential equation is particularly well suited to solution by means of the two-dimensional Laplace transform. Taking the transform of each side, we find

$$\Big[s_1^3 s_2 + s_1^2 s_2^2 + s_1 s_2^3 + (a+2b+d)s_1^2 s_2 + (2a+b+d)s_1 s_2^2 + bs_1^3 + as_2^3 + (ad+2ab+bd+s)s_1 s_2$$

$$+ (ab+bd)s_1^2 + (ab+bd)s_2^2 + (abd+be)s_1 + (abd+ae)s_2 + abe \Big] \hat{Y}(s_1 s_2 = (s_1+s_2+c)\, X(s_1)\, X(s_2)$$

Factoring the polynomials in this expression and solving for $\hat{Y}(s_1, s_2)$, we have

$$\hat{Y}(s_1, s_2) = \frac{s_1 + s_2 + c}{(s_1+b)(s_2+a)[(s_1+s_2)^2 + d(s_1+s_2) + e]}\, X(s_1)\, X(s_2). \tag{15}$$

We now note that

$$H_2(s_1, s_2) \triangleq \frac{s_1 + s_2 + c}{(s_1+b)(s_2+a)[(s_1+s_2)^2 + d(s_1+s_2) + e]}$$

is the transform of the Volterra kernel, $h_2(\tau_1, \tau_2)$ of the system of Fig. XIII-7 when the system is characterized by the integral equation

$$y(t) = \iint h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1 d\tau_2. \tag{16}$$

In fact, taking the inverse transform of (15), we have

$$\hat{y}(t_1, t_2) = \iint h_2(\tau_1, \tau_2)\, x(t_1-\tau_1)\, x(t_2-\tau_2)\, d\tau_1 d\tau_2$$

from which (16) follows by setting $t_1=t_2=t$.

From this example the following observations may be made. Given a system of equations which is the dynamic description of a nonlinear system, we can, by extending the domain of definition from one dimension to two dimensions, find a single linear partial differential equation that also characterizes the system. That is, by extending from one dimension to two dimensions, a one-dimensional nonlinear problem has been converted into a two-dimensional linear problem.

Equations 1-4 and Eq. 14 describe the same situation. A system that is characterized by a single integral equation is equivalently described by a set of several ordinary differential equations and a nondifferential equation. A description by one nonlinear ordinary differential equation does not seem to be possible.

Whenever a system is characterized by an n^{th}-degree Volterra kernel that has a rational transform, a linear partial differential equation with constant coefficients can be found relating the auxiliary output function $\hat{y}(t_1, \ldots, t_n)$ to the input function $x(t)$. If the kernel is of the class that can be realized exactly with a finite number of linear systems and multipliers,[1] then an equivalent description by a set of ordinary differential equations and nondifferential equations can be found.

Although the example and observations presented here have not yet led to the solution of any problems not easily handled by other methods, we feel that the viewpoint presented is unique and may lead to a deeper understanding of the properties of nonlinear systems.

A. M. Bush

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E. USEFUL EXPRESSIONS FOR OPTIMUM LINEAR FILTERING IN WHITE NOISE

In this report, optimum linear filters for extracting a message from an additive white noise are considered. Results that have been derived may be listed as follows:

1. $$H_{opt}(s) = 1 - \frac{N_o^{1/2}}{[S_a(-s^2) + N_o]^+} \tag{1}$$

2. $\overline{\epsilon^2_{min}}$, the minimum mean-square error, is given by

$$\overline{\epsilon^2_{min}} = N_o h_{opt}(0+) = N_o \lim_{s\to\infty} sH_{opt}(s) \tag{2}$$

$$\overline{\epsilon^2_{min}} = \frac{N_o}{2\pi}\int_{-\infty}^{\infty} \log\left[1+\frac{S_a(\omega^2)}{N_o}\right] d\omega. \tag{3}$$

3. $H_{opt}(s)$ is minimum phase.

Equation 2 is especially convenient when the optimum filter is known. On the other hand, Eq. 3 is useful because it makes possible the determination of $\overline{\epsilon^2_{min}}$ without $H_{opt}(s)$.

Equations 1 and 3 have found wide application in the study of analog demodulation theory, their existence being crucial in studies by Develet,[2] Viterbi and Cahn,[3] and Van Trees and Boardman.[4]

Equations 1 and 3 have been derived previously by Yovits and Jackson,[1] and more recently by Viterbi and Cahn.[3] Here, all results are derived by use of a well-known expression for the optimum filter. No minimization arguments are required, as was previously the case.

1. Derivation of the Expression for $H_{opt}(s)$

The optimum linear filter is given by

$$H_{opt}(s) = \frac{1}{[S_a(-s^2)+N_o]^+}\left[\frac{S_a(-s^2)}{[S_a(-s^2)+N_o]^-}\right]_+, \tag{4}$$

where $S_a(\omega^2)$ and N_o are the power densities of the signal and noise, respectively.[5-7] $S_a(\omega^2)$ is taken to be a rational function of ω^2; the superscripts + and − indicate spectral factorization; the subscript + indicates the operation of taking the realizable part of a partial fraction expansion.

It is seen from Eq. 4 that

$$H_{opt}(s) = \frac{1}{[S_a(-s^2)+N_o]^+}\left[\frac{S_a(-s^2)+N_o}{[S_a(-s^2)+N_o]^-} - \frac{N_o}{[S_a(-s^2)+N_o]^-}\right]_+$$

$$= 1 - \frac{1}{[S_a(-s^2)+N_o]^+}\left[\frac{N_o}{[S_a(-s^2)+N_o]^-}\right]_+. \tag{5}$$

The term with the subscript + in Eq. 5 can only be a constant. Under the assumption that $\lim_{\omega\to\infty} S_a(\omega^2) = 0$, the constant is $N_o^{1/2}$. Equation 1 follows upon substitution of this constant in Eq. 5.

2. Derivation of $\overline{\epsilon^2_{min}}$ in Terms of the Optimum Filter

The minimum mean-square error is given by

$$\overline{\epsilon^2_{min}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_a(\omega^2)\ \left|1-H_{opt}(\omega)\right|^2 d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} N_o\left|H_{opt}(\omega)\right|^2 d\omega. \tag{6}$$

From Eq. 1, it follows that

$$\left|1-H_{opt}(\omega)\right|^2 = \frac{N_o}{S_a(\omega^2)+N_o} \tag{7}$$

and

$$\left|H_{opt}(\omega)\right|^2 = -\frac{N_oS_a(\omega^2)}{S_a(\omega^2)+N_o} + 2\left|H_{opt}(\omega)\right| \cos\phi(\omega).$$

By using these two expressions in Eq. 6, $\overline{\epsilon^2_{min}}$ becomes

$$\epsilon^2_{min} = \frac{N_o}{\pi}\int_{-\infty}^{\infty}\left|H_{opt}(\omega)\right| \cos\phi(\omega)\, d\omega.$$

Since $\left|H_{opt}(\omega)\right|$ is an even function and $\phi(\omega)$ is an odd function of ω,

$$\begin{aligned}\overline{\epsilon^2_{min}} &= \frac{N_o}{\pi}\int_{-\infty}^{\infty}\left|H_{opt}(\omega)\right| e^{j\phi(\omega)}\, d\omega \\ &= \frac{N_o}{\pi}\int_{-\infty}^{\infty} H_{opt}(\omega)\, d\omega \\ &= N_o\left[h_{opt}(0-)+h_{opt}(0+)\right].\end{aligned} \tag{8}$$

Since $h_{opt}(0-) = 0$, Eq. 2 follows. The second expression in Eq. 3 is obtained by use of the initial value theorem. From these expressions, it is also observed that for large s, $H_{opt}(s)$ behaves as $\dfrac{h_{opt}(0+)}{a}$.

3. Derivation of $\overline{\epsilon^2_{min}}$ in Terms of the Signal and Noise Spectra

A simple application of contour integration and the residue theorem show that for any function $F(s)$ that is analytic in the right half-plane (RHP) and behaves as $\frac{1}{s^n}$ $(n>1)$ for large s,

$$\int_{-\infty}^{\infty} F(\omega)\, d\omega = 0.$$

Therefore, $\dfrac{N_o}{\pi}\displaystyle\int_{-\infty}^{\infty}\frac{1}{n}H^n_{opt}(\omega)\, d\omega = 0$ for $n > 1$ and, from Eq. 8

$$\overline{\epsilon^2_{min}} = \frac{N_o}{\pi} \int_{-\infty}^{\infty} \left[H_{opt}(\omega) + \frac{1}{2} H^2_{opt}(\omega) + \frac{1}{3} H^3_{opt}(\omega) + \dots \right] d\omega.$$

Since $\log(1-z) = -\left(z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \dots\right)$, it follows that

$$\overline{\epsilon^2_{min}} = -\frac{N_o}{\pi} \int_{-\infty}^{\infty} \log(1-H_{opt}(\omega))\, d\omega$$

$$= -\frac{N_o}{2\pi} \int_{-\infty}^{\infty} \log(1-H_{opt}(\omega))\, d\omega - \frac{N_o}{2\pi} \int_{-\infty}^{\infty} \log(1-H_{opt}(-\omega))\, d\omega \tag{9}$$

$$= -\frac{N_o}{2\pi} \int_{-\infty}^{\infty} \log\left|1-H_{opt}(\omega)\right|^2 d\omega.$$

The desired relation is obtained upon substitution of Eq. 7 and Eq. 9. The derivation used here is similar to that used by Viterbi and Cahn.[3]

4. Minimum Phase Property of $H_{opt}(s)$

$H_{opt}(s)$ is of the form $1-G(s)$, where $G(s)$ is minimum phase and $|G(j\omega)| \leq 1$. A Nyquist plot of $H_{opt}(\omega)$ clearly has no encirclements of the origin. Therefore, the number of RHP poles and zeros of $H_{opt}(s)$ is equal. Since $H_{opt}(s)$ has no RHP poles, it has no RHP zeros and is minimum phase. This result is in agreement with the views expressed by Professor H. B. Lee in a discussion which the writer had with him on the problem.

APPENDIX

$$\left|1-H_{opt}(\omega)\right|^2 = (1-H_{opt}(\omega))(1-H_{opt}(-\omega))$$

$$= 1 + \left|H_{opt}(\omega)\right|^2 - 2\left|H_{opt}(\omega)\right| \cos\phi(\omega),$$

where

$$H_{opt}(\omega) = \left|H_{opt}(\omega)\right| e^{j\phi(\omega)}.$$

Using Eq. 7, we have

$$\left|H_{opt}(\omega)\right|^2 = -\frac{N_o S_a(\omega^2)}{S_a(\omega^2) + N_o} + 2\left|H_{opt}(\omega)\right| \cos\phi(\omega). \tag{10}$$

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F. AN APPLICATION OF VOLTERRA FUNCTIONAL ANALYSIS TO SHUNT-WOUND COMMUTATOR MACHINES

The present report is a continuation, in the form of two appendices, of the report published in Quarterly Progress Report No. 76 (pages 198-208).

APPENDIX A

Demonstration of Uniqueness

Consider the input-output equation (Eq. 11):

$$\omega(t) = \int \overset{\infty}{\underset{-\infty}{\cdots}} \int h_f(\tau_1)\, h_a(\tau_2)\, h_\omega(\tau_3)\, v(t-\tau_1-\tau_3)\, v(t-\tau_2-\tau_3)\, d\tau_1 \ldots d\tau_3$$

$$- G \int \overset{\infty}{\underset{-\infty}{\cdots}} \int h_f(\tau_1)\, h_a(\tau_2)\, h_f(\tau_3)\, h_\omega(\tau_4)\, v(t-\tau_1-\tau_2-\tau_4)\, v(t-\tau_3-\tau_4)$$

$$\cdot\, \omega(t-\tau_2-\tau_4)\, d\tau_1 \ldots d\tau_4$$

For a given input v(t), is the output ω(t) unique? Let us assume that it is not: Assume that for some v(t) there exist two outputs, $\omega_1(t)$ and $\omega_2(t)$, both of which satisfy Eq. 11. Let the difference between these two outputs be

$$\delta(t) = \omega_1(t) - \omega_2(t). \tag{A. 1}$$

Write Eq. 11 for v(t) and $\omega_1(t)$. Write Eq. 11 for v(t) and $\omega_2(t)$. Subtract the latter from the former. The result, with the use of (A. 1), is

$$\delta(t) = -G \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_f(\tau_1)\, h_a(\tau_2)\, h_f(\tau_3)\, h_\omega(\tau_4)\, v(t-\tau_1-\tau_2-\tau_4)$$

$$\cdot\, v(t-\tau_3-\tau_4)\, \delta(t-\tau_2-\tau_4)\, d\tau_1 \ldots d\tau_4. \tag{A. 2}$$

If v(t) exists, then it is everywhere bounded.

$$|v(t)| \leq V, \quad \text{for all } t. \tag{A. 3}$$

By assumption, both $\omega_1(t)$ and $\omega_2(t)$ exist and therefore δ(t) is everywhere bounded.

$$|\delta(t)| \leq D, \quad \text{for all } t. \tag{A. 4}$$

Equations A. 2, A. 3, and A. 4 show that

$$|\delta(t)| \leq GV^2D \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |h_f(\tau_1)\, h_a(\tau_2)\, h_f(\tau_3)\, h_\omega(\tau_4)|\, d\tau_1 \ldots d\tau_4. \tag{A. 5}$$

When the expression for h_f, h_a, h_ω (Eqs. 5, 7, and 10) are substituted in (A.5), the evaluation of the integrals yields

$$|\delta(t)| \leq \frac{G^2V^2}{AR_aR_f^2} D. \tag{A.6}$$

The following lemma has thus been demonstrated.

LEMMA: If $|v(t)| \leq V$ and $|\delta(t)| \leq D$, then $|\delta(t)| \leq kD$, where

$$k = \frac{G^2V^2}{AR_aR_f^2}. \tag{A.7}$$

This lemma is self-reflexive. That is, since $|\delta(t)| \leq kD$, $|\delta(t)| \leq k^2D$, and so forth. Thus

$$|\delta(t)| \leq k^mD, \qquad m = 0, 1, 2, \ldots. \tag{A.8}$$

If $k < 1$, then as m increases without limit, (A.8) shows that

$$|\delta(t)| = |\omega_1(t) - \omega_2(t)| = 0. \tag{A.9}$$

But, as long as the bound on v(t) satisfies Eq. 12,

$$|v(t)| < \frac{R_f}{G}\sqrt{AR_a}, \text{ for all } t,$$

then k is less than one and the solutions to Eq. 11 are unique.

APPENDIX B

Demonstration of Equation 36

Consider Eq. 36:

$$\int_0^\infty \cdots \int |\Omega_{2m}(\tau_1, \ldots, \tau_{2m})| \, d\tau_1 \ldots d\tau_{2m} = \left(\frac{R_f}{G}\right)\left(\frac{G^2}{AR_aR_f^2}\right)^m \quad \text{for } m = 1, 2, 3, \ldots$$

which will be demonstrated by induction.

Initial Case m=1

Observe that h_f, h_a, and h_ω (Eqs. 5, 7, and 10) are non-negative. Thus $g(\alpha, \beta)$, which is formed by an integral of their product (Eq. 20), is also non-negative. The kernel $\Omega_2(\tau_1, \tau_2)$ is formed by a summation of g's (Eq. 23) and is, therefore, non-negative. Thus

$$\int_0^{\infty}\int |\Omega_2(\tau_1, \tau_2)| \, d\tau_1 d\tau_2 = \int_0^{\infty}\int_0^{\infty} \Omega_2(\tau_1, \tau_2) \, d\tau_1 d\tau_2. \tag{B.1}$$

But, by Eq. 23,

$$\int_0^{\infty}\int_0^{\infty} \Omega_2(\tau_1, \tau_2) \, d\tau_1 d\tau_2 = \int_0^{\infty}\int_0^{\infty} g(u, u') \, du du'. \tag{B.2}$$

By Eq. 20 and suitable change of dummy variables,

$$\int_0^{\infty}\int_0^{\infty} g(u, u') \, du du' = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} h_\omega(\tau_1) \, h_f(\tau_2) \, h_a(\tau_3) \, d\tau_1 d\tau_2 d\tau_3. \tag{B.3}$$

Finally, substitution of the expressions for h_f, h_a, and h_ω (Eqs. 5, 7, and 10) yields

$$\int_0^{\infty}\int |\Omega_2(\tau_1, \tau_2)| \, d\tau_1 d\tau_2 = \frac{G}{A} \cdot \frac{1}{R_f} \cdot \frac{1}{R_a}, \tag{B.4}$$

which demonstrates Eq. 36 when $m = 1$.

Inductive Case $m=M+1$

Assume that Eq. 36 is true for $m = M$. We shall then show that it is true for $m = M+1$: Consider Eq. 24 for the case in which $n = 2M + 2$. Due to the fact that the SYM operation does not alter the value of this integration, with appropriate change of dummy variables, we have

$$\int_0^{\infty} \cdots \int |\Omega_{2(M+1)}(\tau_1, \ldots, \tau_{2M+2})| \, d\tau_1 \ldots d\tau_{2M+2} = G \int_0^{\infty} \cdots \int |h_a(\zeta_1) \, h_\omega(\zeta_2) \, h_f(\zeta_3) \, h_f(\zeta_4)|$$

$$|\Omega_{2M}(\sigma_1, \ldots, \sigma_{2M})| d\zeta_1 \ldots d\zeta_4 d\sigma_1 \ldots d\sigma_{2M}$$

$$= G \cdot \frac{1}{R_a} \cdot \frac{G}{A} \cdot \frac{1}{R_f} \cdot \frac{1}{R_f} \cdot \left(\frac{R_f}{G}\right) \cdot \left(\frac{G^2}{AR_aR_f^2}\right)^M$$

$$= \left(\frac{R_f}{G}\right)\left(\frac{G^2}{AR_aR_f^2}\right)^{M+1}. \tag{B.5}$$

Equation B.5 demonstrates that Eq. 36 is true for $m = M+1$ if it is true for $m = M$. Since (B.4) demonstrates Eq. 36 for $m = 1$, it is now clear that Eq. 36 is true for $m = 1, 2, 3, \ldots$.

R. B. Parente