VII. ELECTRODYNAMICS OF MOVING MEDIA*

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A. SPECIAL RELATIVITY AND ASYMMETRIC ENERGY MOMENTUM TENSORS

The principle of virtual work applied to systems with intrinsic angular momentum leads to asymmetric stress tensors. In this report ways are shown for reconciling the relativistic law of angular momentum conservation with the asymmetry of the stress energy tensors.

1. Introduction

The stress tensor of a time dispersive polarizable or magnetizable medium obtained from the principle of virtual work (or principle of virtual power) is nonsymmetric.[1,2] This asymmetry is due in part to the terms $\mu_0 \overline{M} H$ and/or $\overline{P} \overline{E}$, which are asymmetric, if the polarization and magnetization do not align with the electric and magnetic field intensities. From the nonrelativistic point of view, there are no difficulties with such an asymmetry of the stress tensor. Indeed, in a time-dispersive medium there is an intrinsic angular momentum associated with the rotation of the polarization or magnetization vectors. Such an angular momentum requires torques if it is to be changed, and these torques are provided by the antisymmetric part of the stress tensor. Certain difficulties do arise, however, in a relativistic formulation of such media if one tries to write the law of conservation of angular momentum in the conventional four-notation. It is customary to define a four-tensor of third rank describing the angular momentum

$$\theta_{\alpha\beta\gamma} = x_\alpha T_{\gamma\beta} - x_\beta T_{\gamma\alpha}, \tag{1}$$

where $T_{\alpha\beta}$ is the stress energy tensor of the entire system satisfying the equation of motion

$$\partial/\partial x_\alpha T_{\alpha\beta} = 0. \tag{2}$$

The conservation of angular momentum is then written in the form

$$\frac{\partial}{\partial x_\gamma} \theta_{\alpha\beta\gamma} = 0. \tag{3}$$

When (1) and (2) are introduced into (3), the result is

*This work was supported by the Joint Services Electronics Programs (U.S. Army, U.S. Navy, and U.S. Air Force) under Contract DA 36-039-AMC-03200(E).

$$T_{\alpha\beta} - T_{\beta\alpha} = 0. \tag{4}$$

Hence, if one insists on conservation of angular momentum, as well as the specific expression (3), one must conclude that the system must possess a symmetric stress-energy four-tensor. Taking as an example a time dispersive polarizable fluid and constructing the stress energy tensor in the usual way, one arrives at an asymmetric four-tensor for the system.[1,2]

This report will show how it is possible to preserve a law of conservation of angular momentum of the form (3) on the one hand, and the expression for the stress tensor as obtained from the principle of virtual power, on the other hand. At the same time, we shall construct a law of angular momentum conservation which approaches the proper nonrelativistic limit.

2. Fundamental Postulates

We shall require that our theory satisfy the following postulates.

Postulate 1: The angular momentum tensor in four-notation is to be given by

$$\theta_{\alpha\beta\gamma} = x_\alpha T^{(s)}_{\gamma\beta} - x_\beta T^{(s)}_{\gamma\alpha} \tag{5}$$

and satisfies the conservation law

$$\frac{\partial}{\partial x_\gamma} \theta_{\alpha\beta\gamma} = 0, \tag{6}$$

where $T^{(s)}_{\alpha\beta}$ is a stress-energy four-tensor containing the mechanical translational equations of motion in the form

$$\frac{\partial}{\partial x_\alpha} T^{(s)}_{\alpha\beta} = 0. \tag{7}$$

A direct consequence of (6) and (7) is the symmetry of the stress energy tensor $T^{(s)}_{\alpha\beta}$.

Postulate 2: The equations of motion of the entire system are also expressible in terms of the equation

$$\frac{\partial}{\partial x_\alpha} T^{(n)}_{\alpha\beta} = 0, \tag{8}$$

where $T^{(n)}_{\alpha\beta}$ is a nonsymmetric tensor whose three-space part is obtained from the principle of virtual power.

The problem is thus the construction of a symmetric tensor $T^{(s)}_{\alpha\beta}$ from the nonsymmetric tensor obtained from the principle of virtual power in a way that both (7) and (8) are satisfied and at the same time, the usual equation for the conservation of angular momentum is obeyed.

Postulate 3: The vector torque per unit volume, $\bar{\tau}$, constructed from the three-space part of the energy-momentum tensor, according to

$$\bar{\bar{T}}^{(n)}_{\beta\alpha} - \bar{\bar{T}}^{(n)}_{\alpha\beta} \longrightarrow \bar{\tau} \qquad \alpha, \beta = 1, 2, 3 \tag{9}$$

gives the law of conservation of angular momentum $\bar{\sigma}$ per particle in the rest frame:

$$\bar{\tau} = n^0 \left[\frac{d}{dt}\,\bar{\sigma}\right]^0, \tag{10}$$

where the superscripts indicate evaluation in the rest frame.

Let us look briefly at the cause of the asymmetry of the three-space part of the stress-energy tensor. Consider an isotropic time-dispersive polarizable medium. In such a medium the only contribution to the asymmetric three-space part is the term $\bar{P}\bar{E}$.[1,2] Suppose now that we construct the torque per unit volume from

$$T^{(n)}_{\beta\alpha} - T^{(n)}_{\alpha\beta} \tag{11}$$

and look at its one-two component. We obtain

$$T^{(n)}_{\beta\alpha} - T^{(n)}_{\alpha\beta}\Big|_{12} = P_1^0E_2^0 - P_2^0E_1^0 = (\bar{P}^0 \times \bar{E}^0)_3. \tag{12}$$

Apparently, the one-two component of this tensor is the three-component of the vector $\bar{P} \times \bar{E}$ in the rest frame. This is the torque acting on the dipoles of strength $\bar{p}$ and number density n so that the dipole moment per unit volume results in $\bar{P} = n\bar{p}$. This torque, in turn, is equal to the time rate of change of the intrinsic angular momentum of the dipoles. Let us denote the angular momentum per particle by $\bar{\sigma}$. If the number density in the rest frame is n^0, one must have

$$\bar{P}^0 \times \bar{E}^0 = n_0 \frac{d}{dt}(\bar{\sigma}). \tag{13}$$

This equation can be written in four-space notation, once we establish the correct transformation laws for the vector $\bar{\sigma}$. According to Landau and Lifshitz[3] the angular momentum of a body is expressible as the four-tensor of second rank (we interchanged indices to conform to our definition of the force equation).

$$\sigma_{\alpha\beta} = -\frac{i}{c}\int x_\alpha T_{\gamma\beta} - x_\beta T_{\gamma\alpha})\, dS_\gamma. \tag{14}$$

In the rest frame, defined as the frame with no net momentum or energy flow, dS_γ is defined as having a time direction only.

If we apply this formula to a particle, we conclude that its angular momentum is represented by the four-tensor with the components in the rest frame:

$$\sigma_{\alpha\beta} = \begin{bmatrix} 0 & \sigma_3^0 & -\sigma_2^0 & 0 \\ -\sigma_3^0 & 0 & \sigma_1^0 & 0 \\ \sigma_2^0 & -\sigma_1^0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We find a complete analogy between $\sigma_{\alpha\beta}$ and the tensor constructed from the magnetization four-vector

$$\frac{1}{ic} \epsilon_{\alpha\beta\gamma\delta} M_\gamma u_\delta \tag{15}$$

which assumes the above form in the rest frame, with σ_α replaced by M_α. Hence the four-vector angular momentum defined in the rest frame by

$$\sigma_\alpha \equiv [\bar{\sigma}^0, 0] \tag{16}$$

transforms like a magnetization density four-vector.

It is worth considering briefly the implications of this analogy. A magnetization density can be represented by a density of loops of circulating charge currents. The angular momentum is the result of a circulating mass current. Charge-current densities, and mass-current densities do not transform relativistically in the same way. The difference in the transformation laws is compensated for by the fact that M_α contains, in addition, a particle density, whereas σ_α does not contain such a density.

We note that the vector equation (13) is contained in the three-space part of the tensor equation

$$T^{(n)}_{\beta\alpha} - T^{(n)}_{\alpha\beta} = n_0 u_\gamma \frac{\partial}{\partial x_\gamma} \sigma_{\alpha\beta}. \tag{17}$$

A problem arises with the four-components of (17). Take, for example, the four-component in the rest frame of (17) for $\beta = 1, 2, 3$. If one makes the identification (12) and uses the transformation laws for $\sigma_{\alpha\beta}$, one finds in the rest frame (in which usually $T_{\beta 4} = T_{4\beta}$, $\beta = 1, 2, 3$):

$$0 = -\frac{i}{c} \left[\frac{d\bar{v}}{dt}\right]_0 \times \bar{\sigma}^0. \tag{18}$$

This equation puts an inadmissible constraint upon the acceleration. Therefore, one cannot take directly the nonsymmetric tensor as obtained from the principle of virtual power with no modification and hope that it will lead to a consistent equation (17). Modifications are necessary. Here we point out one simple modification that does not produce any additional changes in the principle of virtual power.

3. Modification of Stress-Energy Tensor

We assume that in the rest frame the space-time and time-space parts of the stress-energy four-tensor are symmetric. This has been found true in all examples treated by the authors. We postulate that the time-dispersive medium possesses an additional momentum per unit volume

$$\bar{G}^0 = \frac{1}{c^2}\left[\frac{\partial\bar{v}}{\partial t}\right]^0 \times \bar{\sigma}^0 n^0. \tag{19}$$

The principle of virtual power (of a closed system, with $\phi^0 = 0$),

$$[\nabla\cdot\bar{S}^0]^0 + \left[\frac{\partial}{\partial t} W\right]^0 + W^0[\nabla\cdot\bar{v}]^0 + \frac{\bar{S}^0}{c^2}\cdot\left(\frac{\partial\bar{v}}{\partial t}\right)^0$$

$$= -\bar{\bar{T}}^0 : [\nabla\bar{v}]^0 - \bar{G}_1^0\cdot\left(\frac{\partial\bar{v}}{\partial t}\right)^0, \tag{20}$$

is not affected by such a term because the dot product of the acceleration with the momentum (19) is zero. Therefore, no additional changes have to be made in any of the expressions entering the principle of virtual power if the four-tensor is supplemented by such a momentum density. The addition of the momentum density (19) leaves the stress tensor, the power-flow density, and the energy density of the material system intact. It should be pointed out, however, that the force density on the kinetic system is changed by such a modification. Indeed, the time rate of change of the momentum density and its convective flow have to be subtracted when the kinetic force density is obtained. Hence, a relativistic correction to the force density results even in the rest frame.

We modify the nonsymmetric four-tensor $T_{\alpha\beta}$, obtained from the principle of virtual power (without consideration of intrinsic angular momentum) by the addition of such a momentum density and define the four-tensor

$$T^{(n)}_{\alpha\beta} = T_{\alpha\beta} + \frac{u_\alpha u_\delta}{c^2}\frac{\partial}{\partial x_\gamma}\left(\sigma_{\delta\beta}n^0u_\gamma\right) \tag{21}$$

in which we have included the momentum density (19) by the last term. In terms of the modified nonsymmetric tensor, the conservation of angular momentum (17) is now valid for all values of β or α

$$T^{(n)}_{\beta\alpha} - T^{(n)}_{\alpha\beta} = n_0u_\gamma\frac{\partial}{\partial x_\gamma}(\sigma_{\alpha\beta}). \tag{22}$$

4. Symmetrization of the Stress-Energy Tensor

We shall now construct from the modified stress-energy tensor of (22) a symmetric stress-energy tensor that obeys the same equation of motion (8). This is accomplished by adding to $T^{(n)}_{\alpha\beta}$ the expression

$$\frac{\partial}{\partial x_\gamma} \psi_{\alpha\beta\gamma} = \frac{1}{2} \frac{\partial}{\partial x_\gamma} \left[\sigma_{\alpha\beta} n^0 u_\gamma + \sigma_{\beta\gamma} n^0 u_\alpha - \sigma_{\gamma\alpha} n^0 u_\beta\right], \tag{23}$$

where $\psi_{\alpha\beta\gamma}$ is antisymmetric in $\gamma\alpha$. Because of this antisymmetry, the symmetrized tensor $T^{(s)}_{\alpha\beta} = T^{(n)}_{\alpha\beta} + \frac{\partial}{\partial x_\gamma} \psi_{\alpha\beta\gamma}$ obeys the equation of motion

$$\frac{\partial T^{(s)}_{\alpha\beta}}{\partial x_\alpha} = 0. \tag{24}$$

Next we test that the law of conservation of angular momentum (22) is contained in (6). One obtains

$$T^{(s)}_{\beta\alpha} - T^{(s)}_{\alpha\beta} = T^{(n)}_{\beta\alpha} - T^{(n)}_{\alpha\beta} + \frac{\partial}{\partial x_\gamma} \left(\sigma_{\beta\alpha} n^0 u_\gamma\right) = 0. \tag{25}$$

We see that the law of conservation of angular momentum is indeed contained in (6).

The present discussion has led to results different from those obtained by Meixner.[4] Since the terms in our theory and in Meixner's theory are relativistic, experimental verification of either theory seems out of the question and simplicity of the result is one legitimate criterion to decide between the two.

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