IX. ELECTRODYNAMICS OF MEDIA*

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A. ELECTRODYNAMICS OF QUADRUPOLAR MEDIA

The author has undertaken a study of quadrupolar media for two reasons. First, the availability of high-intensity electromagnetic radiation at optical frequencies makes it possible to observe quadrupolar effects in situations in which they are not swamped by dipolar effects[1] (e.g., frequency-doubling in media with inversion symmetry). Second, the quadrupolar medium provides an interesting example of the application of Chu's postulates to formulations of electrodynamics in the presence of moving media.[2] The question arises as to whether it is necessary to modify Maxwell's equations of the Chu formulation for polarizable and magnetizable media when dealing with a quadrupolar medium, or whether it is sufficient to replace the polarization density $\overline{P}$ by the divergence of the quadrupolar tensor, leaving the equations otherwise unchanged.

In this report we summarize the steps taken to obtain a self-consistent formulation of Maxwell's equations in the presence of a moving quadrupolar medium. These results are used to develop a simple nonrelativistic model of a fluid consisting of noninteracting quadrupolar particles (the particles interact solely through the macroscopic fields that they produce). We generalize the expressions for the quadrupolar fluid to a solid, taking interactions among the quadrupolar particles into account. The force density acting toward acceleration of the volume elements of the solid is obtained.

We start with Maxwell's equations as formulated by Chu,[2] which he has called "the Amperian formulation." The presence of the material medium is taken into account by means of source terms introduced into Maxwell's equations for free space. The reason for using the Amperian formulation, rather than the formulation based on the concept of magnetic-charge dipoles, is that a quadrupolar medium produces circulating currents analogous to the Amperian model of magnetizing currents.

$$\nabla \times \frac{\partial \overline{B}}{\partial t} = 0 \tag{1}$$

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$$\nabla \times \frac{\overline{B}}{\mu_o} - \epsilon_o \frac{\partial \overline{E}}{\partial t} = \overline{J}_Q \tag{2}$$

$$\nabla \cdot \epsilon_o \overline{E} = \rho_Q \tag{3}$$

$$\nabla \cdot \overline{B} = 0. \tag{4}$$

We start with a model of a planar quadrupole shown in Fig. IX-1. The four charges form a parallelogram, so that they do not possess a dipole moment. The charge chosen

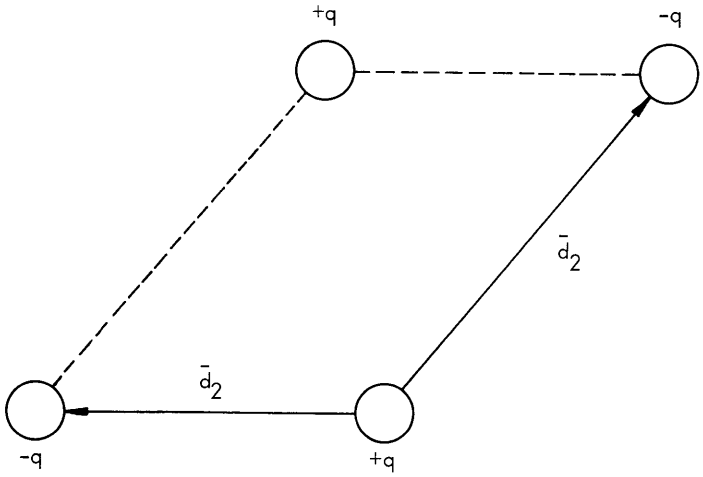

Fig. IX-1. Model of planar quadrupole.

as the origin for the measurement of the vector distances of the remaining three charges is assumed to carry the mass of the quadrupole. This assumption implies that there is no time dispersion in the constitutive law. Suppose that the mass-bearing charge is distributed with a density $n(\bar{r})$ and moves with a velocity $\bar{v}(\bar{r})$. In order to find the source current density $\bar{J}_Q$ and source charge density ρ_Q, we find the charge density and velocity of each of the four charges constituting the quadrupole and add over these. All of the evaluations have to be made to second order in the displacements, $\bar{d}$, of the charges from the mass-bearing charge. The number density at point $\bar{r}$ of a particle displaced a distance $\bar{d}$ from its partner, the latter of which possesses a density distribution $n(\bar{r})$, is given[4,5] to second order in $\bar{d}$ by

$$n - \nabla \cdot (n\bar{d}) + \frac{1}{2} \nabla\nabla : (nd\bar{d}). \tag{5}$$

The currents of the four charge configurations add at any particular point $\bar{r}$ to produce the net current density $\bar{J}_Q$. When the velocities of the four charge distributions are

evaluated, their respective current densities obtained, and then superimposed, one finds for the current density

$$\bar{J}_Q = -\nabla \cdot \left(qn\, \bar{d}_1 \frac{d\bar{d}_2}{dt} + qn\, \bar{d}_2 \frac{d\bar{d}_1}{dt} \right) + \nabla\nabla : (qn\, \bar{d}_1 \bar{d}_2 \bar{v}). \tag{6}$$

Similarly, one finds for the charge density

$$\rho_Q = \nabla\nabla : (qn\, \bar{d}_1 \bar{d}_2). \tag{7}$$

The planar quadrupole model is not sufficient to describe the most general distribution of charge with a quadrupole moment. It is easy to obtain more general expressions than those of Eqs. 6 and 7, by considering each quadrupolar particle to be made up of a superposition of planar quadrupoles. Supposing that each quadrupolar particle is made up of N planar quadrupoles, one obtains a quadrupole tensor density by superposition:

$$\bar{\bar{Q}} = \frac{1}{2} \sum_{i=1}^{N} q^i n \left(\bar{d}_1^i \bar{d}_2^i + \bar{d}_2^i \bar{d}_1^i \right). \tag{8}$$

Equations 6 and 7 can be written in terms of $\bar{\bar{Q}}$ if one introduces the additional definition of the magnetization density

$$\bar{M} = \sum_{i=1}^{N} q^i n \left(\bar{d}_1^i \times \frac{d\bar{d}_2^i}{dt} + \bar{d}_2^i \times \frac{d\bar{d}_1^i}{dt} \right). \tag{9}$$

One then has for the charge density and current density, as produced by moving quadrupolar particles,

$$\rho_Q = \nabla\nabla : Q \tag{10}$$

$$\bar{J}_Q = -\nabla \cdot \left[n \frac{d}{dt} \left(\frac{\bar{\bar{Q}}}{n} \right) \right] + \nabla\nabla : (\bar{\bar{Q}} \bar{v}) + \nabla \times \bar{M}. \tag{11}$$

With these equations the formulation of Maxwell's equations for a system of quadrupolar particles is completed. In order to complete the formulation of the system, it is necessary to find the force density that acts toward acceleration of volume elements of the quadrupolar medium. If one starts with the model of a fluid of noninteracting quadrupolar particles, it is not difficult to find the amount of energy supplied by the electric and magnetic field to the fluid per unit volume, w_Q. This is accomplished by a simple application of the Lorentz law. One finds

$$n\delta\left(\frac{w_Q}{n} \right) = n\nabla\bar{\bar{E}}^{\,o} : \delta\left(\frac{\bar{\bar{Q}}}{n} \right) - \bar{M} \cdot \delta\bar{B}^{\,o}. \tag{12}$$

[The superscript o on $\overline{E}$ and $\overline{B}$ indicates that $\overline{E}$ and $\overline{B}$ are evaluated in the rest frame of the volume element under consideration.]

From the same model one may develop an energy conservation theorem that includes the rate of energy supplied to the particles per unit volume both as internal energy and as kinetic energy, the time rate of change of the energy of the electromagnetic field, and the divergence of the power flow of the electromagnetic field. Also, one finds in such a theorem the divergence of a vector that can be interpreted as the power flow associated with the quadrupolar medium.

$$\overline{S}_Q^o = n\left[\frac{d}{dt}\left(\frac{\overline{\overline{Q}}}{n}\right)\right]^o \cdot \overline{E}^o + \overline{\overline{Q}} \cdot \nabla\overline{v} \cdot \overline{E}^o - \overline{E}^o \times \overline{M}. \tag{13}$$

Equations 12 and 13 can be used as a starting point for the application of the principle of virtual work, or its generalization, the principle of virtual power.[3,4] For this purpose one wants to generalize the expression for the energy density supplied to the fluid per unit volume to describe a solid, rather than a fluid of noninteracting particles. A generalization that suggests itself is

$$nd\left(\frac{w_m}{n}\right) = n\nabla\overline{E}^o : \frac{d}{dt}\left(\frac{\overline{\overline{Q}}}{n}\right) - \overline{M} \cdot \frac{d\overline{B}^o}{dt} + \overline{\overline{t}} : \nabla\overline{v}. \tag{14}$$

Here we have introduced an electromechanical stress tensor $\overline{\overline{t}}$, in order to take the microscopic interactions among the particles of the medium into account. It should be emphasized that the introduction of this tensor is not simply an addition of a mechanical stress tensor because $\overline{\overline{t}}$ may depend upon the magnetic flux density $\overline{B}$ and the quadrupole tensor per particle $\overline{\overline{Q}}/n$. Furthermore, we assume that the power flow density associated with the quadrupolar medium is still properly represented by Eq. 13, which was obtained from the model of the quadrupolar fluid. With this information and with the additional statement that the power-conversion density in the rest frame of the material medium is given by $(\overline{E}+\overline{v}\times\overline{B}) \cdot \overline{J}_Q^o$, where $\overline{J}_Q^o$ is the current density in the rest frame, one can apply the principle of virtual power to obtain the force density that accelerates the volume elements of the quadrupolar fluid. The force density is found to be

$$\overline{f} = \overline{\overline{Q}} : \nabla\nabla\overline{E} + \overline{v} \times (\overline{\overline{Q}} : \nabla\nabla)\overline{B} + n\frac{d}{dt}\left(\frac{\overline{\overline{Q}}}{n}\right) \cdot \times \nabla\overline{B} + (\nabla\overline{B}) \cdot \overline{M}. \tag{15}$$

Two facts should be noted in the preceding results. First of all, Maxwell's equations for a quadrupolar medium are not simple applications of Maxwell's equations for dipolar media, in the sense that it is not possible to replace the polarization density $\overline{P}$, as it appears in equations of motion for the dipolar medium, by the divergence of $\overline{\overline{Q}}$. In Maxwell's equations for a dipolar medium only first-order derivatives of the

velocity field $\overline{v}$ appear. We find that the current density $\overline{J}_Q$ of Eq. 11 contains second-order derivatives of the velocity field. It should also be noted that the quadrupolar medium under consideration has the relatively complicated power-flow density vector of Eq. 13. Hence the quadrupolar medium leads to a novel application of the principle of virtual work. The author's study of the quadrupolar medium was motivated partly by the fact that a quadrupolar medium provided an interesting new application and test of the principle of virtual power.

H. A. Haus

References

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4. E. L. Chu, "On the Concept of Fictitious Surface Charges of an Electron Beam," J. Appl. Phys. 31, 381-388 (1960); see Eqs. (9a) and (9b).
5. P. A. Sturrock, "Generalization of the Lagrange Expansion with Applications to Physical Problems," J. Math. Phys. 1, 405-408 (Sept.-Oct. 1960); see Eq. (5.8).
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B. PULSE DISTORTION IN NONLINEAR OPTICS

A study of the interaction of intense optical radiation with materials that respond nonlinearly has led to recent investigations of the possibility into obtaining electromagnetic shock waves in non-polar liquids such as carbon di-sulfide.[1-3] These would result from a dependence of the phase velocity of light upon the local internal energy density of the closed system containing the pulse and the liquid.

In this report an energy equation is derived, and the Debye approach adapted to a non-polar liquid, reviewed qualitatively. The results are then used to deduce thermodynamic quantities that describe the physical mechanisms of the nonlinear interaction. Computational results are presented, illustrating the interplay between the nonlinear response, and the relaxation of the medium.

1. Energy Equation

It is assumed that the field of the optical pulse can be described in terms of a central frequency, ω, a propagation vector, k, pointing in the z-direction, and complex amplitudes, all of which are slowly varying functions of space and time. Assuming a plane wave, we write

$$\overline{H} = \mathrm{Re}\, \underline{H}(t,z)\, e^{-\left(i\int_0^z k\, dz - i\omega t\right)}; \qquad \underline{H}(t,z) = H(t,z)\, \hat{i}y$$

$$\overline{D} = \mathrm{Re}\, \underline{D}(t,z)\, e^{-\left(i\int_0^z k\, dz - i\omega t\right)}; \qquad \underline{D}(t,z) = D(t,z)\, \hat{i}x$$

$$\overline{E} = \mathrm{Re}\, \underline{E}(t,z)\, e^{-i\left(i\int_0^z k\, dz - i\omega t\right)}; \qquad E(t,z) = E(t,z)\, \hat{i}x$$

for the $\overline{H}$, $\overline{D}$, and $\overline{E}$ fields, respectively. $\hat{i}x$, $\hat{i}y$, and $\hat{i}z$ refer to unit vectors pointing in the x, y, and z directions of a right-handed system.

Maxwell's equations can be employed to obtain a "Poynting theorem" relating the amplitudes and the propagation vector. The real part involving only the amplitudes is

$$\nabla \cdot (\underline{E}\times\underline{H}^* + \underline{E}^*\times\underline{H}) + \mu_o \frac{\partial}{\partial t}(\underline{H}\cdot\underline{H}^*) + \underline{E}\cdot\frac{\partial \underline{D}^*}{\partial t} + \underline{E}^*\cdot\frac{\partial \underline{D}}{\partial t} = 0. \tag{1}$$

The theory of polar molecules can be shown to be consistent with the constitutive relationship[3]

$$\underline{D} = \epsilon(\{\underline{E}\cdot\underline{D}^* + \underline{E}^*\cdot\underline{D}\}, t)\, \underline{E} \tag{2}$$

$$= \epsilon\, \underline{E}.$$

Furthermore, if the envelope of the pulse is slowly varying with respect to the inverse

of the optical frequency, we can write

$$\bar{k} \times \underline{H} - \left(\int_0^z \frac{\partial k}{\partial t} dz - \omega \right) \underline{D} = 0 \tag{3}$$

$$\bar{k} \times \underline{E} + \rho_o \left(\int_0^z \frac{\partial k}{\partial t} dz - \omega \right) \underline{H} = 0 \tag{4}$$

which shows that the phase velocity of the pulse is given by

$$\frac{\left(\omega - \int_0^z \frac{\partial k}{\partial t} dz\right)}{k} = \frac{1}{\sqrt{\mu_o \epsilon}} = v. \tag{5}$$

Moreover, (3) shows that

$$H \cong v D. \tag{6}$$

Together with (2), (6) allows the energy propagation equation (1) to be written

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial z} + \frac{1}{2}\left(\frac{\rho}{\epsilon}\right)\frac{\partial \epsilon}{\partial t} = 0, \tag{7}$$

where

$$\rho \triangleq \frac{\underline{E} \cdot \underline{D}^*}{2}. \tag{8}$$

The explicit formula for the dielectric constant can be deduced from the Debye theory, under the assumption of an anisotropic molecular polarizability. For axially symmetric molecules that polarize instantaneously, this gives to lowest order

$$\frac{\partial}{\partial t}(\epsilon - \epsilon_o) = -\left(\frac{\epsilon - \epsilon_o}{\tau} - \frac{\beta \rho}{\tau}\right); \quad \beta = \frac{K}{kT}, \tag{9}$$

where ϵ_o is the linear dielectric constant. K is a constant depending upon the polarizabilities and the density, k is Boltzmann's constant, T is the temperature in degrees Kelvin, and τ is the relaxation time associated with viscous damping.

Because $v = \frac{1}{\sqrt{\mu_o \epsilon}}$, if we let $v_o = \frac{1}{\sqrt{\mu_o \epsilon_o}}$, (7) and (9) can also be written

$$\frac{\partial \rho}{\partial t} + v_o \frac{\partial \rho}{\partial z} + (v - v_o)\frac{\partial \rho}{\partial z} + \rho \frac{\partial}{\partial z}(v - v_o) - \left(\frac{\rho}{v}\right)\frac{\partial v}{\partial t} = 0 \tag{10}$$

$$\frac{\partial}{\partial t}(v - v_o) = -\left(\frac{(v - v_o)}{\tau} + \frac{\alpha \rho}{\tau}\right); \quad \alpha = \left(\frac{v_o \beta}{2}\right). \tag{11}$$

For $\tau = 0$, $(v-v_o) = \delta v = -\alpha\rho$, so that the energy equation (7), with the term $\delta v^2 \frac{\partial \rho}{\partial z}$ neglected, becomes

$$\frac{\partial \rho}{\partial t} + (v_o + 3\delta v) \frac{\partial \rho}{\partial z} = 0. \tag{12}$$

In this particular case, the propagation of the quantity ρ is three times as sensitive to the nonlinearity as is the phase.[4]

Equation 12, according to the theory of characteristics, implies that ρ is a constant along the lines

$$z = (v_o - 3\alpha\rho_o)t$$

where $\rho = \rho_o$ and $z = z_o$, for $t = 0$. If ρ'_o is infinitesimally greater than ρ_o and corresponds to z'_o, its corresponding characteristic crosses that corresponding to ρ_o at the point z, for a value of t equal to

$$t = \frac{1}{3\alpha} \left\{ \frac{\partial \rho}{\partial z} \right\}^{-1}$$

This equation for $\frac{\partial \rho}{\partial z}$, having its maximum value in space along the initial ρ curve, gives the time of shock formation.

2. Thermodynamics of Pulse Evolution

Even though the system exhibits changes at the optical frequency, the particular form of the dielectric constant, as well as that of the energy flow equation, indicates that with respect to the interaction of the field with the medium such rapid changes are absent. This is true as long as the envelope changes are slow compared with the inverse of the optical frequency. If these changes are much slower than the relaxation time of the medium, too, the latter does achieve a thermodynamic equilibrium with the field. Reversible thermodynamics can then be applied.

The electromagnetic work done on the medium, dL, as obtained from (1), noting from (7) that the spatial flow of energy as given by the Koynting vector is simply $\frac{\partial \rho v}{\partial z}$; is given by

$$dL' = \mu_o d \frac{(H H^*)}{2} + \frac{E dD^*}{2} + \frac{E^* dD}{2} = d\rho + \frac{1}{2} \left(\frac{\rho}{\epsilon} \right) d\epsilon \tag{13}$$

Choosing ρ and T as independent variables, the differential heat change for the thermodynamic system is given by

$$dQ = dU - L = d(U-\rho) - \frac{1}{2}\left(\frac{\rho}{\epsilon}\right)\frac{\partial \epsilon}{\partial \rho}\, d\rho - \frac{1}{2}\left(\frac{\rho}{\epsilon}\right)\frac{\partial \epsilon}{\partial T}\, dT. \tag{14}$$

For a reversible process $dQ = TdS$, and thus we may write

$$\frac{\partial}{\partial \rho}(U-\rho) = \frac{1}{2}\rho\frac{\partial}{\partial \rho}(\ln \epsilon) + T\frac{\partial S}{\partial \rho}. \tag{15}$$

The appropriate Maxwell relationship can be obtained from the differential of the free energy, which for the reversible process is

$$dF = -SdT + dL$$

$$= \left(-S + \frac{1}{2}\frac{\rho}{\epsilon}\frac{\partial \epsilon}{\partial T}\right) dT + \left(1 + \frac{1}{2}\frac{\rho}{\epsilon}\frac{\partial \epsilon}{\partial \rho}\right) d\rho. \tag{16}$$

The differentiability condition then implies that

$$\left.\frac{\partial S}{\partial \rho}\right|_T = \frac{1}{2}\left.\left(\frac{\partial}{\partial T}(\ln \epsilon)\right)\right|_T. \tag{17}$$

The total internal energy from (15) is then equal to

$$U = \rho + \int_0^\rho \frac{d\rho'}{2}\left(\rho'\frac{\partial}{\partial \rho'}(\ln \epsilon) + T\frac{\partial}{\partial T}(\ln \epsilon)\right) + C_o(T-T_o), \tag{18}$$

where C_o is the heat capacity at constant B and D, for B and D equal to zero, and T_o is a reference temperature.[5]

The entropy is given by (17):

$$S = \frac{1}{2}\int_0^\rho d\rho' \frac{\partial}{\partial T}(\ln \epsilon) + C_o \ln\left(\frac{T}{T_o}\right). \tag{19}$$

In the Debye approach, ϵ is a function of $\frac{\rho}{kT}(\tau = 0)$ so that the integral in (18) is zero, which implies that the internal energy increase in the interacting system is equal to ρ plus the internal energy increase attributable to a temperature increase.

From (18) and (19), other thermodynamic functions can be obtained. For instance, the free energy, which is equal to U - TS, is

$$F = \rho - \int_0^\rho d\rho' \left(\rho'\left.\frac{\partial \ln \epsilon}{\partial \rho'}\right|_T\right) + C_o\left((T-T_o) - T\ln\left(\frac{T}{T_o}\right)\right). \tag{20}$$

To lowest order in ρ, S and F are

$$S = C_o \ln\left(\frac{T}{T_o}\right) - \frac{1}{4}\frac{K}{kT^2\epsilon_o}\rho^2 \tag{21}$$

$$F = \frac{1}{2}\left(2 + \frac{\epsilon - \epsilon_o}{2\epsilon_o}\right)\rho + C_o\left((T-T_o) - T \ln\left(\frac{T}{T_o}\right)\right). \tag{22}$$

These can be expressed in terms of the variables H, B, D, and E by employing the constitutive relationship for $(\epsilon-\epsilon_o)$ in terms of ρ, and by attributing half of the linear ρ terms that arise to the magnetic stored energy. Noting that the electromagnetic variables have at most a constant phase (ϵ is real), which can be assumed to be zero, thereby making these variables real, we get

$$\left.\frac{\partial F}{\partial \frac{D}{\sqrt{2}}}\right|_{B,T} = \left(\frac{E}{\sqrt{2}}\right) \qquad \left.\frac{\partial F}{\partial T}\right|_{B,D} = -S.$$

$$\left.\frac{\partial F}{\partial \frac{B}{\sqrt{2}}}\right|_{D,T} = \left(\frac{H}{\sqrt{2}}\right)$$

In the light of these results, the optical interaction can be described as follows. As the pulse begins to penetrate the medium, it polarizes the molecules and tends to align them along the field direction. This alignment under isothermal conditions ($T = T_o$), according to (20), is equivalent to a decrease in the entropy of the system, and thus represents a heat flow out of the system. The resultant loss in internal energy, $T \int dS$, is compensated for by a portion of the electromagnetic work done on the system. The remainder of the work, which just turns out to be ρ, contributes to an increase in the internal energy.

Positions located on the lagging wing of the pulse are behaving oppositely. Because of a decreasing torque, molecular alignment is decreasing. Heat is being absorbed, which, in turn, contributes to an increase in the internal energy. The electromagnetic work done on the system is also decreasing, however, and the internal energy increase resulting from a decreasing alignment compensates for a portion of this decreasing electromagnetic work; once again, ρ is left as the internal energy.

Thus, while the pulse is within the medium, a portion of the electromagnetic energy contributes to an increase in the internal energy, while the remainder is stored in molecular alignment. As the pulse leaves the medium, the electromagnetic field has once again retrieved the energy stored in alignment; consequently, in this case no loss can occur. Only a change in shape of the pulse profile is possible.

This implies that at any point the total energy available for electromagnetic work is given by $U - T \int ds$ under isothermal conditions. This is precisely the free energy when T equals T_o.

Now when relaxation is present, the basic relationship

$$\Delta U = \Delta Q + \Delta W$$

must still apply, since it expresses a conservation of energy, even for an irreversible process. Moreover, the internal energy of the system must still be a function of state of the system. Thus, as before, u must be equal to ρ. Q and W need not be functions of the state, however, although the entropy is. Thus it is reasonable to assume that the entropy is still given by (21), and that the energy available for electromagnetic work is given by the free energy, in the case of an isothermal process.

In order to justify these claims, consider the electromagnetic work done on the medium, $\int dL$. Upon substitution of (ρ/ϵ) from (9) in (13) and integrating where possible, keeping only lowest order terms in ρ, we obtain

$$W \approx \rho + \frac{1}{4\epsilon_o\beta}\{\epsilon-\epsilon_o\}^2 + \frac{\tau}{2\beta}\int_1^{(\epsilon/\epsilon_o)}\left\{\frac{\partial}{\partial t}\left(\ln\left(\frac{\epsilon}{\epsilon_o}\right)\right)\right\} d\left(\frac{\epsilon}{\epsilon_o}\right). \tag{24}$$

The second term is just the reversible heat of alignment considered previously. The last term is the electromagnetic work done to overcome viscous forces. β is positive for a passive material. In this case there must also be a term in ΔQ representing the generation of heat arising from viscous damping, which compensates for the work done against viscous forces. Consequently,

$$\Delta Q = T\Delta S - \frac{\tau}{2\beta}\int_1^{(\epsilon/\epsilon_o)}\left\{\frac{\partial}{\partial t}\left(\ln\left(\frac{\epsilon}{\epsilon_o}\right)\right)\right\} d\left(\frac{\epsilon}{\epsilon_o}\right). \tag{25}$$

As postulated, then, we have

$$U = \rho. \tag{26}$$

Returning to the energy flow equation (7); it can be written, by using (24),

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho v}{\partial z} + \frac{1}{2}\frac{(\epsilon-\epsilon_o)}{\beta\epsilon_o}\frac{\partial\epsilon}{\partial t} + \frac{\tau}{2\beta}\frac{1}{\epsilon}\left(\frac{\partial(\epsilon-\epsilon_o)}{\partial t}\right) = 0. \tag{27}$$

Consider a small control volume of the system. Equation 27 implies that the flow of the electromagnetic contribution to the internal energy is equal to its rate of increase within the small-volume element, minus the rate of heat extraction attributable to electromagnetic alignment of the molecules, plus the rate of dissipation within the volume. If the process is isothermal, this can be written

$$\frac{\partial\rho v}{\partial x} + \frac{\partial F}{\partial t} + (\text{Rate of Loss}) = 0. \tag{28}$$

Furthermore, if the loss is zero, the rate of heat extraction resulting from alignment

can be written in terms of the internal energy term that results in (12). This shows that the internal energy decrease resulting from the alignment causes an effective decrease in the speed of propagation of the internal energy, changing it to $(v_o + 3\delta v)$. In this case the effective flow velocity of the internal energy through the boundary of a control element obeys the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left\{\left(v_o + \frac{3}{2}\delta v\right)\rho\right\} = 0. \tag{29}$$

3. Computational Results

Figures IX-2, IX-3, and IX-4 show numerical solutions for the two equations (10) and (11). These bear out the theoretical results obtained above. Figure IX-2 shows that as a Gaussian pulse penetrates the nonlinear medium, a portion of the electromagnetic work goes into alignment and the remaining increases the internal energy. The decrease in phase velocity causes the internal energy pulse to steepen on the lagging wing and a shock is eventually formed. Even though relaxation is present, steepening occurs indefinitely, although at a slower rate than when τ is zero. The total electromagnetic contribution to the internal energy is also seen to decrease as the pulse traverses the medium, and electromagnetic energy is dissipated as heat.

Figures IX-3 and IX-4 illustrate pulse deformation for the case of an initial modulating amplitude formed by the beating of two frequencies, thereby obtaining a periodic pulse profile. They indicate that the behavior is highly dependent upon the ratio of the relaxation time to the period of the initial amplitude. For a small ratio, a sequence of sawteeth which gradually merge is obtained. For a larger ratio, however, in which case the relaxing velocity from one cycle can dominate the initial portion of the succeeding cycle, pulse steepening occurs both at the beginning and at the end of each cycle, pulse steepening occurs both at the beginning and at the end of each cycle. In the latter, the velocity profile tends to a constant and therefore a state of zero dissipation.

Kelly has calculated that in the case of Cs_2 in the self-trapped[5,6] region of a mode-locked laser pulse, the steepening should occur in the order of a centimeter or two.

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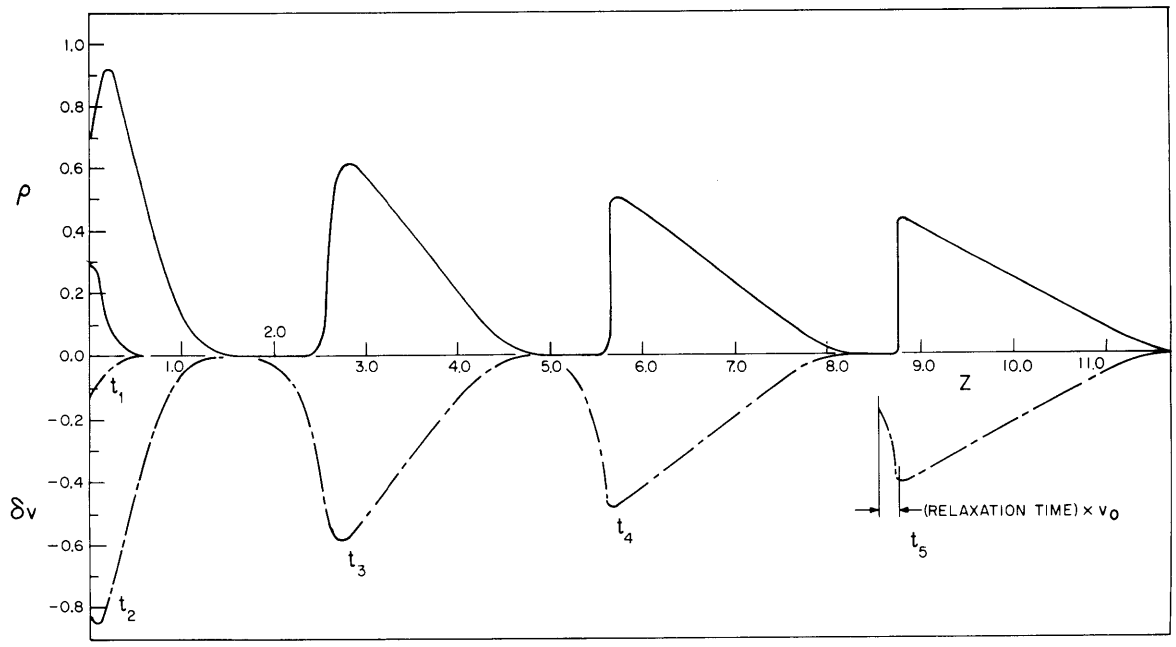

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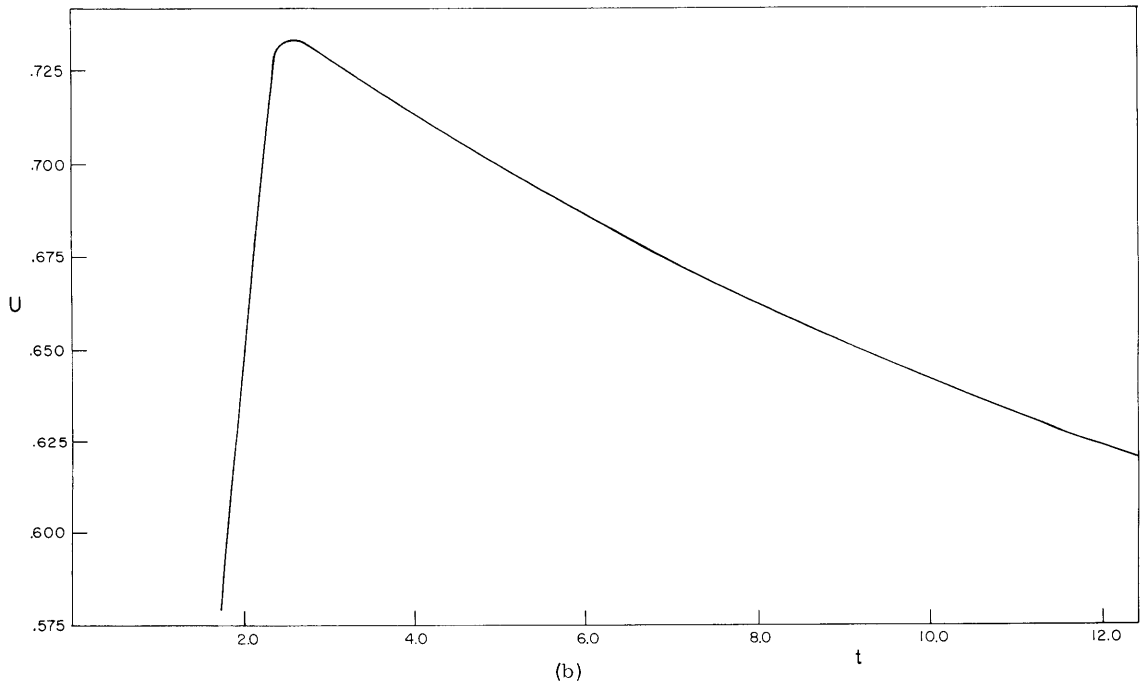
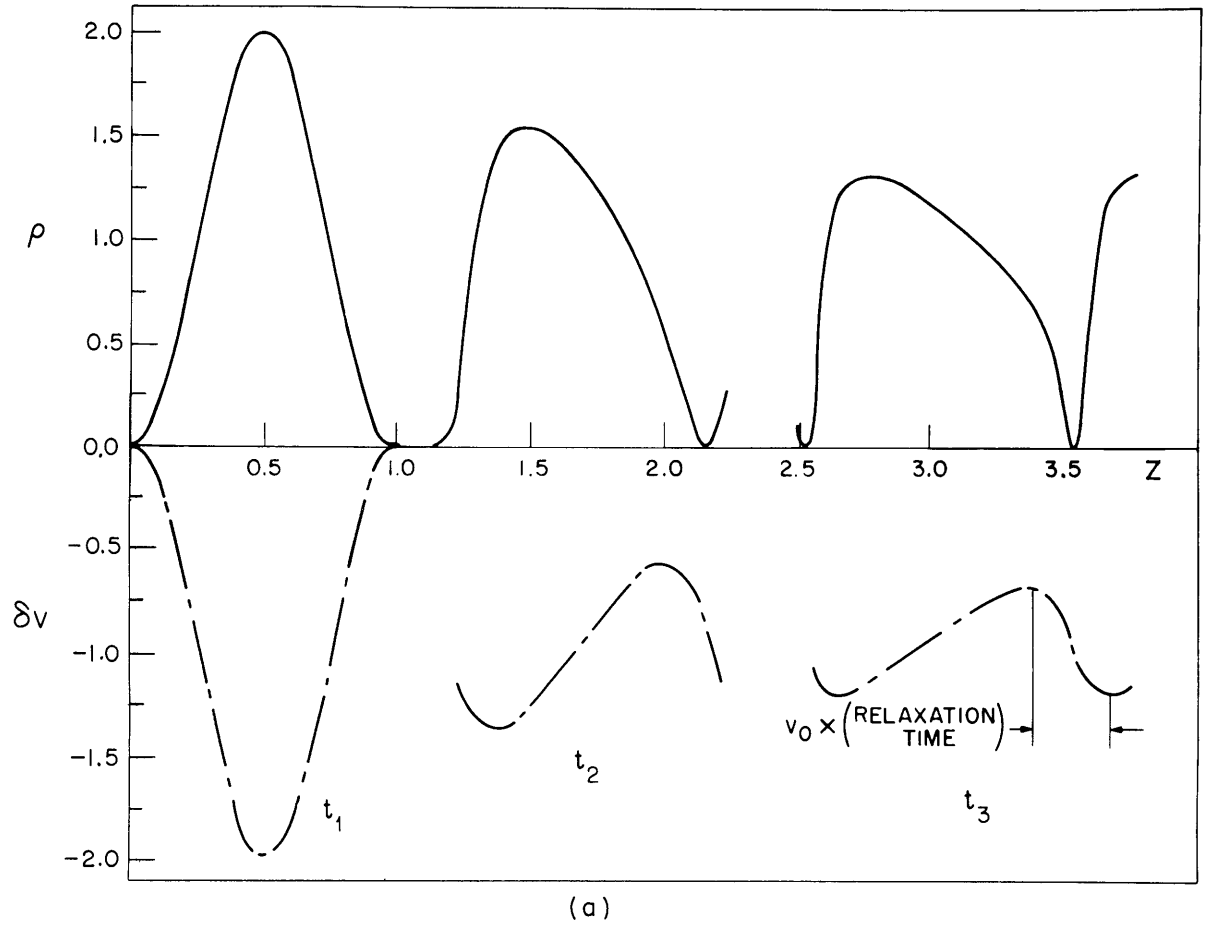


(b)

Fig. IX-2. (a) Propagation of a pulse in the stationary frame. The initial pulse just before entering the medium is Gaussian, with the height and width normalized to unity. $\frac{v_2\rho_o}{v_o} = .2$, $\left(\frac{t_s v_o}{\ell} = .291\right) \cdot \tau = .229$, $t_s \cdot T_1 = .89\ t_s$, $T_2 = 1.77\ t_s$, $T_3 = 5.22\ t_s$, $T_4 = 8.72\ t_s$, $T_5 = 12.29\ t_s$.

(b) Integrated energy density, $u = \int_{-\infty}^{\infty} \rho\ dz$, as a function of time (normalized to the shock time).



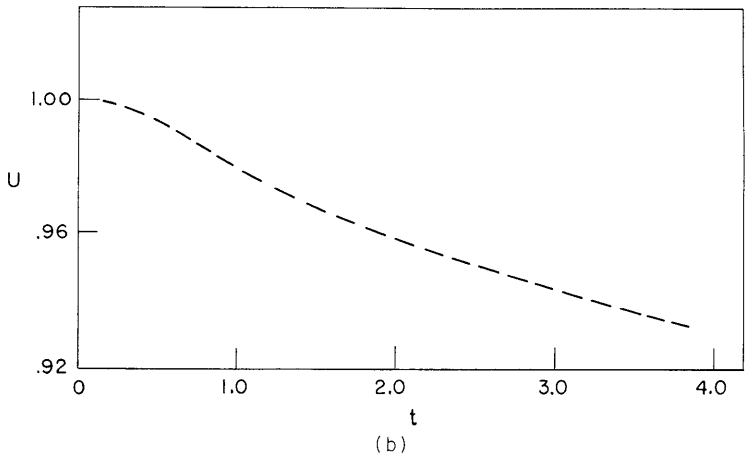


Fig. IX-3. (a) Propagation of a pulse in the stationary frame. The initial pulse is sinusoidal, of the form $(1 + \sin 2\pi z)$. Distance is normalized to modulation wavelengths. $\frac{v_2 \rho_o}{v_o} = .075$, $\left(\frac{t_s v_o}{\ell} = .707\right) \cdot \tau = .354\ t_s$, $T_1 = 0$, $T_2 = 1.74\ t_s$, $T_3 = 4.56\ t_s$.

(b) Integrated energy density as a function of time (normalized to the shock time).

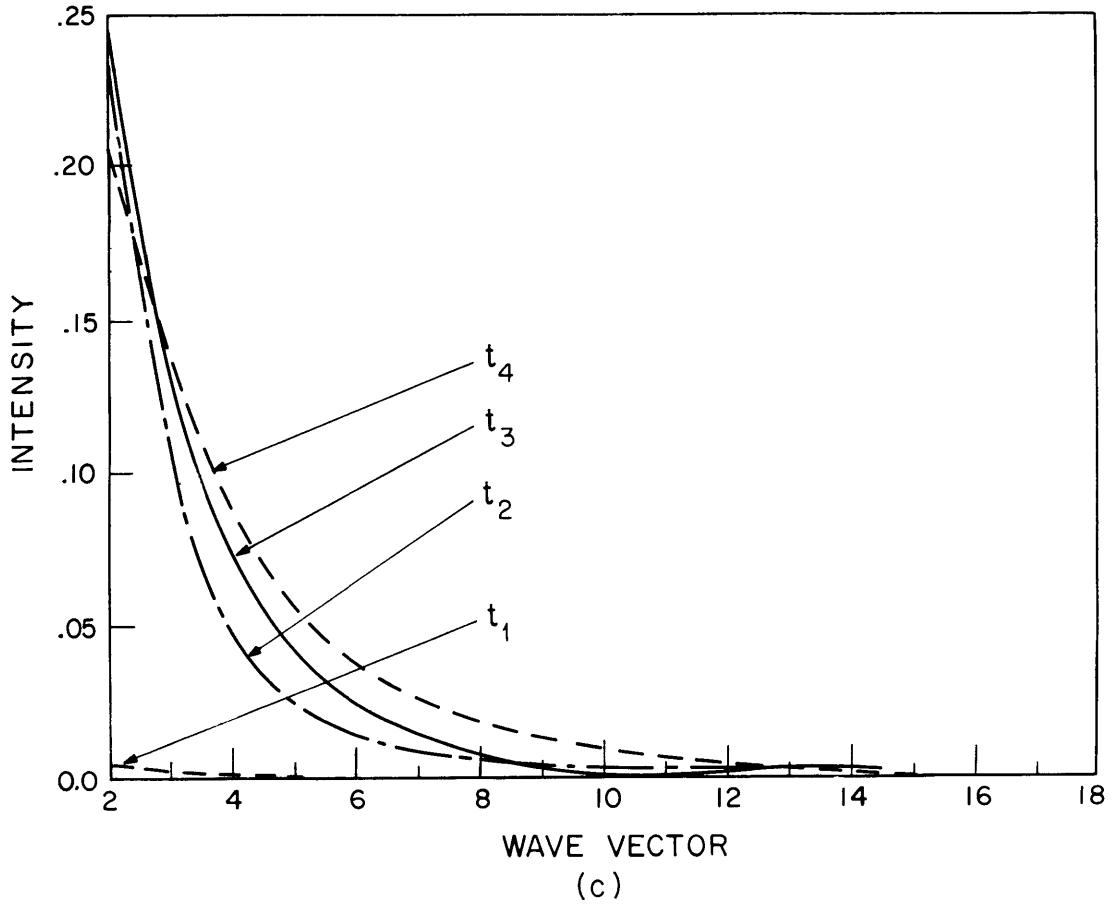


Fig. IX-3. (c) Absolute value of the Fourier transform of the pulse at various times. The wave vector is in units of the initial modulation wave vector. Amplitudes are normalized to the initial amplitude of the fundamental. $T_1 = 3.65^{-3} t_s$, $T_2 = 1.79\, t_s$, $T_3 = 2.55\, t_s$, $T_4 = 3.39\, t_s$.

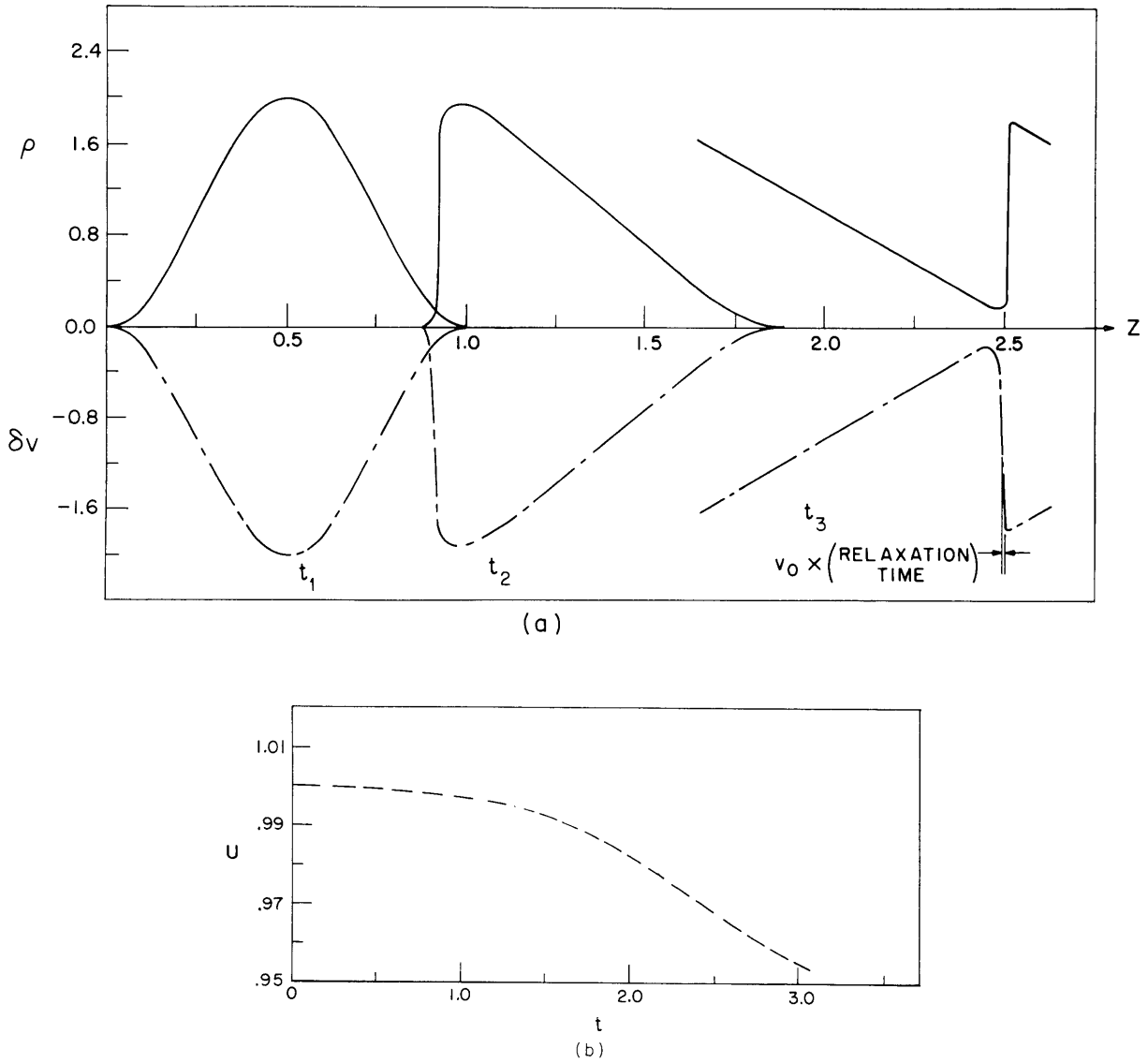


Fig. IX-14. (a) Propagation of a pulse in the stationary frame. Initial pulse in sinusoidal. of the form $(1 + \sin 2\pi z)$. Distance is given in modulation wavelengths. $\frac{v_2 \rho_o}{v_o} = .1 \left(\frac{t_s v_o}{\ell} = .531 \right) \cdot \tau = .0094\ t_s$, $T_1 = 0$, $T_2 = 1.66\ t_s$, $T_3 = 3.13\ t_s$.

(b) Internal energy increase contributed by the electromagnetic stored energy as a function of time (normalized to the shock time).

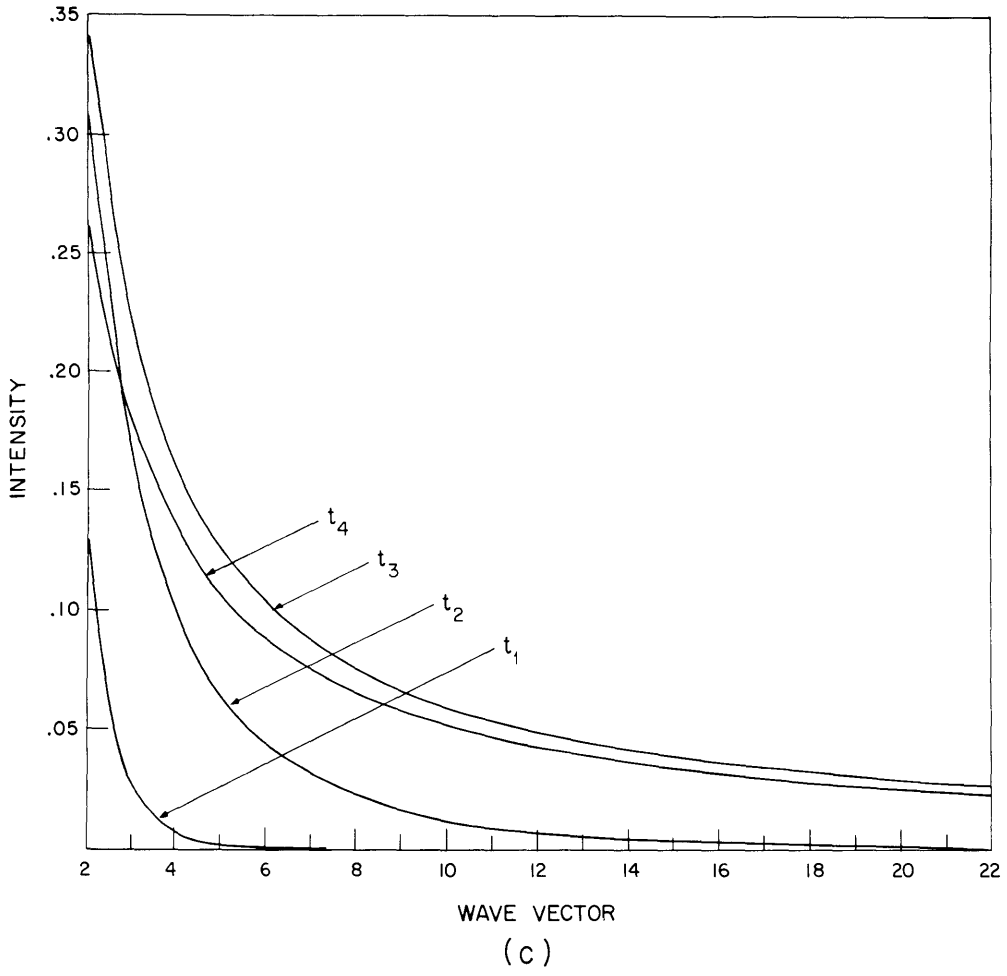


Fig. IX-4. (c) Absolute value of the Fourier transform of the pulse at various times. The wave vector is in units of the initial modulation wave vector. Amplitudes are normalized to the initial amplitude of the fundamental. $T_1 = .367\ t_s$, $T_2 = 1.10\ t_s$, $T_3 = 1.85\ t_s$, $T_4 = 3.13\ t_s$.

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