XXXII. NEUROPHYSIOLOGY*

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RESEARCH OBJECTIVES AND SUMMARY OF RESEARCH

1. Basic Theory

Research on the functional organization of the reticular core of the central nervous system continues, in collaboration with Dr. William L. Kilmer of Michigan State University.

Our problem is to construct a theory for the reticular system which is compatible with known neuroanatomy and neurophysiology, and which will lead to testable hypotheses concerning its operation.[1,2]

Our first and second approaches to this problem[3] were outlined in Quarterly Progress Report No. 76 (page 313).

We can report that we are embarked on a kind of iterative net statistical decision theory[4] that is comprehensive, versatile, and penetrating enough to stand a reasonable chance of success.

The computer modeling is being done at the Instrumentation Laboratory, M. I. T., by members of Louis L. Sutro's group.

W. S. McCulloch

References

1. W. S. McCulloch and W. L. Kilmer, "Introduction to the Problem of the Reticular Formation," in Automata Theory (Academic Press, Inc., New York, 1966).
2. W. S. McCulloch, "What's in the Brain That Ink May Character?," Proceedings of the 1964 International Congress for Logic, Methodology and Philosophy of Science, Held in Jerusalem, August 26-September 2, 1964 (North-Holland Publishing Company, Amsterdam, 1965).

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3. W. L. Kilmer, "On Dynamic Switching in One-Dimensional Iterative Logic Networks," Inform. Contr. 6, 399-415 (1963).

4. W. L. Kilmer, "Topics in the Theory of One-Dimensional Iterative Networks," Inform. Contr. 7, 180-199 (1964).

2. Project Plans

a. Sensory Processes and Multiplexing

The past year's work has suggested to us that the firing pattern of a single neuron which, in the histogram, shows a bimodal or trimodal distribution, conveys information of different sorts with each of the modes. We have spent some time searching for the kind of stimulation needed to separate effects on the different modes, but are not yet able to give a completely satisfactory account of what is happening. There is enough evidence, however, that we have had to consider the characteristics of a system capable of handling information that is distributed or partitioned in the pulse-interval domain. This has led to the making of a new theory of nervous action, an account of which will soon appear.

Color Vision in Amphibia and Reptiles. We have undertaken to study the coding of color in the retinas of frog and turtle, extending the work of Dr. Muntz in this laboratory (4 years ago).

Taste. We shall attempt a study of taste similar to the one that we did on smell.

b. Learning Process

In consequence of the theory of nervous action which we have recently developed, we are studying the notion of the change of probability of invasion into a branch of a single fiber. The work will be done initially on dorsal root-dorsal column system in the cat. We shall try to see if the probability of invasion into the branches at a bifurcation of an axon can be altered permanently in one direction or another by the application of a current across the bifurcation favoring the invasion of one branch more than the other. This is a far-shot experiment, but we feel obliged to do it.

c. Instrumentation

1. We are applying our real-time analyzer of pulse intervals to the study of speech, and for this purpose are devising some new analogue equipment such as a peak picker-outer to take envelopes and a wave-shape detector that works in real time.

2. For the medical profession we are devising an oscillator whose frequency is an exponential function of an applied voltage. This device transforms secular voltage swings such as EKG into a sliding tone that has the same melodic line independent of pitch, i. e., the tune one hears is independent of the DC bias low-applied signal. We have already tried something like this, and it turns out to be very quickly learned for making fine diagnostic distinctions on EKG. We envision a stethoscopelike instrument to replace the ordinary pen recordings of EKG so that screening of patients can be done without accumulation of paper.

3. We are attempting to build an inexpensive low-voltage oscilloscope using crossed galvanometers with 5-kc bandwidth, and a fluorescing paper on which the light spot is cast. One galvanometer gives vertical deflection, the other horizontal deflection.

d. Computer Approach to Diagnosis

Gordon Nelson, a graduate student, during the past two years, has devised a method for handling the diagnostic groupings of a population of rats by similarities of trajectories in time of the course of a combination of 17 independent measures made on the animals. The program was elegantly simple, and in the end the results were discriminations far higher and more reliable than could be made by any of the people – pediatricians, biologists, students – who handled the animals daily. He is now going to use the same scheme to build an automatic neurological diagnosis machine working in the realm of those diseases that are accompanied by disorders in motion of the body.

J. Y. Lettvin

References

1. M. Takata, W. F. Pickard, J. Y. Lettvin, and J. W. Moore, "Ionic Conduction Changes in Lobster Axon Membrane where Lanthanum Is Substituted for Calcium" (J. Gen. Physiol., in press).
2. R. C. Gesteland, J. Y. Lettvin, and S-H. Chung, "A Code in the Nose," Bionics Symposium, Dayton, Ohio, 1966 (in press).

3. Proposed Research

The work for the coming year will continue an analysis of the organization of the somesthetic system. In the past, we have concentrated on the methods by which cells in the spinal cord handle information that has come in over the dorsal roots. We have unravelled the way in which six stages of abstraction and analysis are organized with respect to each other and to some descending control systems. As a by-product of this research, information has been obtained about synaptic transmission and about sensory processes, particularly those leading to pain reactions. This analysis will continue.

In addition to the system in the spinal cord which receives impulses from the periphery, there is a second more recently evolved system that also receives similar information. The method of handling information in the recent system, the dorsal column-medial lemniscus system, contrasts in many important respects from the method. The relative roles of these two systems in sensory analysis and behavior will be studied.

P. D. Wall

A. ON A CALCULUS FOR TRIADAS

1. Introduction

De Morgan, obstructed by his terminology, thought the construction of a logic of relations impossible. A quarter of a century later, C. S. Peirce initiated it. Repeated attempts to understand him failed because in every paper he changed his terminology. It was not until we attempted to formulate family relations in Gilstrap's matricial calculus that he and we were able to understand Peirce, who had actually invented such a calculus and extended it to three-dimensional arrays which we call "mints." It is now clear what he had done and what stopped him. He also used a symbolism in molecular diagrams which is transparent. Finally, he interpreted these in terms of sentences containing n blanks to be filled by the names of things in the universe of discourse. Whether these be real or imaginary is immaterial to this calculus, which therefore can cope with intension, not merely extension, and hence is of value in psychophysiological contexts. Many theorems not involving negation can now be proved, but negation is not simple and we are struggling to discover its multifarious consequences. At the moment, we want to present the following useful results.

2. Triadas

A triada is a structure of any kind involving three elements or members of a given set at a time. For example, "a gives b to c" is a triada, G, involving the objects a, b, and c. Peirce suggested different ways to develop a calculus for triadas, i. e., "an art of drawing inferences." For cases in which triadas are of the nature of the previously mentioned example, i. e., of the nature of a sentence or phrase with three blanks that are to be filled by particular members of a given set, a calculus may be developed that is similar to the calculus of functional propositions of three arguments – or you have Boolian tensors of rank 3 – but that is richer in possibilities and consequences. One of the ways to develop such a calculus is to consider two kinds of variables or symbols, one for the elements of the set where the triadas apply (here lower-case letters are used), and the other for the triadas themselves (represented here by upper-case letters). A calculus involving only upper-case letters will be called a "proper calculus for triadas."

In the process of constructing the calculus, operations on or among triadas are defined which have a definite meaning. The object of the calculus is then to combine the operations and to obtain conclusions or theorems about the combined operations of triadas. We concern ourselves here only with closed operations, i. e., operations on or among triadas, which again generate triadas.

3. Definitions and Operations

A triada is a sentence or phrase with three blanks that are to be filled with specific names of objects, or members of a given set, in order for the sentence to have meaning. For example, if in the sentence "a gives b to c," we delete the names a, b, and c, we end with the triada "____ gives ____ to ____." We denote by i, j, and k the first, second, and third blanks, respectively. Furthermore, we represent the triada by G_{ijk}, i. e., G_{ijk} means "____ gives ____ to ____." If we want to express the fact that the particular member a gives the particular member b to the particular one c, we shall write G_{abc}. Therefore, the subscripts are regarded as variables, as are the blanks, Somewhere in the calculus we shall be able to delete subscripts without confusion, to obtain the calculus proper.

Two triadas are said to be equal if they have the same meaning, i. e., they originate equivalent sentences, when applied to any three objects in the same order. We represent the equality of two triadas by separating them with the sign =. In any expression in which triadas appear, any of them can be replaced by an equivalent one. For example, the triadas "____gives____to____" and "____is given to____ by____" are not equal because when applied to objects a, b, and c in this order the resulting sentences do not have the same meaning; however, the triadas "____ gives ____to ____" and "____is identical to the one who gives____to____" are equal.

We now distinguish three kinds of closed operations. These are unary operations, involving one triada; binary, or nonrelative, involving two triadas; and triadic, or relative, involving three triadas.

a. Unary Operations

Rotation is the clockwise rotation of the order of the blanks in the triada one step. For example, let G_{ijk} be "____gives____to____." Its rotation, represented by $\overset{\frown}{G}_{ijk}$, is the triada "____is given by____the gift____." According to the definition of equality, we may write

$$\overset{\frown}{G}_{ijk} = G_{kij}$$

which indicates that if G applies to objects a, b, and c in this order, then $\overset{\frown}{G}$ applies to them in the order c, a, b.

Reflection, where the first and third blanks interchange positions, for example, the reflection of G_{ijk} is the triada "____is given____ by____," that we represent by $\overset{\smile}{G}_{ijk}$, that is, we may write

$$\overset{\smile}{G}_{ijk} = G_{kji}.$$

By iteratively applying each unary operation to a triada, it is easy to see that

$$G_{ijk} = \overset{\smile}{\overset{\smile}{G}}_{ijk} \text{ and } G_{ijk} = \overset{\frown}{\overset{\frown}{\overset{\frown}{G}}}_{ijk}.$$

Since, in these expressions, subscripts are the same on both sides of the equality sign and they appear in the same order, we may delete them without confusion, to obtain

$$\overset{\smile}{\overset{\smile}{G}} = G \text{ and } \overset{\frown}{\overset{\frown}{\overset{\frown}{G}}} = G.$$

b. Binary Operations (or Nonrelative Operations)

Nonrelative Product: The nonrelative product of two triadas is a triada obtained after joining the two original triadas with the logical connective "and," and making the subscripts in both triadas the same. For example, let G_{ijk} mean "____ gives____to ____" and let L_{ijk} mean "____ lies in between____ and____." The nonrelative product, represented by $G_{ijk} \cdot L_{ijk}$, is the triada "____ gives ____to ____ and the first lies between the second and the third." It follows that $G_{ijk} \cdot L_{ijk} = L_{ijk} \cdot G_{ijk}$.

Nonrelative Sum: The nonrelative sum of two triadas is the triada obtained after joining the two original triadas with the logical connective "or" (inclusive or), and making the subscripts in both triadas the same. For example, the nonrelative sum of G_{ijk} and L_{ijk} is the triada "____ gives ____ to ____ or the first lies in between the second and the third." We represent it by $G_{ijk} + L_{ijk}$. It is clear that $G_{ijk} + L_{ijk} = L_{ijk} + G_{ijk}$.

c. Triadic Operations (or Relative Operations)

Now we introduce the existential quantifier Σ (read "there is some ...") and the universal quantifier Π (read "all," or "everybody" or "everything"). Application of a quantifier to a triada gives a lower structure (a structure with a lower number of blanks). For example, $\Sigma_i G_{ijk}$ reads "there is some who gives ___ to ___," that is, a diadic structure. In order to obtain a closed operation, we could define an "open" or "external" product or sum to obtain a higher structure, and then reduce it to a triada, by applying one or the two quantifiers one or more times. For example, let "and" be the open operation between L_{ijk} and G_{emn} such that $L_{ijk} \cdot G_{emn}$ means "____ lies in between ____ and ____, and ____ gives ____ to ____," that is, a hexada. If we now "contract" by application of the Σ quantifier, we obtain the triada

$$\sum_{iem} L_{ijk} \cdot G_{emn}.$$

This reads "there is some individual who lies in between ____ and ____, and someone gives something to ____."

More interesting are the combinations of triadas with some elements, or blanks, in common, that is, having colligative terms. Such is the case of the so-called relative products and sum for binary, or diadic, relations. For triadas, let us write the product with one colligative term

$$L_{ijk} \cdot G_{kem}$$

that reads "____ lies in between ____ and ____ who gives ____ to ____," that is, a pentadic structure. If we now contract upon the repeated index, by means of the Σ quantifier, we obtain

$$\sum_k L_{ijk} \cdot G_{kem},$$

that is, the tetrada "____ lies in between ____ and someone who gives ____ to ____." If the operation between L_{ijk} and G_{kem} were a sum, we would obtain first the pentada

$$L_{ijk} + G_{kem}$$

that reads "____ lies in between ____ and ____ or this gives ____ to ____." By contracting now on the repeated index by means of the Π quantifier, we obtain

$$\prod_k L_{ijk} + G_{kem},$$

that is, "take any individual; then, either ____ lies in between ____ and this individual or this gives ____ to ____." This is similar to the relative sum of diadas.

The combination of triadas with colligative terms is amenable (Peirce) to clear graphical representation. For example, Diagram XXXII-1 represents the two triadas G_{ijk} and L_{kem}.

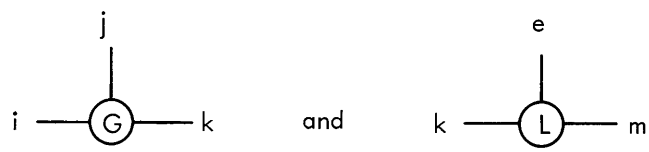

Diagram XXXII-1.

The two operations described above could be graphically represented by Diagram XXXII-2, in which the colligative term appears as a "common bound," and the number of blanks left is the number of "free bounds."

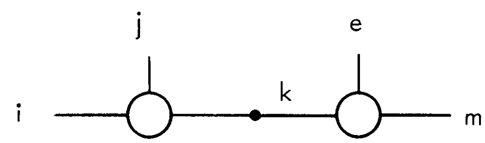

Diagram XXXII-2.

For convenience, we shall define closed relative products and sums among triadas in which the contraction or generalization by the quantifiers is realized upon repeated indexes, and in which each repeated index repeats only once. This permits the use of the above-mentioned type of graph as a means for visualizing the relative operations, and, at the same time, provides us with another tool to prove theorems. It turns out that many of the combinations of open operations which finally result in triadas are particular cases of closed products and sums defined with those rules. Briefly, the rules for forming relative operations of triadas, which permit the use of the above-mentioned graphs, may be stated as follows.

(i) Each repeated index repeats only once.

(ii) Quantifiers act on repeated indexes.

It follows from the graphs that at least three triadas are necessary to verify a closed operation. There are three different ways in which the triadas could be connected (see Diagram XXXII-3):

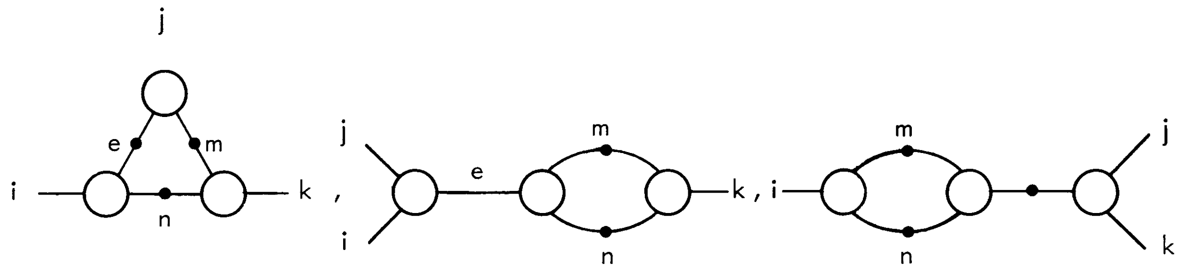

Diagram XXXII-3.

These lead to the relative products and sums that are defined below.

Relative Products

Δ Product of three triadas A, B, and C is the triada

$$\sum_{nem} A_{nie} \cdot B_{ejm} \cdot C_{mkn}$$

which we represent by $\underset{\bullet}{\Delta}$ (ABC).

$\succ$ Product of the triadas A, B, and C is the triada

$$\sum_{emn} A_{ije} \cdot B_{emn} \cdot C_{nmk}$$

which we represent by $\underset{\bullet}{\succ}$ (ABC).

$\prec$ Product of the triadas A, B, and C is the triada

$$\sum_{emn} A_{iem} \cdot B_{men} \cdot C_{njk}$$

which we represent by $\underset{\bullet}{\prec}$(ABC).

For example, let G be the triada "____gives____to____"; let L be "____lies in between____and____"; and let T be "____thinks____is____." Then, $\underset{\bullet}{\Delta}$(GLT) reads "someone gives____to somebody who lies in between____and some other who thinks ____is the first," or "there are three individuals such that the first gives____to the second, this lies in between ____ and the third, and this thinks ____ is the first."

Relative Sums

Δ Sum of three triadas A, B, and C is the triada

$$\prod_{nem} A_{nie} + B_{ejm} + B_{mkn}$$

which we represent by $\underset{+}{\Delta}$ (ABC)

$\succ$ Sum of three triadas A, B, and C is the triada

$$\prod_{nem} A_{ije} + B_{emn} + C_{nmk}$$

which we represent by $\underset{+}{\succ}$ (ABC).

$\prec$ Sum of three triadas A, B, and C is the triada

$$\prod_{emn} A_{iem} + B_{men} + C_{njk}$$

which we represent by $\underset{+}{\prec}$ (ABC).

For example, $\underset{+}{\Delta}$ (GLT) reads "take any three individuals; then, either the first gives ____ to the second, or the second lies in between ____ and the third, or the third thinks ____ is the first."

Resume of Closed Operations for Triadas

Unary
- Rotation, $\widehat{A}$
- Reflection, $\breve{A}$

Binary
- Nonrelative Product $A \cdot B$
- Nonrelative Sum $A + B$

Triadic
- Relative Products
  - $\underset{\cdot}{\Delta}$(ABC)
  - $\underset{\cdot}{\succ}$(ABC)
  - $\underset{\cdot}{\prec}$(ABC)
- Relative Sums
  - $\underset{+}{\Delta}$(ABC)
  - $\underset{+}{\succ}$(ABC)
  - $\underset{+}{\prec}$(ABC)

4. Immediate Theorems

By combining the closed operations among triadas, we can prove the set of equalities, or theorems, that follow.

First, let P_{ijk} be the triada that results from the nonrelative product of A_{ijk} and B_{ijk}. That is,

$$P_{ijk} = A_{ijk} \cdot B_{ijk}.$$

Rotation of P_{ijk} gives

$$\widehat{P}_{ijk} = P_{kij} = A_{kij} \cdot B_{kij} = \widehat{A}_{ijk} \cdot \widehat{B}_{ijk},$$

that is,

$$\widehat{P}_{ijk} = \widehat{A}_{ijk} \cdot \widehat{B}_{ijk}.$$

Since subscripts now appear in the same order, we may delete them to obtain

$$\widehat{A \cdot B} = \widehat{A} \cdot \widehat{B}. \tag{1}$$

Similarly, we can prove that

$$\widehat{A + B} = \widehat{A} + \widehat{B}. \tag{2}$$

By the same method, we can prove that

$$\smile{A \cdot B} = \breve{A} \cdot \breve{B} \tag{3}$$

$$\smile{A + B} = \breve{A} + \breve{B}. \tag{4}$$

Let Q_{ijk} be the triada that results from the operation $\underset{\cdot}{\Delta}(ABC)$, that is,

$$Q_{ijk} = \sum_{emn} A_{nim} \cdot B_{mje} \cdot C_{ekn}.$$

Rotation of Q_{ijk} gives

$$\widehat{Q}_{ijk} = Q_{kij}.$$

From the definition of Δ product, we have

$$Q_{kij} = \sum_{emn} A_{nkm} \cdot B_{mie} \cdot C_{ejn}.$$

Since the "and" operation is commutative, we have

$$Q_{kij} = \sum_{emn} B_{mie} \cdot C_{ejn} \cdot A_{nkm}.$$

That is,

$$\hat{Q}_{ijk} = \sum_{emn} B_{mie} \cdot C_{ejn} \cdot A_{nkm}.$$

The subscripts that are not affected by the quantifier appear in the same order in both sides of the last equation. Therefore, we may write

$$\hat{Q} = \underset{\cdot}{\Delta}(BCA).$$

That is,

$$\overset{\frown}{\underset{\cdot}{\Delta}(ABC)} = \underset{\cdot}{\Delta}(BCA). \tag{5}$$

The reflection of Q_{ijk} gives

$$\breve{Q}_{ijk} = Q_{kji}.$$

From the definition of Δ product, we have

$$Q_{kji} = \sum_{emn} A_{nkm} \cdot B_{mje} \cdot C_{ein}.$$

That is,

$$Q_{kji} = \sum_{emn} C_{ein} \cdot B_{mje} \cdot A_{nkm}.$$

From the definition of reflection,

$$\breve{Q}_{ijk} = Q_{kji} = \sum_{emn} \breve{C}_{nie} \cdot \breve{B}_{ejm} \cdot \breve{A}_{mkn}.$$

By deleting subscripts, we obtain

$$\breve{Q} = \underset{\cdot}{\Delta}(\breve{C}\,\breve{B}\,\breve{A}).$$

That is,

$$\overset{\smile}{\underset{\cdot}{\Delta}(ABC)} = \underset{\cdot}{\Delta}(\breve{C}\,\breve{B}\,\breve{A}). \tag{6}$$

By similar procedures, it is possible to show that

$$\overset{\smile}{\underset{\cdot}{\succ}(ABC)} = \underset{\cdot}{\prec}(\breve{C}\,\breve{B}\,\breve{A}) \tag{7}$$

$$\underset{\bullet}{\prec}\overset{\smile}{(ABC)} = \underset{\bullet}{\succ}(\breve{C}\,\breve{B}\,\breve{A}). \tag{8}$$

Similarly, we can prove that

$$\underset{+}{\Delta}\overset{\frown}{(ABC)} = \underset{+}{\Delta}(BCA) \tag{9}$$

$$\underset{+}{\Delta}\overset{\smile}{(ABC)} = \underset{+}{\Delta}(\breve{C}\,\breve{B}\,\breve{A}) \tag{10}$$

$$\underset{+}{\succ}(ABC) = \underset{+}{\prec}(\breve{C}\,\breve{B}\,\breve{A}) \tag{11}$$

$$\underset{+}{\prec}\overset{\smile}{(ABC)} = \underset{+}{\succ}(\breve{C}\,\breve{B}\,\breve{A}). \tag{12}$$

5. Constant Triadas

We define five particular triadas that we shall use in the calculus.

a. Universal triada, I_{ijk}, or simply I, is the triada "____, ____ and ____ are individuals." It has the following properties: Let A be any triada; then $A + I = I$ and $A \cdot I = A$. It is clear that $\breve{I} = I$ and $\hat{I} = I$.

b. Null triada, θ, or θ_{ijk}, is the triada "neither____ nor____ nor ____are individuals." Let A be any triada; then $A + \theta = A$ and $A \cdot \theta = \theta$. Also, $\breve{\theta} = \theta$ and $\hat{\theta} = \theta$.

c. Left and Right Identities, denoted by I_λ and I_ρ, respectively, are the following: I_λ is the triada "____ is an individual and ____ is identical to ____"; I_ρ is the triada "____ is identical to ____, and ____ is an individual." It follows that

$$\breve{I}_\lambda = I_\rho \quad ; \quad \breve{I}_\rho = \hat{I}_\lambda \text{ and } I_\lambda = I_\rho. \tag{13}$$

Let A be any triada; then

$$\underset{\bullet}{\Delta}(I_\lambda A I_\rho) = A. \tag{14}$$

For example, let A be "____ gives ____ to ____". $\underset{\bullet}{\Delta}(I_\lambda A I_\rho)$ reads "there are three individuals such that, the first is an individual and ____ is identical to the second, this gives ____ to the third, the third is identical to ____, and the first is an individual." That is the same as "____ gives ____ to ____".

d. Central Identity, I_c, is, by definition, $I_c = \hat{I}_\rho$. It follows that

$$\hat{I}_c = I_\lambda \text{ and } \breve{I}_c = I_c. \tag{15}$$

THEOREM. Let R by any triada. Then

$$\hat{R} = \underset{\bullet}{\Delta}(R I_\rho I_\lambda). \tag{16}$$

Proof. According to Eq. 14, $R = \underset{\bullet}{\Delta}(I_\lambda R I_\rho)$. By rotating both members, we obtain

$$\hat{R} = \overset{\frown}{\Delta(I_\lambda R I_\rho)}.$$

And, by applying Eq. 5, $\underset{\cdot}{\Delta}(\widehat{I_\lambda R I_\rho}) = \underset{\cdot}{\Delta}(R I_\rho I_\lambda)$.

THEOREM. Let A, B, and C be any three triadas. Then

$$\underset{\bullet}{\Delta}[\underset{\bullet}{\Delta}(\widehat{B}\,\overset{\frown}{\breve{A}}\, I_c) I_\lambda\, \widehat{C}] = \underset{\bullet}{\prec}(ABC). \tag{17}$$

This theorem could be proved by operating on subscripts, in a form similar to the proofs of Eqs. 5 and 6. It can also be proved by means of a graph. The proof by means of a graph is illustrated in the following diagrams.

The graph for $\underset{\bullet}{\Delta}[\underset{\bullet}{\Delta}(\widehat{B}\,\overset{\frown}{\breve{A}}\, I_c) I_\lambda\, \widehat{C}]$ is shown in Diagram XXXII-4.

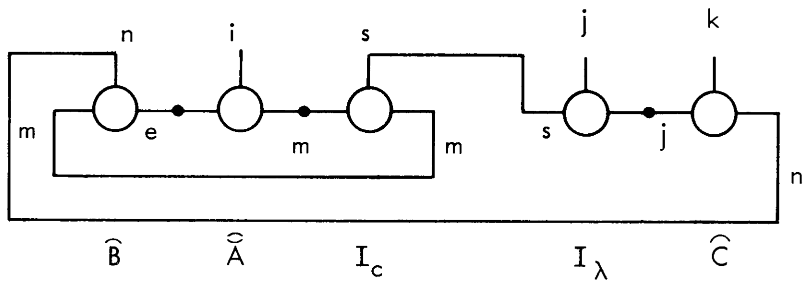

Diagram XXXII-4.

The graph for $\underset{\bullet}{\succ}(ABC)$ is shown in Diagram XXXII-5.

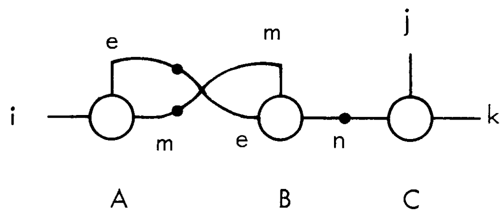

Diagram XXXII-5.

Because of the nature of the identities I_c and I, both graphs are the same. The introduction of the subscript s in the first does not affect this, since it is equivalent to saying that "someone is an individual."

THEOREM. Let A, B, and C be any three triadas. Then

$$\underset{\bullet}{\Delta}[\hat{\hat{A}}\, I_\rho\, \underset{\bullet}{\Delta}(I_c\, \check{\hat{C}}\, \hat{\hat{B}})] = \underset{\bullet}{\succ}(ABC). \tag{18}$$

<u>Proof.</u> Let R, S, and T be any three triadas.

According to Eq. (17), we have

$$\underset{\cdot}{\prec}(RST) = \underset{\cdot}{\Delta}[\underset{\cdot}{\Delta}(\overset{\frown}{S}\,\overset{\frown}{\breve{R}}\,I_c)I_\lambda\,\overset{\frown}{T}].$$

By reflecting both sides, and iteratively applying Eqs. 8 and 6, we obtain

$$\overset{\smile}{\underset{\cdot}{\prec}(RST)} = \underset{\cdot}{\succ}(\breve{T}\,\breve{S}\,\breve{R}) = \Delta[\breve{\overset{\frown}{T}}\,\breve{I}_\lambda\,\Delta(\breve{I}_c\,\breve{\overset{\frown}{\breve{R}}}\,\breve{\overset{\frown}{S}})].$$

But $\breve{I}_\lambda = I_\rho$ and $\breve{I}_c = I_c$. Therefore

$$\underset{\cdot}{\succ}(\breve{T}\,\breve{S}\,\breve{R}) = \underset{\cdot}{\Delta}[\breve{\overset{\frown}{T}}\,I_\rho\,\underset{\cdot}{\Delta}(I_c\,\breve{\overset{\frown}{\breve{R}}}\,\breve{\overset{\frown}{S}})].$$

Let $A = \breve{T}$, $B = \breve{S}$, and $C = \breve{R}$. Then

$$\overset{\frown}{\overset{\frown}{T}} = \overset{\frown}{A},\quad \breve{\overset{\frown}{S}} = \overset{\frown}{\overset{\frown}{B}},\quad \text{and}\quad \breve{\overset{\frown}{\breve{R}}} = \breve{\overset{\frown}{C}}.$$

By substitution, we finally prove the theorem.

From theorems (16), (17), and (18), it follows that rotation ($\frown$) and the triadic products $\underset{\cdot}{\succ}$ and $\underset{\cdot}{\prec}$ are reducible to Δ products.

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