

## XII. GASEOUS ELECTRONICS\*

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### A. ELECTRON COLLISIONAL EFFECTS ON WAVE PROPAGATION IN A NONMAGNETIZED PLASMA

To account properly for the effect of electron-atom collisions on the propagation of waves in plasmas, it is necessary to utilize the Boltzmann collision integral.<sup>1</sup> This integral assumes only two-body collisions, and therefore is valid only for collisions involving short-range forces such as those occurring in electron-atom collisions.

Our treatment utilizes the Boltzmann collision integral and assumes that the electrons collide with particles of much greater mass. Thus, this treatment applies to electron-atom collisions. The calculation determines the form of the dispersion relation for the electron plasma wave in the presence of electron-atom collisions, and is a direct application of the method developed by Allis.<sup>1</sup>

Generally, the Krook model<sup>2,3</sup> is used to account for collisions when studying collisional effects on the electron plasma wave. The advantage of this method is that it can be applied readily to problems in which the electron temperature is high; for example, when the Larmor orbit is comparable with the wavelength perpendicular to  $\vec{B}$ , the mean-free path is long compared with the wavelength parallel to  $\vec{B}$ , and the thermal motion of the electrons parallel to  $\vec{B}$  introduces a significant Doppler shift of the plasma-wave frequency. It has the disadvantage that it is only an approximation to the collisional effect and, although it can be so formulated that it does not violate the conservation of plasma particles, it cannot be readily formulated in such a way that it also conserves plasma particle energy and accounts correctly for the angular and velocity dependence of the electron-atom scattering cross section. The method used in this report is cumbersome to apply to plasmas with high electron temperatures, but it does account exactly for the collisional effects. The results of the two methods will be compared at the end of the calculation.

The Boltzmann equation for the electron distribution function,  $f$ , in the absence of an applied magnetic field is

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$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_r f - \frac{e}{m} \vec{E} \cdot \vec{\nabla}_v f = \left( \frac{\partial f}{\partial t} \right)_c, \quad (1)$$

where  $e/m$  is the electron charge-to-mass ratio,  $\vec{E}$  is the electric field, and  $\left( \frac{\partial f}{\partial t} \right)_c$  represents the Boltzmann collision integral. The distribution function is expanded in Legendre polynomials,  $P_\ell(\cos \theta)$ , where  $\theta$  is the angle between  $\vec{E}$  and the electron velocity,  $\vec{v}$ :

$$f = \sum_{\ell=0}^{\infty} f^\ell(r, v, t) P_\ell(\cos \theta). \quad (2)$$

This expression is substituted in both the right- and left-hand sides of (1), and the terms comprising the coefficients of each  $P_\ell$  on each side of the equation are equated separately; this generates an infinite set of equations for the  $f^\ell$  (the ratio of the electron-to-atom mass is set equal to zero, also):

$$\frac{\partial f^0}{\partial t} + \frac{v}{3} \frac{\partial f^1}{\partial z} - \frac{eE}{3mv^2} \frac{\partial}{\partial v} v^2 f^1 = 0 \quad (3)$$

$$\frac{\partial f^1}{\partial t} + v \frac{\partial}{\partial z} \left[ f^0 + \frac{2}{5} f^2 \right] - \frac{eE}{m} \left[ \frac{\partial f^0}{\partial v} + \frac{2}{5v^3} \frac{\partial}{\partial v} v^3 f^2 \right] = -\nu_{c1} f^1$$

...

$$\begin{aligned} \frac{\partial f^\ell}{\partial t} + v \frac{\partial}{\partial z} \left[ \frac{\ell}{2\ell-1} f^{\ell-1} + \frac{\ell+1}{2\ell+3} f^{\ell+1} \right] - \frac{eE}{m} \left[ \frac{\ell}{2\ell-1} v^{\ell-1} \frac{\partial}{\partial v} \frac{f^{\ell-1}}{v^{\ell-1}} \right. \\ \left. + \frac{\ell+1}{2\ell+3} \frac{1}{v^{\ell+2}} \frac{\partial}{\partial v} v^{\ell+2} f^{\ell+1} \right] = -\nu_{c\ell} f^\ell. \end{aligned}$$

It has been assumed that the perturbed distribution function does not depend on the  $x$  and  $y$  spatial coordinates, and the  $E$  field is assumed to be parallel to the  $z$ -axis. This is consistent with the effect of plasma wave propagation ( $\vec{E} \parallel \vec{k}$ ) on the distribution function, in the absence of a magnetic field. The quantities  $\nu_{c\ell}$  are weighted collision frequencies arising from the Boltzmann collision integral. They are given by

$$\nu_{c\ell} = 2\pi v n_g \int_{\theta=0}^{\theta=\pi} \sigma(\theta) (1 - P_\ell(\cos \theta)) \sin \theta \, d\theta, \quad (4)$$

where  $\sigma(\theta)$  is the differential scattering cross section,  $v$  the electron velocity, and  $n_g$  the volume density of the gas atoms.

In the absence of the perturbing wave only the spherically symmetric, time-independent part of  $f$ ,  $f_0^0(v)$ , is nonzero. The unperturbed distribution function,  $f_0^0$ , is so normalized that  $\int f_0^0 d^3v = n_0$ , where  $n_0$  is the unperturbed electron density. Denoting

the time-dependent part of  $f^0$  by  $f_1^0$ , we obtain the linearized form of Eqs. 3:

$$\frac{\partial f_1^0}{\partial t} + \frac{v}{3} \frac{\partial f^1}{\partial z} = 0 \quad (5)$$

$$\frac{\partial f^1}{\partial t} + v \frac{\partial}{\partial z} \left[ f_1^0 + \frac{2}{5} f^2 \right] - \frac{eE}{m} \frac{\partial f_0^0}{\partial v} = -v_{c1} f^1$$

...

$$\frac{\partial f^\ell}{\partial t} + v \frac{\partial}{\partial z} \left[ \frac{\ell}{2\ell-1} f^{\ell-1} + \frac{\ell+1}{2\ell+3} f^{\ell+1} \right] = -v_{c\ell} f^\ell \quad (\ell > 1).$$

We next assume that all of the perturbed quantities and the electric field vary as  $e^{-i(\omega t - kz)}$ ; thus we obtain for Eqs. 5

$$-i\omega f_1^0 + \frac{ikv}{3} f^1 = 0 \quad (6)$$

$$(-i\omega + v_{c1}) f^1 + ikv \left( f_1^0 + \frac{2}{5} f^2 \right) = + \frac{eE}{m} \frac{\partial f_0^0}{\partial v}$$

...

$$(-i\omega + v_{c\ell}) f^\ell + ikv \left[ \frac{\ell}{2\ell-1} f^{\ell-1} + \frac{\ell+1}{2\ell+3} f^{\ell+1} \right] = 0 \quad (\ell > 1).$$

This infinite set of equations can be solved for  $f_1^0$  to obtain

$$f_1^0 = \frac{-i \frac{eE}{m} \frac{kv}{3} \frac{\partial f_0^0}{\partial v}}{i\omega(i\omega - v_{c1}) + \frac{(kv)^2}{3} - \frac{4}{15} \frac{i\omega(kv)^2}{(-i\omega + v_{c2} + D)}}, \quad (7)$$

where  $D$  is a fraction of the form

$$D = \frac{9(kv)^2}{35 \left( -i\omega + v_{c3} + \frac{16(kv)^2}{63(-i\omega + v_{c4} + \dots)} \right)}, \quad (8)$$

and the general term in the fraction is

$$-i\omega + v_{c\ell-1} + \frac{\ell^2(kv)^2}{(4\ell^2 - 1) \left( -i\omega + v_{c\ell} + \frac{(\ell+1)^2(kv)^2}{(4(\ell+1)^2 - 1)(-i\omega + v_{c\ell+1} + \dots \text{etc.})} \right)}.$$

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We may now calculate the electron density perturbation,  $n_1$ , which is due to the wave:

$$n_1 = n_0 \int_0^\infty f_1^0 4\pi v^2 dv. \quad (9)$$

Poisson's equation for the electric field is

$$\nabla \cdot E = ikE = -\frac{n_1 e}{\epsilon_0}, \quad (10)$$

and combining (9) and (10), we obtain the dispersion relation for the wave

$$1 = \frac{4\pi}{3} \omega_p^2 \int_0^\infty \frac{v^3 \frac{\partial f_0^0}{\partial v} dv}{i\omega(i\omega - \nu_{c1}) + \frac{(kv)^2}{3} - \frac{4}{15} \frac{i\omega(kv)^2}{(-i\omega + \nu_{c2} + D)}}, \quad (11)$$

where  $\omega_p = \sqrt{\frac{n_0 e^2}{m\epsilon_0}}$  is the electron plasma frequency.

If  $f_0^0$  is Maxwellian, it follows that  $\frac{\partial f_0^0}{\partial v} = -\frac{v}{v_0^2} f_0^0$ , where  $v_0^2 = \frac{kT}{m}$ , and is of the order of the mean-square electron velocity. In this case Eq. 11 becomes

$$1 = -\frac{4\pi}{3} \frac{\omega_p^2}{v_0^2} \int_0^\infty \frac{v^4 f_0^0 dv}{i\omega(i\omega - \nu_{c1}) + \frac{(kv)^2}{3} - \frac{4}{15} \frac{i\omega(kv)^2}{(-i\omega + \nu_{c2} + D)}}. \quad (12)$$

For a cold plasma in which  $(kv)^2 \ll \omega(\omega^2 + \nu_{c1}^2)^{1/2}$ , only the collision frequency  $\nu_{c1}$  enters significantly into the damping. To account correctly for first-order temperature effects, it is seen from the denominator of (12) that the collision frequency  $\nu_{c2}$  is also important. The last of the three terms in the denominator (see Eqs. 6) is absent if  $f^2$  is assumed to be negligible and, therefore,  $f^2$  must not be neglected in this treatment if the dispersion relation is to be correct to first order in temperature. It is seen

that this treatment does not converge rapidly unless  $\frac{(kv)^2}{\omega(\omega^2 + \nu_{c1}^2)^{1/2}} \ll 1$  and, therefore,

is not readily applied to cases in which the wave phase velocity is not much greater than the electron thermal velocity. Setting  $D = 0$  is equivalent to assuming  $f^\ell = 0$  for  $\ell \geq 3$

in Eqs. 6, and will give us the correct first-order dependence on temperature of Eq. 12.

By setting  $f_0^o = \left(\frac{1}{2\pi v_0^2}\right)^{3/2} e^{-v^2/2v_0^2}$  and  $D = 0$ , Eq. 12 becomes

$$1 = -\frac{4\pi}{3} \frac{\omega_p^2}{v_0^2} \left(\frac{1}{2\pi v_0^2}\right)^{3/2} \int_0^\infty \frac{e^{-v^2/2v_0^2} v^4 dv}{i\omega(i\omega - \nu_{c1}) + \frac{(kv)^2}{3} \left(1 + \frac{4}{5} \frac{i\omega}{i\omega - \nu_{c2}}\right)}. \quad (13)$$

Bers and Musha<sup>4</sup> have calculated the plasma wave dispersion relation, using the Krook model and assuming a collisional relaxation rate,  $\nu_c$ , that is independent of velocity. Their result for zero magnetic field is

$$1 = -\frac{\frac{\omega_p^2}{k^2 v_0^2} [1 + \xi_0 Z(\xi_0)]}{1 + \frac{i\nu_c}{\omega + i\nu_c} \xi_0 Z(\xi_0)}, \quad (14)$$

where  $\xi_0 = \frac{\omega + i\nu_c}{kv_0 \sqrt{2}}$ , and  $Z(\xi_0)$  is the plasma dispersion function.<sup>5</sup> For low temperature, the asymptotic expansion of  $Z(\xi_0)$  may be used. With Landau damping neglected, this is

$$Z(\xi_0) = -\frac{1}{\xi_0} \left(1 + \frac{1}{2\xi_0^2} + \frac{3}{4\xi_0^4} + \dots\right). \quad (15)$$

Substituting (15) in (14) and saving terms to first order in temperature in the numerator and denominator, we obtain, after some algebraic manipulations,

$$1 = \frac{\frac{\omega_p^2}{\omega(\omega + i\nu_c)} \left[1 + \frac{3k^2 v_0^2}{(\omega + i\nu_c)^2}\right]}{1 - \frac{i\nu_c}{\omega} \frac{k^2 v_0^2}{(\omega + i\nu_c)^2}}. \quad (16)$$

To first order in temperature Eq. 16 becomes

$$1 = \frac{\omega_p^2}{\omega(\omega + i\nu_c)} \left[1 + \frac{3k^2 v_0^2}{(\omega + i\nu_c)^2} \left(1 + \frac{i\nu_c}{3\omega}\right)\right]. \quad (17)$$

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It is only possible to check Eq. 17 against Eq. 13 if we assume  $\nu_{c1}$  and  $\nu_{c2}$  to be velocity-independent in Eq. 13. Making this assumption, evaluating Eq. 13 to first order in temperature, and again neglecting the effect of Landau damping gives

$$1 = \frac{\omega_p^2}{\omega(\omega + i\nu_{c1})} \left[ 1 + \frac{3(kv_0)^2}{(\omega + i\nu_{c1})(\omega + i\nu_{c2})} \left( 1 + \frac{5}{9} \frac{i\nu_{c2}}{\omega} \right) \right]. \quad (18)$$

If we consider a gas for which the scattering is isotropic, then  $\nu_{c1} = \nu_{c2}$ . For this case, Eqs. 17 and 18 agree very closely. For low-temperature plasmas and for  $\nu_c \ll \omega$ , this small error in the Krook model will not be important. In a plasma for which the electron scattering cross section is not isotropic and is strongly velocity-dependent, for instance, in an argon plasma, a deviation from the Krook model predictions of measured wave lengths and collisional damping effects might be observable.

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References

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