COMMUNICATION SCIENCES

AND

ENGINEERING

XVII. STATISTICAL COMMUNICATION THEORY*

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A. DISCRETE ANALYSIS OF HOMOMORPHIC DECONVOLUTION

A method for separating convolved signals based on the canonic system of Fig. XVII-1 has been previously discussed.[1] Some preliminary analytical results for continuous signals and a brief discussion of the problems involved in implementing such a system have

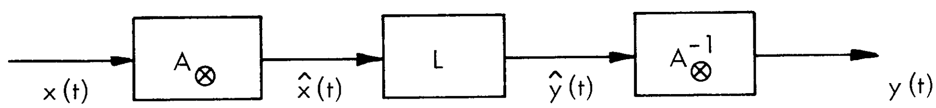

Fig. XVII-1. Canonic form for deconvolution filter.

also been given.[2] Since such a system must be implemented by using a digital computer, it is of importance to consider the characterization of the system of Fig. XVII-1 when all of the signals are discrete time functions.

In this report we shall discuss the discrete characterization of the system $A_{\otimes}$. This is a nonlinear system that is characterized by the property that if $x = x_1 \otimes x_2$, then $\hat{x} = \hat{x}_1 + \hat{x}_2$, with $\otimes$ denoting discrete convolution. If $X(z)$ and $\hat{X}(z)$ denote the two-sided z-transforms of $x(n)$ and $\hat{x}(n)$, respectively, this property can be realized through the requirement that

$$\hat{X}(z) = \log X(z),$$

provided that when $z = e^{j\omega}$, $\arg[X(z)]$ is continuous in the interval $-\pi < \omega < \pi$. Since the z-transform is periodic in ω, $\arg[X(z)]$ must also be periodic in ω. It is not necessary to associate the region of convergence of $X(z)$ with $\hat{X}(z)$.

We shall make use of just two choices for the region of convergence of $\hat{X}(z)$. One

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choice is the region outside all poles and zeros of $X(z)$. This results in a sequence, $\hat{x}(n)$, which is zero for $n < 0$. The other choice is an annular region that includes the unit circle. This choice results in a sequence, $\hat{x}(n)$, which always approaches 0 as n approaches $\pm\infty$.

It will be assumed that $X(z)$ is of the form

$$X(z) = Az^r \frac{\prod_{k=1}^{m} \left(1-a_k z^{-1}\right)}{\prod_{i=1}^{p} \left(1-b_i z^{-1}\right)} \tag{1}$$

which can be rewritten in the form

$$X(z) = \frac{z^{r-m_o+p_o} A(-1)^{m_o-p_o} \prod_{k=1}^{m_o} a_k \prod_{k=1}^{m_o} \left(1-a_k^{-1}z\right) \prod_{k=m_o+1}^{m} \left(1-a_k z^{-1}\right)}{\prod_{i=1}^{p_o} b_i \prod_{i=1}^{p_o} \left(1-b_i^{-1}z\right) \prod_{i=p_o+1}^{p} \left(1-b_i z^{-1}\right)},$$

where the first m_o zeros and p_o poles are assumed to be outside the closed contour that is used for determining $\hat{x}(n)$, and the rest inside. For $X(z)$ of the form of Eq. 1 and for which the angle is computed as discussed above, it can be shown by integrating by parts that

$$\hat{x}(0) = \log \left[\frac{A \prod_{k=1}^{m_o} |a_k|}{\prod_{i=1}^{n_o} |b_i|}\right] + \sigma(r-m_o+p_o), \tag{2a}$$

and

$$\hat{x}(n) = Ke^{\sigma n} \frac{\cos \pi n}{n} + c(n) \qquad n = \pm 1, \pm 2, \dots . \tag{2b}$$

The function $c(n)$ is defined by

$$c(n) = -\frac{1}{2\pi j} \oint_C z \frac{X'(z)}{X(z)} z^{n-1}\, dz, \tag{3}$$

where C is a circular contour defined by $z = e^{\sigma+j\omega}$, which lies in the region of convergence of $\hat{X}(z)$. The constant K in Eq. 2b is given by

$$K = r + p_o - m_o.$$

It can also be shown that

$$K = r + m_i - p_i + p - m,$$

where m_i and p_i are the number of nonzero zeros and poles inside the contour C.

By applying the properties of z-transforms, it can be shown that c(n) also satisfies the equation

$$c(n) = \frac{1}{n} \sum_{k=-\infty}^{\infty} kc(k) \frac{x(n-k)}{x(0)} \qquad n \neq 0.$$

In the special case for which $x(n) = 0$ for $n < 0$, $x(0) \neq 0$, and the contour C in Eq. 3 encloses all of the poles and zeros of X(z), so that $\hat{x}(n) = 0$, $n < 0$, the constant K in Eq. 2 is zero and therefore $\hat{x}(n) = c(n)$. Under these conditions, $\hat{x}(n)$ satisfies the recursion formula

$$\hat{x}(n) = \frac{x(n)}{x(0)} - \frac{1}{n} \sum_{k=0}^{n-1} k\hat{x}(k) \frac{x(n-k)}{x(0)} \qquad n > 0 \tag{4a}$$

$$\hat{x}(0) = \log A = \log x(0). \tag{4b}$$

It should also be noted that the inverse operation, corresponding to the characterization of the system $A_{\otimes}^{-1}$, can easily be obtained simply by solving Eqs. 4 for x(n).

If $x(n) = 0$, $n < 0$ and $x(0) \neq 0$, Eq. 4 can be used to compute $\hat{x}(n)$. The computation is practical only in the case in which X(z) has no poles or zeros outside the unit circle because if this is not true, $\hat{x}(n)$ will grow at least as fast as $\frac{a^n}{n}$, where a is some number greater than 1. It should be mentioned, however, that multiplication of x(n) by β^n ($\beta<1$) will sometimes result in a new function whose transform has no poles or zeros outside the unit circle. It can also be shown that if X(z) is a rational function, the effect of multiplying x(n) by β^n is simply to multiply the output $\hat{x}(n)$ by β^n.

In many cases of interest it may not be possible or desirable to use Eq. 4 in the calculation of $\hat{x}(n)$. In such cases, the inverse discrete Fourier transform of the logarithm of the discrete Fourier transform of x(n) must be computed. If N denotes the number of points in the discrete Fourier transform, then the resulting sequence, $\tilde{\hat{x}}(n)$, is related to $\hat{x}(n)$ by the equation

$$\tilde{\hat{x}}(n) = \sum_{a=-\infty}^{\infty} \hat{x}(n+aN) \qquad n = 0, 1, \ldots, N-1,$$

corresponding to aliasing of the sequence $\hat{x}(n)$. This aliasing can be made less noticeable by exponentially weighting x(n) or by augmenting the sequence x(n) by terminating in zeros.

R. W. Schafer, A. V. Oppenheim

References

1. A. V. Oppenheim, "Nonlinear Filtering of Convolved Signals," Quarterly Progress Report No. 80, Research Laboratory of Electronics, M.I.T., January 15, 1966, pp. 168-175.

2. R. W. Schafer, "A New Approach to Echo Removal," Quarterly Progress Report No. 84, Research Laboratory of Electronics, M.I.T., January 15, 1967, pp. 194-201.

B. UNITY FEEDBACK TWO-STATE MODULATOR CONFIGURATIONS

1. Introduction

The transfer function for the two-state modulation configuration described by Bose[1] has been derived in detail.[2] It has been shown that the dynamic ramp response, within an error term, is of the form

$$\overline{y}(t) = -x(t) - \tau x'(t). \tag{1}$$

The frequency response of the modulator, then, as inferred from the ramp response, goes as

$$\frac{\overline{Y}(s)}{X(s)} = -(1+\tau s) \tag{2}$$

which, as in the case of a continuous amplifier with feedback, is the inverse of the feedback network response.

The system frequency response can be made flat by placing an RC filter identical to the feedback network in front of the modulator loop. The same effect is achieved in the circuit of Fig. XVII-2, in which the RC feedback filter has been located in the place where it also acts to filter the input waveform.

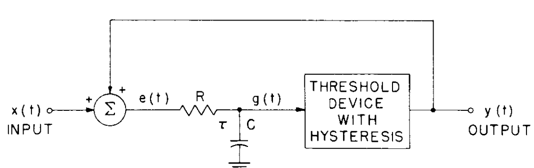

Fig. XVII-2. Unity feedback two-state modulation system.

In Fig. XVII-2 the hysteresis switch with delay, previously used as the forward element of the modulator loop, has been replaced by an arbitrary threshold device with hysteresis. This has been done to permit application of the analysis to more general effects that are present in a physical realization, such as switching transients and variations in the switching levels. The threshold device has two states or modes of operation: the charge state, entered when its input g(t) drops below w_c, during which the output will be designated $y_c(t)$; and the discharge state, entered when g(t) rises above w_d, during which the output will be designated $y_d(t)$. The only contraints on the device behavior to insure recurrent alternation of states or cyclic operation of the modulator are that x(t) plus $y_c(t)$ be sufficiently positive that g(t) tends to increase or "charge," eventually reaching w_d, and that x(t) plus $y_d(t)$ be sufficiently negative that g(t) tends to decrease or "discharge," eventually reaching w_c. A switch with hysteresis and delay, for example, meets this requirement, provided the input is bounded by $h_d - w_c < x < h_c - w_d$.

A general transfer expression for this system configuration will be derived. For static input, the system is shown to be identical to the RC feedback configuration[2] if ideal hysteresis with delay is assumed. For dynamic inputs — again, under the assumption of ideal hysteresis with delay — it can be shown that a flat frequency response is obtained with error equivalent to the static error. Finally, an improved system will be introduced for which the transfer function predicts zero error, even for the arbitrary threshold device described above.

2. Analysis

The transfer function for the circuit of Fig. XVII-2 will now be derived. For the RC circuit,

$$e(t) = g(t) + \tau g'(t). \tag{3}$$

The input to the nonlinear element has the form sketched in Fig. XVII-3. As long as this waveform is cyclic, the desired modulation will obtain.

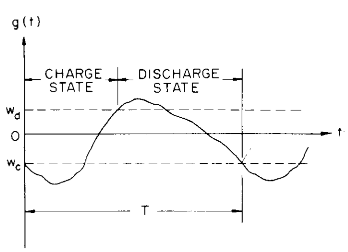

Fig. XVII-3. Typical input to the threshold device.

Define a cycle of operation to be from one w_c state transition to the next. Then, choosing t = 0 at the start of a cycle, we have

$$g(T) = g(0). \tag{4}$$

From (3), since e(t) = x(t) + y(t),

$$\int_0^T [x(t)+y(t)]\, dt = \int_0^T g(t)\, dt + \tau g(t)\Big|_0^T. \tag{5}$$

Rearranging terms, using (4), and dividing by the period T yields

$$\frac{1}{T}\int_0^T y(t)\, dt = -\frac{1}{T}\int_0^T x(t)\, dt + \frac{1}{T}\int_0^T g(t)\, dt. \tag{6}$$

The output mean over the cycle is equal to the negative of the input mean over the cycle to an error term in g(t), independently of the form of x(t) and y(t), as long as cyclic behavior is maintained. For efficiency, y(t) should be a switching waveform, and, to enable accurate reconstruction of the input waveform, the modulator parameters should be such that the cycle time is short compared with the rate of change of x(t); however, neither of these considerations is involved in the derivation of Eq. 6.

As indicated by Eq. 6, the modulator output in the unity feedback configuration is referenced to the mean of the input during a cycle, rather than depending only on the input values at the switching instants as was the case with the RC feedback configuration. The RC network continuously looks at the input waveform and effectively stores its behavior over the entire cycle in g(t).

3. Investigation of the Error Term

In the preceding development arbitrary parameters that were appropriate for cyclic operation were assumed. It is intuitively clear that a narrow hysteresis window, bounded input and output, and small delays will limit the excursion of g(t) to a restricted range and permit a bound on the error term. For typical operation this bound is exceedingly loose because the approximate symmetry of g(t) reduces the value of its integral over a cycle. Expressions for the modulator error term have been developed for the case in which the forward path is an ideal switch with hysteresis and delay, and the input is static or characterized by its midpoint and slope over the cycle.

A typical errror waveform g(t) over a cycle with static input is sketched in Fig. XVII-4. Notice that g(t) is precisely the static error waveform evaluated in a previous report,[2] in which the RC feedback modulator configuration was discussed. The average of g(t) was shown to be equal to its midpoint, which can be evaluated in terms of the modulator parameters, plus a term expressing the discrepancy between mean and midpoint. This

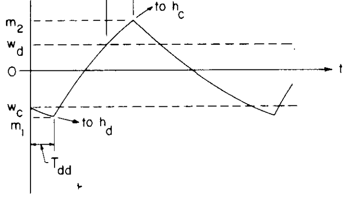

Fig. XVII-4. Error waveform, static input.

result, for the symmetric case, is reproduced in Eq. 7.

$$\overline{y} = -x_o e^{-\frac{T_d}{\tau}} \left\{ 1 + \frac{2}{3} \frac{\left[\left(1 - e^{-\frac{T_d}{\tau}} \right) + \frac{w}{h} e^{-\frac{T_d}{\tau}} \right]^2}{1 - \left[\frac{x_o e^{-\frac{T_d}{\tau}}}{h} \right]^2} \right\}. \tag{7}$$

The error function for ramp input can be derived by characterizing the ramp as altering the effective midpoint and half-width of the exponential window function g(t), which is the same procedure previously used to discuss the ramp response of the RC modulator configuration.[2] The detailed mathematics will be omitted here; the result supports our intuitive claim that the device should exhibit flat frequency response by predicting the same transfer expression as for the static case. The "static" input value of Eq. 7 is simply replaced by the mean of the ramp during the modulator cycle.

4. Integral Modulator

In the derivation presented in this report, the error term was found to be the integral of g(t) over a cycle. If the RC network is replaced by an integrator, as drawn in Fig. XVII-5,

$$e(t) = \tau g'(t), \tag{8}$$

and the error term in the derivation vanishes! Equation 6 becomes simply

$$\frac{1}{T} \int_0^T y(t)\, dt = -\frac{1}{T} \int_0^T x(t)\, dt. \tag{9}$$

Independent of the form of y(t) — as long as it is appropriate to produce cyclic operation — and independent of the parameters of the hysteresis element and the nature

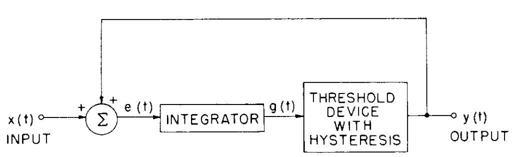

Fig. XVII-5. Integral two-state modulation system.

of the input, the average of the output over a cycle is precisely the negative of the mean of the input over the cycle.

A qualitative explanation of this type of modulator operation is easy to provide. The output of the integrator represents the input mean plus the output mean, and the hysteresis element guarantees that this output is cyclic. Then, over a cycle, the sum of the means, the output error, is brought to zero.

In practice, the realization of the integrator may not be ideal. The transfer function of the integrator may then be represented in the form.

$$e(t) = Ag(t) + \tau g'(t), \tag{10}$$

where A is zero ideally, approximately .001 for a typical operational amplifier realization of an integrator, and equal to one for the simple RC approximation to an integrator. We have shown that the modulator error resulted from the integral of g(t) over a cycle. In the analysis of this error, A was taken as unity (an RC approximation) and error expressions were derived. These same error expressions can be applied to the case of incomplete realization of the integrator in an integral modulation scheme simply by multiplying the error expressions by the factor A. Of course, for an ideal integrator A is zero and there will be no error, and for most practical realizations A is sufficiently small that the error may be neglected.

J. E. Schindall

References

1. A. G. Bose, Quarterly Progress Reports No. 66 and No. 67, Research Laboratory of Electronics, M.I.T., July 15 and October 15, 1962, pp. 187-189 and 115-119.
2. J. E. Schindall, Quarterly Progress Report No. 84, Research Laboratory of Electronics, M.I.T., January 15, 1966, pp. 184-194.