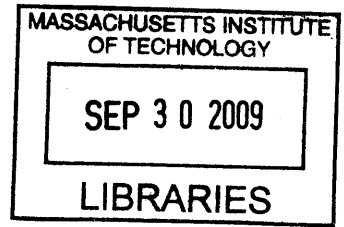# Towards Unifying Multi-Resolution and Multi-Description
A Distortion-Diversity Perspective

by

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Abstract

We consider codec structures that exploit diversity in both source coding and channel coding components. We propose to study source-channel schemes using the tradeoff between end-to-end distortion level and the outage probability as our performance metric, namely distortion-diversity tradeoff. In the high SNR regime, within the distortion-diversity tradeoff framework, we are able to differentiate two source-channel schemes, one based on multi-resolution (MR) and the other based on multi-description (MD), that have been previously determined to have the same average distortion exponent. We then propose a triple-level source-channel scheme that unifies the MR-based and the MD-based schemes. In particular, we demonstrate that the triple-level scheme dominates the MD-based and the MR-based schemes within the distortion-diversity tradeoff framework.

We then extend the distortion-diversity tradeoff to the low SNR regime. We compare the distortion performance of the MR-based scheme and MD-based scheme with separate source-channel decoder that achieve constant levels of outage probability. The performance comparison between the two source-channel schemes is mixed, which naturally links the low outage probability and the high outage probability cases. In particular, the MD-based scheme with separate source-channel decoder preserves the interface between source coding component and channel coding component. The fact that MD-based scheme could outperform MR-based scheme while preserving the source-channel interface suggests that bit rates may not be a complete characterization of the source-channel interface.

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Chapter 1

Introduction

The main challenge of wireless communication is the fading characteristic of wireless channels, which represents the fluctuation of channel quality over time and frequency (see [1] and reference therein for an overview of fading). Researchers usually denote the information of fading processes as channel state information (CSI). CSI at the receiver could be obtained through various methods using training sequences [2]. CSI at the transmitter is usually obtained via feedback channels. The CSI availability significantly affects certain performance metrics [3], such as channel capacity and error exponent. The impact of CSI is also affected by the latency requirement imposed by upper-level applications. If the delay limit of upper-level applications is not significantly larger than the channel variation period, then the channel quality uncertainty may not be sufficiently averaged out for the traditional ergodic analysis [1] to apply. A common alternative to the ergodic analysis is the outage formulation [4]. In [5], the outage framework was extended to study multiple-input multiple-output (MIMO) channel models in the high SNR regime. In particular, the authors of [5] use *diversity order* to measure how fast the error probability decreases with SNR exponentially as SNR increases and *multiplexing gain* to measure how fast the channel code rate increases with log SNR as SNR increases. The authors of [5] then proceed to characterize a fundamental tradeoff between the diversity order and the multiplexing gain for point-to-point communication over MIMO channels. They name this tradeoff the *diversity-multiplexing tradeoff*. In [6], the authors further extend the

diversity-multiplexing tradeoff to characterize multiple access channels.

Another network setting that has attracted considerable amount of research effort is broadcast channels [7], where a single transmitter simultaneously transfers information to multiple receivers. The diversity-multiplexing tradeoff framework [5] still applies to each transmitter-receiver pair individually. However, different receivers may have disparate preferences regarding which diversity-multiplexing point to operate at. The naive approach to address this issue would be devoting devoting a separate codec structure for each transmitter-receiver pair, specifically tuned to accommodate its preference. Not only is this approach costly in terms of the number of encoders the transmitter needs to carry, but also it does not scale well in large networks. The results in [5] do not tell us whether multiple points on the optimal diversity-multiplexing tradeoff could be simultaneously achieved using a single encoder (and multiple decoders at different receivers). Existing research work that addresses this problem is roughly subdivided into two areas, one focusing on channel coding techniques and the other focusing on source coding techniques.

From channel coding perspective, the authors of [8] proposed the idea of *diversity-embedded code*, which is a high-rate space-time code with an embedded high-diversity code embedded. In [9], the authors study the diversity-embedded code from the perspective of diversity-multiplex tradeoff. They label a channel as *successively refinable* if there exists a diversity-embedded code such that both the high-rate component code and the high-diversity component code operates on the optimal diversity-multiplexing tradeoff boundary. In [9], the authors show that channels with one degree of freedom (examples include single-input single-output (SISO) channel, single-input multiple-output (SIMO) channels and multiple-input single-output (MISO) channels) are successively refinable. This result generalizes the diversity-multiplexing tradeoff [5] by showing that, for certain channel models, there exists a codec structure that could simultaneously achieve multiple optimal diversity-multiplexing operating points. However, in [10], the authors prove that channels with more than one degree of freedom are not successively refinable. It is worth pointing out that the channel coding technique used in [9] is superposition code (SPC) [11], [12].

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Source coding techniques have been traditionally treated in the rate-distortion framework [13]. The conventional single-description source code [14] has been extended in many different directions. In [15], the authors categorize a source as *successively refinable* if there exists a source code that could be interrupted in the middle while still achieves the optimal rate-distortion boundary. The authors lay down the necessary and sufficient condition to determine whether a source is successively refinable. This type of codes is called successive refinement in [15] [16] or more generally MR codes [17]. Another extension of the traditional source codes is MD [18], which was originally proposed in [19], [20], [21] and [22]. In [23], a particular multiple description code is constructed and its achievable rate-distortion region characterized, which turns out to be optimal for certain special case, such as the double-description problem with Gaussian source and mean square error distortion [21]. These advanced source coding techniques share the same gist that they intelligently add redundancy into the encoded messages to accommodate contingent scenarios, such as loss of certain message. In [24], the authors consider using these source coding techniques to improve the reliability of certain source reconstructions. Though both MR code and MD code share the feature of embedded diversity components, they are structurally different and provide disparate performance guarantee. The MR code has a *sequential* structure while the MD code resembles a *parallel* structure. Taking double-description MR and MD as an example. The double-description MR code provides no performance guarantee if only the second description is received, while double-description MD guarantees the performance for all three contingency scenarios (both descriptions are received, either one description is received).

In this paper, we consider codec structures that exploit diversity in both source coding and channel coding. These source-channel schemes have traditionally been studied using average end-to-end distortion as performance metric [25]. There are two drawbacks inherent in the average distortion approach. Firstly, distortion is usually a delay-limited notion to begin with. However, the traditional average distortion approach usually averages out the stochastic effect of fading coefficients, which does not fit the delay-limited scenario. Secondly, the traditional average distortion approach

is not sufficient to differentiate the performance of certain source-channel schemes. In particular, in [25], the authors show that the two source-channel schemes based on MR an MD respectively achieve the same average distortion exponent. Therefore, we propose to study source-channel schemes using the tradeoff between end-to-end distortion level and the outage probability as our new performance metric, namely *distortion-diversity tradeoff*. In essence, the distortion-diversity tradeoff represents the distribution function of the end-to-end distortion level. This contrasts the traditional average distortion performance metric, which is the average value of the end-to-end distortion. The distortion-diversity tradeoff characterization of source-channel schemes reveals more operational intuitions than the average distortion performance metric does. In particular, within the distortion-diversity tradeoff framework, we are able to differentiate the two source-channel schemes that [25] determined to have the same average distortion exponent. We then set out to unify the MD-based and the MR-based source-channel schemes. In particular, we propose a triple-level source-channel scheme that unifies the two double-level schemes. We demonstrate that the triple-level scheme indeed dominates the MD-based and the MR-based schemes from the perspective of distortion-diversity tradeoff.

We then turn our attention to the low SNR scenario. We first show that, for our channel model, the Alamouti scheme [26] is equivalent to the MR-based scheme. We then proceed to compare the MR-based scheme and MD-based scheme with separate source-channel decoder for the following three cases.

- In the low outage probability case, the distortion exponent measures how fast the end-to-end distortion approaches 1 as SNR decreases, while the diversity order measures how fast the outage probability approaches 0 as SNR decreases. We show that, for this case, the MD-based scheme with separate source-channel decoder always outperforms the MR-based scheme from the perspective of distortion-diversity tradeoff;

- In the high outage probability case, the distortion exponent still measures how fast the end-to-end distortion approaches 1 as SNR decreases, while the diversity

order measures how fast the outage probability approaches 1 as SNR decreases. We show that, for this case, the MR-based scheme always outperforms the MD-based scheme with separate source-channel decoder from the perspective of distortion-diversity tradeoff;

- In the constant outage probability case, the distortion coefficient measures how fast the end-to-end distortion approaches 1 as SNR decreases. In this case, the performance comparison between the two source-channel schemes is mixed, which links the low outage probability and the high outage probability cases.

Unlike the MD-based scheme considered in the high SNR regime, the MD-based scheme with separate source-channel decoder preserves the interface between source coding component and channel coding component. Moreover, the fact that MD-based scheme sometimes outperforms the Alamouti scheme demonstrates that specially designed channel codes may not lead to the optimal end-to-end performance. Instead, designing codec structure within the framework of distortion-diversity tradeoff is more appropriate.

The remaining part of this paper is organized as follows: in Section 2, we introduce the system setting, including the two channel models we consider; in Section 3, we introduce our performance metrics and the distortion-diversity tradeoff framework; in Section 4, we review several source coding techniques; in Section 5, we study the single-level source-channel scheme in the distortion-diversity framework; in Section 6, we study and compare the two double-level source-channel schemes, i.e. the MD-based and the MR-based schemes; in Section 7, we propose a triple-level source-channel scheme and show that it dominates the two double-level schemes in the distortion-diversity framework; in Section 8, we focus on the low SNR regime and characterize the performance of several source-channel schemes in three different cases. in Section 9, we outline future research directions that could stem from our work.

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Chapter 2

System Overview

We consider wireless broadcast networks composed of a single source and multiple users. The users are categorized into multiple classes based on their *quality* preferences which will be defined in the following section. Let M be the number of user groups in the broadcast network, where $M = 1, 2, \cdots$. One example of the wireless broadcast network comes into existence when video streaming to multiple portable devices, such as PDAs, laptops and etc. The users may be categorized according to
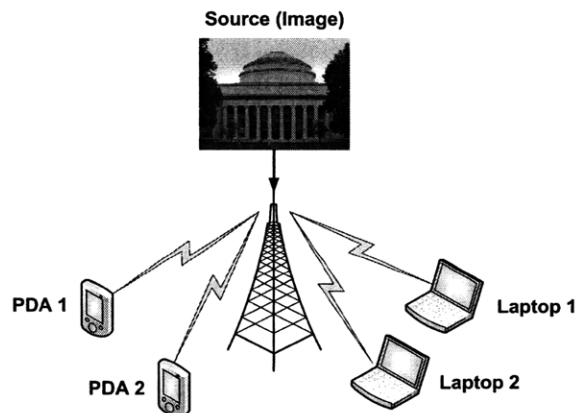
Figure 2-1: Example: Multi-Class Broadcast Network

their requirements for quality of services, such as image resolution and frame rate. For example, the PDA users may prefer high frame rate while the laptop users may prefer high image resolution. The transmitter, such as a base-station tower, is usually armed with multiple antennas. Moreover, a growing number of portable devices are

equipped with more than one antennas. Naturally, multiple-input multiple output (MIMO) channel is a good model for the link between each transmitter-receiver pair. In this document, we do not consider general MIMO channel models. However, we do assume channel models that capture a key characteristic of MIMO channel model, that is multiple degrees-of-freedom (DoF). Furthermore, we assume that the links for different transmitter-receiver pairs undergo independent fading processes. Assuming limited mobility for users, the fading process is modeled to follow the block fading block.

2.1   Parallel Channel Model

In the parallel channel model, each transmitter-receiver link is composed of two independent subchannels as follows,

$$\mathbf{y}_i[n] = \mathbf{h}_i \mathbf{x}_i[n] + \mathbf{w}_i[n], \quad i = 1, 2, \, n = 1, \cdots, N, \tag{2.1}$$

where the subscript is the subchannel index while the number in the bracket is the time index. $\mathbf{h}_1$ and $\mathbf{h}_2$ denotes fading coefficients for subchannel 1 and 2, respectively. Note that N is the length of fading block. We assume that N is long enough for the information theoretic quantity, such as mutual information, to apply. N itself would not enter the analysis in later sections. Following the block fading model, we assume that both $\mathbf{h}_1$ and $\mathbf{h}_2$ are fixed from time 1 up until time N, i.e. within the length-N fading block. Moreover, we assume that both $\mathbf{h}_1$ and $\mathbf{h}_2$ are randomly chosen according to the distribution of $\mathcal{CN}(0,1)$. We assume that the receivers perfectly track the instantaneous channel fading coefficients while the sender only knows the statistical information of the channel fading coefficients. The additive noise $\mathbf{w}_{l,i}[n]$ are i.i.d. $\mathcal{CN}(0,1)$. The input power constraint is SNR on each transmit antenna. Therefore, SNR could also be interpreted as the received signal-to-noise ratio at the receiver.

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2.2 2×1 MIMO Channel Model

In the 2×1 MIMO channel model, each transmitter-receiver link is expressed as follows,

$$\mathbf{y}[n] = \mathbf{h}_1 \mathbf{x}_1[n] + \mathbf{h}_2 \mathbf{x}_2[n] + \mathbf{w}[n], \quad n = 1, \cdots, N, \tag{2.2}$$

where the subscript is the transmit antenna index and the number in the bracket is the time index. $\mathbf{h}_1$ and $\mathbf{h}_2$ are fading coefficients between transmit antenna 1 and 2 and the receive antenna. The fading coefficients are randomly chosen according to the distribution of $\mathcal{CN}(0,1)$. Similar to the parallel channel model, the block length N is assumed to be large enough. We assume that the receivers perfectly track the instantaneous channel fading coefficients while the sender only knows the statistical information of the channel fading coefficients. The additive noises $\mathbf{w}[n]$ are i.i.d. $\mathcal{CN}(0,1)$. The input power constraint is $\mathsf{SNR}/2$ on each transmit antenna. Therefore, SNR could still be interpreted as the received signal-to-noise ratio at the receiver.

Chapter 3

Performance Metric

The general structure of a source-channel scheme is as follows (the detailed structure of encoder and decoder will be introduced in later sections),

1. The sender encodes a source sequence $\underline{s}$ into channel input sequences $\underline{x}_1$ and $\underline{x}_2$;

2. The sender transmits $\underline{x}_1$ and $\underline{x}_2$ over antenna 1 and 2, respectively;

3. Each receiver reconstructs the source sequence as $\hat{s}$ according to his own channel output sequence(s).

The quality of each receiver's source reconstruction is characterized by its accuracy and reliability. Taking the video streaming application in Figure 2-1 as an example, accuracy corresponds to the image resolution while reliability is interpreted as the probability of successfully loading one frame. The accuracy and reliability performance metric used in this paper is different from the corresponding notions in the traditional joint source-channel coding research. In particular, the source-channel schemes studied in this paper proceeds frame by frame. Outage happens if certain frame is not decodable and the entire frame is dropped. This approach is a suitable model for MPEG and other video compression systems. In the following, we formalize our definitions of accuracy and reliability.

3.1 High SNR Regime

In the high SNR regime, we consider the asymptotic scenario where SNR increases to infinity. The accuracy is measured by *distortion exponent*, which is defined as the base-SNR exponentially decreasing speed of the average distortion at high SNR,

$$d = \lim_{SNR \to \infty} \frac{-\log \mathcal{E}\left[\,d(\underline{s}, \underline{\hat{s}})\,\right]}{\log SNR},$$

where the expectation is taken over the random source sequence $\underline{s}$ and $d(\underline{s}, \underline{\hat{s}})$ is the per-symbol distortion between $\underline{s}$ and $\underline{\hat{s}}$. The reliability is measured by the *diversity order*, which is defined as the base-SNR exponentially decreasing speed of the outage probability at high SNR,

$$\delta = \lim_{SNR \to \infty} \frac{-\log \mathcal{P}\left[\,\mathcal{O}(\underline{\hat{s}})\,\right]}{\log SNR},$$

where $\mathcal{P}\left[\,\mathcal{O}(\underline{\hat{s}})\,\right]$ is the probability of $\mathcal{O}(\underline{\hat{s}})$, the outage event that the receiver fails to reconstruct the source sequence as $\underline{\hat{s}}$.

3.2 Low SNR Regime

In the low SNR regime, we consider the asymptotic scenario where SNR decreases to 0. The performance metric is different for the following three cases.

3.2.1 Low Outage Probability Case

The accuracy is measured by the *distortion exponent*, which is defined as the base-SNR exponent of the end-to-end distortion as SNR approaches 0,

$$d = \lim_{SNR \to 0} \frac{\log\left(1 - \mathcal{E}\left[\,d(\underline{s}, \underline{\hat{s}})\,\right]\right)}{\log SNR}, \tag{3.1}$$

where the expectation is taken over the random source sequence $\underline{s}$ and $d(\underline{s}, \underline{\hat{s}})$ is the per-symbol distortion between $\underline{s}$ and $\underline{\hat{s}}$. The reliability is measured by the *diversity*

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order, which is defined as the base-SNR exponentially decreasing speed of the outage probability as SNR approaches 0,

$$\delta = \lim_{\text{SNR} \to 0} \frac{\log \mathcal{P} \left[\mathcal{O}(\hat{\underline{s}}) \right]}{\log \text{SNR}}, \tag{3.2}$$

where $\mathcal{P} \left[\mathcal{O}(\hat{\underline{s}}) \right]$ is the probability of $\mathcal{O}(\hat{\underline{s}})$, the outage event that the receiver fails to reconstruct the source sequence as $\hat{\underline{s}}$.

3.2.2 High Outage Probability case

The accuracy is measured by the *distortion coefficient*, which is defined as the coefficient of $\text{SNR} \log \frac{1}{\text{SNR}}$ in the end-to-end distortion as SNR approaches 0,

$$d = \lim_{\text{SNR} \to 0} \frac{1 - \mathcal{E} \left[d(\underline{s}, \hat{\underline{s}}) \right]}{\text{SNR} \log \frac{1}{\text{SNR}}}, \tag{3.3}$$

where the expectation is taken over the random source sequence $\underline{s}$ and $d(\underline{s}, \hat{\underline{s}})$ is the per-symbol distortion between $\underline{s}$ and $\hat{\underline{s}}$. The reliability is measured by the *diversity order*, which is defined as the base-SNR exponent of the outage probability as SNR approaches 0,

$$\delta = \lim_{\text{SNR} \to 0} \frac{\log \left(1 - \mathcal{P} \left[\mathcal{O}(\hat{\underline{s}}) \right] \right)}{\log \text{SNR}}, \tag{3.4}$$

where $\mathcal{P} \left[\mathcal{O}(\hat{\underline{s}}) \right]$ is the probability of $\mathcal{O}(\hat{\underline{s}})$, the outage event that the receiver fails to reconstruct the source sequence as $\hat{\underline{s}}$.

3.2.3 Constant Outage Probability Case

The accuracy is measured by the *distortion coefficient*, which is defined as the coefficient of SNR in the end-to-end distortion as SNR approaches 0,

$$d = \lim_{\text{SNR} \to 0} \frac{1 - \mathcal{E} \left[d(\underline{s}, \hat{\underline{s}}) \right]}{\text{SNR}}, \tag{3.5}$$

where the expectation is taken over the random source sequence $\underline{s}$ and $d(\underline{s}, \hat{\underline{s}})$ is the per-symbol distortion between $\underline{s}$ and $\hat{\underline{s}}$. We use the outage probability as our reliability

measure, which is called *diversity order*,

$$\delta = \mathcal{P}\left[\,\mathcal{O}(\hat{\underline{s}})\,\right],\tag{3.6}$$

where $\mathcal{P}\left[\,\mathcal{O}(\hat{\underline{s}})\,\right]$ is the probability of $\mathcal{O}(\hat{\underline{s}})$, the outage event that the receiver fails to reconstruct the source sequence as $\hat{\underline{s}}$.

3.3 Distortion-Diversity Tradeoff

There is a natural tradeoff between the two performance metric associated with each user, i.e. the distortion exponent/coefficient and the diversity order. The more stringent the distortion requirement is, the more information bits are need to describe the source sequence and then pushed through the communication link, and therefore the transmission is more like to exceed the link capacity, resulting in an outage. We name this fundamental tradeoff the *distortion-diversity tradeoff* and use it characterize and compare the performance various source-channel schemes.

Recall that, in a wireless broadcast network, multiple users are characterized according to their quality preference. In particular, the quality of each transmitter-receiver pair could be measured by the distortion-diversity point the user chooses to operate at. Certainly, each user would like to operate at the optimal distortion-diversity boundary permitted by the underlying source-channel scheme. However, each user select the operating distortion-diversity point individually using a utility function. We assume the utility functions remain the same within the same class of users while differ across multiple classes. Therefore, a single distortion-diversity pair is associated with each class of receivers. For a broadcast network of M classes of receivers, the performance of the source-channel scheme is characterized by the distortion-diversity vector $(d_1, \delta_1, \cdots, d_M, \delta_M)$. The distortion-diversity tradeoff is the closure of all achievable distortion-diversity vectors.

Chapter 4

Source Coding Background

Source $\underline{s}$ is a sequence of independent and identically distributed (i.i.d.) random variables, with marginal distribution of $p(s)$. Let $\underline{\hat{s}}$ be a reconstructed version of $\underline{s}$. We assume that $\underline{s}$ and $\underline{\hat{s}}$ share the same alphabet $\mathcal{S}$. The symbol-wise distortion measure is a function that maps $\mathcal{S} \times \mathcal{S}$ to $\mathbb{R}^+$, and the sequence-wise distortion $d(\underline{s}, \underline{\hat{s}})$ is defined as the average per symbol distortion. In this section, we review several types of block source code, using n as the block length.

4.1 Single-Description Source Code

A single-description source code is composed of a source code rate R_s $(nats/ss)$[1], a source codebook $\mathcal{C}_n$, an source encoder $\phi_n(\cdot)$ and a source decoder $\psi_n(\cdot)$.

- The source codebook $\mathcal{C}_n$ is an $e^{nR_s} \times n$ matrix of complex numbers. The e^{nR_s} rows of $\mathcal{C}_n$ correspond to the e^{nR_s} reconstruction sequences, denoted as $\underline{\hat{s}}(1), \cdots, \underline{\hat{s}}(e^{nR_s})$;

- The source encoder $\phi_n(\cdot)$ is a function that maps $\mathcal{S}^n$ to $\left\{ 1, \cdots, e^{nR_s} \right\}$;

- The source decoder $\psi_n(\cdot)$ is a function that maps $\left\{ 1, \cdots, e^{nR_s} \right\}$ to $\mathcal{S}^n$.

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[1]ss stands for *source symbol*

The average distortion achieved by the above single-description source code is

$$D(\phi_n, \psi_n) = \mathcal{E}\left[d(\underline{s}^n, \psi_n(\phi_n(\underline{s}^n)))\right],$$

where the expectation is taken over the random source sequence $\underline{s}^n$. A rate-distortion pair (R_s, D) is achievable if there exists a sequence of rate-R_s single-description source code, indexed by block length n, such that,

$$\lim_{n \to \infty} D(\phi_n, \psi_n) \le D.$$

The minimum rate required to achieve a certain distortion is characterized by the following rate distortion function,

$$R_s(D) = \inf_{(R_s,D)\text{ achievable}} R_s.$$

Theorem 4.1.1 (Single-Description Source Code Rate Distortion Function [14]). *For i.i.d. source* s *with marginal distribution of $p(s)$ and bounded symbol-wise distortion function $d(\cdot, \cdot)$, the rate distortion function is*

$$R_s(D) = \min_{p(\hat{s}|s):\, \sum_{(s,\hat{s})} p(s)p(\hat{s}|s)d(s,\hat{s})\le D} I(s; \hat{s}). \tag{4.1}$$

Corollary 4.1.2. *For i.i.d. unit-variance complex Gaussian source with $p(s) = \mathcal{CN}(0,1)$ and squared-error distortion measure, the rate distortion function is,*

$$R(D) = \ln \frac{1}{D}, \tag{4.2}$$

where we assume that $D \le 1$.

4.2   MR Source Code

A double-description MR source code is composed of two source code rates $R_{s,b}$, $R_{s,r}$, a base source codebook $\mathcal{C}_{n,b}$, $e^{nR_{s,b}}$ refinement source codebooks $\mathcal{C}_{n,r}(1), \cdots, \mathcal{C}_{n,r}(e^{nR_{s,b}})$,

a source encoder $\phi_n(\cdot)$ and two source decoders $\psi_{n,b}(\cdot)$, $\psi_{n,r}(\cdot)$.

- The base source codebook $\mathcal{C}_{n,b}$ is an $e^{nR_{s,b}} \times n$ matrix of complex numbers. The $e^{nR_{s,b}}$ rows of $\mathcal{C}_{n,b}$ correspond to the $e^{nR_{s,b}}$ base source reconstruction sequences, denoted as $\underline{\hat{s}}_b(1), \cdots , \underline{\hat{s}}_b(e^{nR_{s,b}})$;

- The refinement source codebooks $\mathcal{C}_{n,r}(1), \cdots , \mathcal{C}_{n,r}(e^{nR_{s,b}})$ are $e^{nR_{s,r}} \times n$ matrices of complex numbers. Refinement source codebook $\mathcal{C}_{n,r}(i_b)$ is associated with base source description i_b. The $e^{nR_{s,r}}$ rows of $\mathcal{C}_{n,r}(i_b)$ correspond to the $e^{nR_{s,r}}$ refinement source reconstruction sequences, denoted as $\underline{\hat{s}}_r(i_b, 1), \cdots , \underline{\hat{s}}_r(i_b, e^{nR_{s,b}})$;

- The source encoder $\phi_n(\cdot)$ is a function that maps $\mathcal{S}^n$ to $\left\{1, \cdots , e^{nR_{s,b}}\right\} \times \left\{1, \cdots , e^{nR_{s,r}}\right\}$;

- The base source decoder $\psi_{n,b}(\cdot)$ is a function that maps $\left\{1, \cdots , e^{nR_{s,b}}\right\}$ to $\mathcal{S}^n$;

- The refinement source decoder $\psi_{n,r}(\cdot)$ is a function that maps $\left\{1, \cdots , e^{nR_{s,b}}\right\} \times \left\{1, \cdots , e^{nR_{s,r}}\right\}$ to $\mathcal{S}^n$.

The average distortion achieved by the base source decoder is

$$D_b(\phi_n, \psi_{n,b}) = \mathcal{E}\left[d(\underline{s}^n, \psi_{n,b}(\phi_n(\underline{s}^n)))\right].$$

The achieved distortion achieved by the refinement source decoder is

$$D_r(\phi_n, \psi_{n,r}) = \mathcal{E}\left[d(\underline{s}^n, \psi_{n,r}(\phi_n(\underline{s}^n)))\right].$$

The rate-distortion quadruple (R_b, R_r, D_b, D_r) is achievable if there exists a sequence of rate-$(R_{s,b}, R_{s,r})$ double-description MR source codes, indexed by block length n, such that,

$$\lim_{n\to\infty} D_b(\phi_n, \psi_{n,b}) \le D_b,$$
$$\lim_{n\to\infty} D_r(\phi_n, \psi_{n,r}) \le D_r.$$

The achievable rate-distortion region is the closure of all achievable rate-distortion quadruples (R_b, R_r, D_b, D_r).

Theorem 4.2.1 (Double-Description MR Source Code Rate-Distortion Region [15]). *For i.i.d. source* s *with marginal distribution of* $p(s)$ *and bounded symbol-wise distortion function* $d(\cdot, \cdot)$, *we evaluate the single-description source code rate distortion function* (4.1) *as distortion levels* D_b *and* D_r *($D_b \geq D_r$),*

$$R_b = R_s(D_b), \quad R_r = R_s(D_r).$$

The rate-distortion quadruple (R_b, R_r, D_b, D_r) *is achievable if and only if there exist auxiliary random variables* $\hat{s}_b, \hat{s}_r$ *with the conditional distribution* $p(\hat{s}_b, \hat{s}_r | s)$ *satisfying,*

$$p(\hat{s}_b, \hat{s}_r | s) = p(\hat{s}_r | s)\, p(\hat{s}_b | \hat{s}_r),$$

and

$$I(s; \hat{s}_b) = R_b, \quad \mathcal{E}\left[d(s, \hat{s}_b)\right] \leq D_b,$$
$$I(s; \hat{s}_r) = R_r, \quad \mathcal{E}\left[d(s, \hat{s}_r)\right] \leq D_r.$$

Remark 4.2.2. *In [15], the authors show that Gaussian source with squared-error distortion satisfies Theorem 4.2.1. For unit-variance complex Gaussian source, to achieve the distortions levels of* D_b *and* D_r, *we use the auxiliary random variables* $\hat{s}_b, \hat{s}_r$ *defined as follows (illustrated in Figure 4-1),*

$$s = \hat{s}_r + w_r,$$
$$\hat{s}_r = \hat{s}_b + w_b,$$

where $\hat{s}_b, w_b, w_r$ *are independently distributed as* $\mathcal{CN}(0, 1 - D_b)$, $\mathcal{CN}(0, D_b - D_r)$ *and* $\mathcal{CN}(0, D_r)$, *respectively.*

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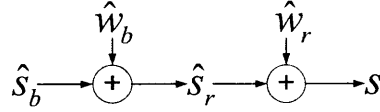
Figure 4-1: Successive Refinement Test Channel

4.3 Symmetric MD Source Code

A symmetric MD source code is composed of a source code rate R_s, three source code-
books $\mathcal{C}_{n,c}, \mathcal{C}_{n,1}, \mathcal{C}_{n,2}$, a source encoder $\phi_n(\cdot)$ and three source decoders $\psi_{n,1}(\cdot), \psi_{n,2}(\cdot), \psi_{n,c}(\cdot, \cdot)$.

- The two side source codebooks $\mathcal{C}_{n,1}, \mathcal{C}_{n,2}$ both are $e^{nR_s} \times n$ matrices of complex numbers. The e^{nR_s} rows of $\mathcal{C}_{n,1}$ correspond to the e^{nR_s} side 1 source reconstruction sequences, denoted as $\hat{\underline{s}}_1(1), \cdots, \hat{\underline{s}}_1(e^{nR_s})$. Similarly, the e^{nR_s} side 2 source reconstruction sequences are denoted as $\hat{\underline{s}}_2(1), \cdots, \hat{\underline{s}}_2(e^{nR_s})$;

- The central source codebook $\mathcal{C}_{n,c}$ is a $e^{2nR_s} \times n$ matrix of complex numbers. The e^{2nR_s} rows of $\mathcal{C}_{n,c}$ correspond to the e^{2nR_s} central source reconstruction sequences, denoted as $\hat{\underline{s}}_c(i_1, i_2)$, $i_1, i_2 = 1, \cdots, e^{nR_s}$;

- The source encoder $\phi_n(\cdot)$ is a function that maps $\mathcal{S}^n$ to $\left\{1, \cdots, e^{nR_s}\right\} \times \left\{1, \cdots, e^{nR_s}\right\}$;

- The two side source decoders $\psi_{n,1}(\cdot), \psi_{n,2}(\cdot)$ are functions that map $\left\{1, \cdots, e^{nR_s}\right\}$ to $\mathcal{S}^n$;

- The central source decoder $\psi_{n,c}(\cdot, \cdot)$ is a function that maps $\left\{1, \cdots, e^{nR_s}\right\} \times \left\{1, \cdots, e^{nR_s}\right\}$ to $\mathcal{S}^n$.

For symmetric MD source codes, the average distortion levels achieved by the two side source decoders are the same, defined as follows,

$$D_s(\phi_n, \psi_{n,1}, \psi_{n,2}) = \mathcal{E}\left[d(\underline{s}^n, \psi_{n,i}(\phi_n(\underline{s}^n)))\right], \quad i = 1, 2.$$

The average distortion level achieved by the central source decoder is

$$D_c(\phi_n, \psi_{n,c}) = \mathcal{E}\left[d(\underline{s}^n, \psi_{n,c}(\phi_n(\underline{s}^n)))\right].$$

The rate distortion triple (R_s, D_s, D_c) is achievable if there exists a sequence of rate-R_s symmetric MD source codes, indexed by block length n, such that,

$$\lim_{n \to \infty} D_s(\phi_n, \psi_{n,1}, \psi_{n,2}) \leq D_s,$$

$$\lim_{n \to \infty} D_c(\phi_n, \psi_{n,c}) \leq D_c.$$

The achievable rate-distortion region is the closure of all achievable rate-distortion triples (R_s, D_s, D_c). Since the optimal rate-distortion region is not known except for several special cases, we only consider the following MD source code, proposed in [23].

Theorem 4.3.1 (Simplified El-Gamal-Cover MD Source Code [23]). *For i.i.d. source $\underline{s}$ with marginal distribution of $p(s)$ and bounded symbol-wise distortion function $d(\cdot, \cdot)$, the rate-distortion triple R_s, D_s, D_c is achievable if there exist auxiliary random variables $\mathbf{s}_1$ and $\mathbf{s}_2$ and functions $g_1(\cdot), g_2(\cdot), g_c(\cdot, \cdot)$ such that,*

$$R_s > \max\left\{ I(\mathbf{s}; \mathbf{s}_1), I(\mathbf{s}; \mathbf{s}_1), \frac{1}{2}\left(I(\mathbf{s}; \mathbf{s}_1, \mathbf{s}_2) + I(\mathbf{s}_1; \mathbf{s}_2)\right) \right\}, \tag{4.3}$$

and

$$D_s \geq \max\left\{ \mathcal{E}\left[d(\mathbf{s}, g_1(\mathbf{s}_1))\right], \mathcal{E}\left[d(\mathbf{s}, g_2(\mathbf{s}_2))\right] \right\},$$

$$D_c \geq \mathcal{E}\left[d(\mathbf{s}, g_c(\mathbf{s}_1, \mathbf{s}_2))\right].$$

Remark 4.3.2. *For unit-variance complex Gaussian source and squared-error distortion, the simplified symmetric El-Gamal-Cover source code uses the following code rate, auxiliary random variables and reconstruction functions.*

- *The auxiliary random variables are defined as follows (also illustrated in Figure 4-2),*

$$\mathbf{s}_1 = \mathbf{s} + \mathbf{w}_1,$$

$$\mathbf{s}_2 = \mathbf{s} + \mathbf{w}_2,$$

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where $(\mathbf{w}_1, \mathbf{w}_2)$ *is independent of* $\mathbf{s}$ *and*

$$(\mathbf{w}_1, \mathbf{w}_2) \sim \mathcal{CN} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix} \right) ;$$
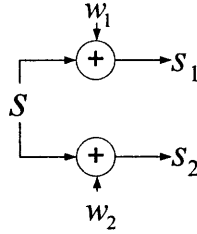
Figure 4-2: MD Test Channel

- *The reconstruction functions are*

$$
\begin{aligned}
g_1(\mathbf{s}_1) &= \frac{1}{1+\sigma^2}\mathbf{s}_1 \,, \\
g_2(\mathbf{s}_2) &= \frac{1}{1+\sigma^2}\mathbf{s}_2 \,, \\
g_c(\mathbf{s}_1, \mathbf{s}_2) &= \frac{1}{2+(1+\rho)\sigma^2}(\mathbf{s}_1 + \mathbf{s}_2) \,;
\end{aligned}
$$

- *The source code rate* R_s *satisfies the following condition,*

$$R_s > \ln \left(\frac{1}{\sqrt{1-\rho^2}} \frac{1+\sigma^2}{\sigma^2} \right) .$$

- *The achieved distortion levels are*

$$
\begin{aligned}
D_s &= \frac{\sigma^2}{1+\sigma^2} \,, \\
D_c &= \frac{(1+\rho)\sigma^2}{2+(1+\rho)\sigma^2} \,,
\end{aligned}
$$

The rate-distortion region of MD for Gaussian source and squared-error distortion has been completely characterized [23] [21]. For the symmetric case, the rate-

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distortion region simplifies to the following lemma.

Lemma 4.3.3 (Symmetric MD, Gaussian source and squared-error distortion, Rate-Distortion Region [24]). *Let an i.i.d. unit-variance Gaussian source be described by two descriptions both of which have rate R. The distortions D_s and D_c corresponding to observations of one or both descriptions. The achievable distortion region for a fixed rate R is*

$$D_c \geq \max\left\{a,\, e^{-2R}\right\},$$

$$D_s \geq \max\left\{e^{-R},\, \frac{1+a}{2} - \frac{1-a}{2}\sqrt{1 - \frac{e^{-2R}}{a}}\right\},$$

where $a \in \left[e^{-2R},\, \frac{e^{-R}}{2-e^{-R}}\right]$.

4.4 Symmetric MD with Common Refinement Source Code

A symmetric MD with common refinement source code is composed of two source code rates R_s, R_r, four source codebooks $\mathcal{C}_{n,1}, \mathcal{C}_{n,2}, \mathcal{C}_{n,c}, \mathcal{C}_{n,r}$, a source encoder $\phi_n(\cdot)$ and four source decoders $\psi_{n,1}(\cdot), \psi_{n,2}(\cdot), \psi_{n,c}(\cdot, \cdot), \psi_{n,r}(\cdot, \cdot, \cdot)$.

- The two side source codebooks $\mathcal{C}_{n,1}, \mathcal{C}_{n,2}$ both are $e^{nR_s} \times n$ matrices of complex numbers. The e^{nR_s} rows of $\mathcal{C}_{n,1}$ correspond to the e^{nR_s} side 1 source reconstruction sequences, denoted as $\hat{\underline{s}}_1(1), \cdots, \hat{\underline{s}}_1(e^{nR_s})$. Similarly, the e^{nR_s} side 2 source reconstruction sequences are denoted as $\hat{\underline{s}}_2(1), \cdots, \hat{\underline{s}}_2(e^{nR_s})$;

- The central source codebook $\mathcal{C}_{n,c}$ is a $e^{2nR_s} \times n$ matrix of complex numbers. The e^{2nR_s} rows of $\mathcal{C}_{n,c}$ correspond to the e^{2nR_s} central source reconstruction sequences, denoted as $\hat{\underline{s}}_c(i_1, i_2)$, $i_1, i_2 = 1, \cdots, e^{nR_s}$;

- The refinement source codebook $\mathcal{C}_{n,r}$ is a $e^{nR_r} \times n$ matrix of complex numbers. The e^{nR_r} rows of $\mathcal{C}_{n,r}$ correspond to the e^{nR_r} refinement source reconstruction sequences, denoted as $\hat{\underline{s}}_r(1), \cdot, \hat{\underline{s}}_r(e^{nR_r})$;

- The source encoder $\phi_n(\cdot)$ is a function that maps $\mathcal{S}^n$ to $\left\{1, \cdots, e^{nR_s}\right\} \times \left\{1, \cdots, e^{nR_s}\right\}$;

- The two side source decoders $\psi_{n,1}(\cdot), \psi_{n,2}(\cdot)$ are functions that maps $\left\{1, \cdots, e^{nR_s}\right\}$ to $\mathcal{S}^n$;

- The central source decoder $\psi_{n,c}(\cdot, \cdot)$ is a function that maps $\left\{1, \cdots, e^{nR_s}\right\} \times \left\{1, \cdots, e^{nR_s}\right\}$ to $\mathcal{S}^n$;

- The refinement decoder $\psi_{n,r}(\cdot, \cdot)$ is a function that maps $\left\{1, \cdots, e^{nR_r}\right\}$ to $\mathcal{S}^n$.

For symmetric MD with common refinement source codes, the average distortion levels achieved by the two side source decoders are the same, defined as follows,

$$D_s(\phi_n, \psi_{n,1}, \psi_{n,2}) = \mathcal{E}\left[d(\underline{s}^n, \psi_{n,i}(\phi_n(\underline{s}^n)))\right], \quad i = 1, 2.$$

The average distortion level achieved by the central source decoder is

$$D_c(\phi_n, \psi_{n,c}) = \mathcal{E}\left[d(\underline{s}^n, \psi_{n,c}(\phi_n(\underline{s}^n)))\right].$$

The average distortion level achieved by the refinement source decoder is

$$D_r(\phi_n, \psi_{n,r}) = \mathcal{E}\left[d(\underline{s}^n, \psi_{n,r}(\phi_n(\underline{s}^n)))\right].$$

The rate distortion pentuple $(R_s, R_r, D_s, D_c, D_r)$ is achievable if there exists a sequence of rate-(R_s, R_r) symmetric MD with common refinement source codes, indexed by block length n, such that,

$$\lim_{n \to \infty} D_s(\phi_n, \psi_{n,1}, \psi_{n,2}) \leq D_s,$$
$$\lim_{n \to \infty} D_c(\phi_n, \psi_{n,c}) \leq D_c,$$
$$\lim_{n \to \infty} D_r(\phi_n, \psi_{n,r}) \leq D_r.$$

The achievable rate-distortion region is the closure of all achievable rate-distortion pentuple $(R_s, R_r, D_s, D_c, D_r)$. Since the optimal rate-distortion region is not known

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except for several special cases, we only consider the following MD with common refinement source code, proposed in [23].

Theorem 4.4.1 (El-Gamal-Cover MD With Common Refinement Source Code [23]). *For i.i.d. source $\underline{s}$ with marginal distribution of $p(s)$ and bounded symbol-wise distortion function $d(\cdot,\cdot)$, the rate-distortion pentuple $(R_s, R_r, D_s, D_c, D_r)$ is achievable if there exist auxiliary random variables s_1, s_2 and s_r and functions $g_1(\cdot)$, $g_2(\cdot)$, $g_c(\cdot,\cdot)$, $g_r(\cdot,\cdot,\cdot)$, such that,*

$$R_s \; > \; \max\left\{ I(s; s_1),\, I(s; s_1),\, \frac{1}{2}\left(I(s; s_1, s_2) + I(s_1; s_2)\right) \right\},$$
$$R_r \; > \; I(s; s_r \,|\, s_1, s_2),$$

and

$$D_s \; \geq \; \max\left\{ \mathcal{E}\left[d(s,\, g_1(s_1)) \right],\, \mathcal{E}\left[d(s,\, g_2(s_2)) \right] \right\},$$
$$D_c \; \geq \; \mathcal{E}\left[d(s,\, g_c(s_1, s_2)) \right],$$
$$D_r \; \geq \; \mathcal{E}\left[d(s,\, g_r(s_1, s_2), s_r) \right].$$

Remark 4.4.2. *For unit-variance complex Gaussian source and squared-error distortion, the symmetric El-Gamal-Cover source code uses the following code rates, auxiliary random variables and reconstruction functions.*

- *The auxiliary random variables are defined as follows (also illustrated in Figure),*

$$s_r \; = \; s + w_r,$$
$$s_1 \; = \; s_r + w_1,$$
$$s_2 \; = \; s_r + w_2,$$

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where $(\mathbf{w}_r, \mathbf{w}_1, \mathbf{w}_2)$ is independent of s *and*

$$(\mathbf{w}_r, \mathbf{w}_1, \mathbf{w}_2) \sim \mathcal{CN} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_b^2 & \rho\sigma_b^2 \\ 0 & \rho\sigma_b^2 & \sigma_b^2 \end{bmatrix} \right) ;$$
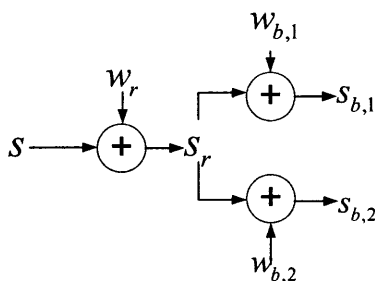
Figure 4-3: MD with Common Refinement Test Channel

- *The reconstruction functions are*

$$
\begin{aligned}
g_1(\mathbf{s}_1) &= \frac{1}{1 + \sigma_r^2 + \sigma_b^2}\mathbf{s}_1 , \\
g_2(\mathbf{s}_2) &= \frac{1}{1 + \sigma_r^2 + \sigma_b^2}\mathbf{s}_2 , \\
g_c(\mathbf{s}_1, \mathbf{s}_2) &= \frac{1}{2 + 2\sigma_r^2 + (1 + \rho)\sigma_b^2}(\mathbf{s}_1 + \mathbf{s}_2) , \\
g_r(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_r) &= \frac{1}{1 + \sigma_r^2}\mathbf{s}_r
\end{aligned}
$$

- *The source code rates satisfy the following conditions,*

$$
\begin{aligned}
R_{s,b} &> \frac{1}{2}\ln\frac{(1 + \sigma_r^2 + \sigma_b^2)^2}{(\sigma_r^2 + \sigma_b^2)^2 - (\sigma_r^2 + \rho\sigma_b^2)^2} , \\
R_{s,r} &> \ln\frac{1 + \sigma_r^2}{\sigma_r^2} - \ln\left(\frac{(1 + \sigma_r^2 + \sigma_b^2)^2 - (1 + \sigma_r^2 + \rho\sigma_b^2)}{(\sigma_r^2 + \sigma_b^2)^2 - (\sigma_r^2 + \rho\sigma_b^2)}\right) .
\end{aligned}
$$

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- *The achieved distortion levels are*

$$D_s = \frac{\sigma_r^2 + \sigma_b^2}{1 + \sigma_r^2 + \sigma_b^2},$$

$$D_c = \frac{\sigma_r^2 + \frac{1+\rho}{2}\sigma_b^2}{1 + \sigma_r^2 + \frac{1+\rho}{2}\sigma_b^2},$$

$$D_r = \frac{\sigma_r^2}{1 + \sigma_r^2}.$$

Chapter 5

Single-Class Broadcast Networks

In a single-class broadcast network, all receivers belong to the same class and seek to operate at the same distortion-diversity point. We consider a single-level source-channel scheme which proceeds as follows (we use D to denote the target distortion level),

1. Source codebook and channel codebook generation

   - We generate a rate-R_s single-description source code as described in Section 4.1. The codebook $\mathcal{C}_s$, a set of e^{NR_s} length-N sequences, is distributed to both the transmitter and the receiver. According to Theorem 4.1.1, the source code rate R_s needs to satisfy the following condition,

$$R_s > -\log D \,;$$

   - We then generate a rate-R_s zero-error channel code [13] which could be decoded error-free is the channel capacity is above the code rate R_s. The channel codebook $\mathcal{C}_c$, a set of e^{NR_s} $2 \times N$ matrices, is also distributed to both the transmitter and the receiver. Each column of the codewords is generated independently according to the Gaussian distribution $\mathcal{CN}(\mathsf{SNR}\, I_2)$. The mutual information between the channel inputs $(\mathbf{x}_1, \mathbf{x}_2)$ and the chan-

nel outputs $(\mathbf{y}_1, \mathbf{y}_2)$ is

$$R_c(\mathbf{h}_1, \mathbf{h}_2) = I(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2 \,|\, \mathbf{h}_1, \mathbf{h}_2) = \sum_{i=1,2} \log\left(1 + |\mathbf{h}_i|^2 \mathsf{SNR}\right) ;$$

2. The sender uses the single-description source encoder (reviewed in Section 4.1) to convert the length-N source sequence $\underline{s}$ to a length-NR_s bit sequence, which we denote as the source description i. The sender then transmit the ith space-time channel codeword $(\underline{x}_1(i), \underline{x}_2(i))$ over subchannel 1 and subchannel 2, respectively;

3. The receiver takes the following steps to reconstruct the source sequence,

   - If $R_c(\mathbf{h}_1, \mathbf{h}_2) > R_s(d)$, we receiver retrieves the description i without error (assuming the block length N is large enough) and outputs the corresponding source reconstruction sequence in the source codebook;

   - If $R_c(\mathbf{h}_1, \mathbf{h}_2) < R_s(d)$, the receiver declares outage and outputs the length-N zero sequence as the source reconstruction sequence.

The outage probability is characterized as follows,

$$\begin{aligned}
\mathcal{P}^{out}(d) &= \mathcal{P}\left[R_c(\mathbf{h}_1, \mathbf{h}_2) < R_s(d)\right] \\
&= \mathcal{P}\left[\sum_{i=1,2} \log\left(1 + |\mathbf{h}_i|^2 \mathsf{SNR}\right) < d \log \mathsf{SNR}\right] \\
&= \mathcal{P}\left[\prod_{i=1,2} \left(1 + |\mathbf{h}_i|^2 \mathsf{SNR}\right) < \mathsf{SNR}^d\right].
\end{aligned}$$

Let $|\mathbf{h}_i|^2 = \mathsf{SNR}^{-\alpha_i}$, then at high SNR, we have $(1 + |\mathbf{h}_i|^2 \mathsf{SNR}) \doteq \mathsf{SNR}^{(1-\alpha_i)^+}$, where $\doteq$ denotes exponential equality in the high SNR regime, i.e. if $f(\mathsf{SNR}) \doteq g(\mathsf{SNR})$, then

$$\lim_{\mathsf{SNR} \to \infty} \frac{\log f(\mathsf{SNR})}{\log g(\mathsf{SNR})} = 1.$$

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The above outage probability can then be written as

$$
\begin{aligned}
\mathcal{P}^{out}(d) & \doteq \mathcal{P}\left[\prod_{i=1,2} \mathsf{SNR}^{(1-\alpha_i)^+} < d \log \mathsf{SNR}\right] \\
& \doteq \mathcal{P}\left[\sum_{i=1,2}(1-\alpha_i)^+ < d\right] \\
& \doteq \mathsf{SNR}^{-\delta(d)},
\end{aligned}
$$

where

$$
\begin{aligned}
\delta(d) & = \min_{\sum_{i=1,2}(1-\alpha_i)^+ < d, \alpha_i \geq 0} \sum_{i=1,2} \alpha_i \\
& = 2 - d,
\end{aligned}
$$

which follows from an argument similar to Theorem 4 in [5]. Therefore, the single-level source-channel scheme achieves the distortion-diversity tradeoff as follows[1],

$$
\delta(d) = 2 - d, \quad 0 < d \leq 2. \tag{5.1}
$$

In figure 5-1, we illustrate the distortion-diversity tradeoff (5.1).
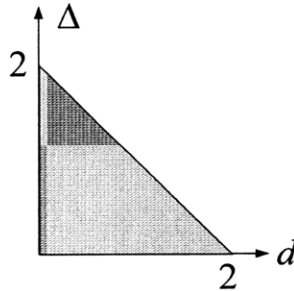
Figure 5-1: Single-Level Source-Channel Scheme, Distortion-Diversity Tradeoff

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[1]We exclude the $(d, \delta) = (0, 2)$ point, since $d = 0$ can be achieved without any information transfer from the sender to the receiver, thus infinite reliability. However, this case is not very interesting for our study of the distortion-diversity tradeoff.

Chapter 6

Double Class Broadcast Networks

In double class broadcast networks, receivers in the two classes seek to operate at two different distortion-diversity points, though each of them still desires to operate on the single-level distortion-diversity boundary illustrated in Figure 5-1. Ideally, we would like to have a single encoding component at the transmitter while still accommodate different distortion-diversity operating points for different receivers. Equivalently, we would like to have achieve multiple distortion-diversity operating points using a single codec structure. Certainly, both distortion-diversity operating points should be as close to the optimal distortion-diversity tradeoff as possible. If both distortion-diversity operating points could be pushed on to the optimal distortion-diversity boundary, we call that the distortion-diversity tradeoff is *successively refinable* However, it is not immediately clear whether this objective could be achieved within a single codec structure. If not, how close could we get to the ideal successively refinable scenario? To address this problem, we study two double-level source-channel schemes in the following two sections.

6.1 MR-Based Source-Channel Scheme

The MR-based source-channel scheme is composed of a double-description MR source code (reviewed in Section 4.2) and a super-position channel code. Suppose that the distortion exponents requested by the two classes of receivers are d_p an d_f, resp. For

each SNR, the double-level source-channel scheme achieves two distortion levels of $D_p = \mathsf{SNR}^{-d_p}$ and $D_f = \mathsf{SNR}^{-d_f}$. The double-level MR based source-channel scheme proceeds as follows,

1. The double-description MR source encoder converts the length-N source sequence $\underline{s}$ into a length-$NR_{s,b}$ bit sequence (base description i_b) and a length-$NR_{s,r}$ bit sequence (refinement description i_r). According to Theorem 4.2.1, the source code rates (R_b, R_r) required to achieve the distortion levels of D_p, D_f are

$$
\begin{aligned}
R_{s,b}(d_p, d_f) &= -\log D_p = d_p \log \mathsf{SNR}, \\
R_{s,r}(d_p, d_f) &= -\log D_f + \log D_p = (d_f - d_p) \log \mathsf{SNR};
\end{aligned}
$$

2. The super-position channel encoder with power split ratio of β: the base description encodes and modulates i_b with power (per subchannel) $\mathsf{SNR} - \mathsf{SNR}^{1-\beta} \doteq \mathsf{SNR}$ $(0 < \beta < 1)$ into $(\underline{x}_{b,1}, \underline{x}_{b,2})$; the refinement description encodes and modulates i_r with power $\mathsf{SNR}^{1-\beta}$ into $(\underline{x}_{r,1}, \underline{x}_{r,2})$; the channel input sequences $(\underline{x}_1, \underline{x}_2)$ are the sum of $(\underline{x}_{b,1}, \underline{x}_{b,2})$ and $(\underline{x}_{r,1}, \underline{x}_{r,2})$;

3. The successive interference cancelation decoder first decodes for i_b treating $(\underline{x}_{r,1}, \underline{x}_{r,2})$ as noise; if i_b is decoded successfully, the decoder then encodes and modulates the decoded i_b and subtracts it from the channel output sequences $(\underline{y}_1, \underline{y}_2)$; the decoder then decodes for i_r;

4. Assuming a Gaussian input distribution of covariance matrix $\mathsf{SNR}\, I$ for the base description channel code and a Gaussian input distribution of covariance matrix $\mathsf{SNR}^{1-\beta} I$ for the refinement description channel code, we obtain

$$
\begin{aligned}
R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \beta) &= I((\mathbf{x}_{b,1}, \mathbf{x}_{b,2})\,;\,(\mathbf{y}_1, \mathbf{y}_2)\,|\,\mathbf{h}_1, \mathbf{h}_2) = \sum_{i=1,2} \log \frac{1 + \mathsf{SNR}|\mathbf{h}_i|^2}{1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2}, \\
R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \beta) &= I((\mathbf{x}_{r,1}, \mathbf{x}_{r,2})\,;\,(\mathbf{y}_1, \mathbf{y}_2)\,|\,(\mathbf{x}_{b,1}, \mathbf{x}_{b,2}), \mathbf{h}_1, \mathbf{h}_2) = \sum_{i=1,2} \log \left(1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2\right);
\end{aligned}
$$

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5. The receiver reconstructs the source sequence as follows,

- If $R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \beta) > R_{s,b}(d_p, d_f)$ and $R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \beta) > R_{s,r}(d_p, d_f)$, the receiver outputs the reconstruction sequence corresponding to (i_b, i_r);

- If $R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \beta) > R_{s,b}(d_p, d_f)$ and $R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,r}(d_p, d_f)$, the receiver declares full outage and outputs the reconstruction sequence corresponding to i_b;

- If $R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,b}(d_p, d_f)$, the receiver declares partial outage and outputs the length-N all zero sequence as the source reconstruction sequence.

The partial outage probability is characterized as follows,

$$
\begin{aligned}
\mathcal{P}_p^{out}(d_p, d_f, \beta) &= \mathcal{P}\left[R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,b}(d_p, d_f)\right] \\
&= \mathcal{P}\left[\sum_{i=1,2} \log\left(\frac{1 + \mathsf{SNR}|\mathbf{h}_i|^2}{1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2}\right) < d_p \log \mathsf{SNR}\right] \\
&= \mathcal{P}\left[\prod_{i=1,2} \frac{1 + \mathsf{SNR}|\mathbf{h}_i|^2}{1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2} < \mathsf{SNR}^{d_p}\right].
\end{aligned}
$$

Let $|\mathbf{h}_i|^2 = \mathsf{SNR}^{-\alpha_i}$, then at high SNR, we have $\frac{1+\mathsf{SNR}|\mathbf{h}_i|^2}{1+\mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2} \doteq \mathsf{SNR}^{(1-\alpha_i)^+ - (1-\beta-\alpha_i)^+}$. The partial outage probability can then be written as

$$
\begin{aligned}
\mathcal{P}_p^{out}(d_p, d_f, \beta) &= \mathcal{P}\left[\prod_{i=1,2} \frac{1 + \mathsf{SNR}|\mathbf{h}_i|^2}{1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2} < \mathsf{SNR}^{d_p}\right] \\
&\doteq \mathcal{P}\left[\prod_{i=1,2} \mathsf{SNR}^{(1-\alpha_i)^+ - (1-\beta-\alpha_i)^+} < \mathsf{SNR}^{d_p}\right] \\
&\doteq \mathcal{P}\left[\sum_{i=1,2} \left((1-\alpha_i)^+ - (1-\beta-\alpha_i)^+\right) < d_p\right] \\
&\doteq \mathsf{SNR}^{-\delta_p(d_p, d_f, \beta)},
\end{aligned} \tag{6.1}
$$

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where we have applied the Laplace principle in the last step and

$$\delta_p(d_p, d_f, \beta) \triangleq \min_{\alpha_i \geq 0, \sum_{i=1,2}((1-\alpha_i)^+ - (1-\beta-\alpha_i)^+) < d_p} \sum_{i=1,2} \alpha_i,$$

is the partial diversity order. Since the receiver is not able to output a full reconstruction (declare either partial outage or full outage) if either $R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,b}(d_p, d_f)$ or $R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,r}(d_p, d_f)$ happens, the full outage probability can be expressed as follows,

$$
\begin{aligned}
\mathcal{P}_f^{out}(d_p, d_f, \beta) &= \mathcal{P}\left[\{R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,b}(d_p, d_f)\} \cup \{R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,r}(d_p, d_f)\}\right] \\
&\doteq \mathsf{SNR}^{-\delta_p(d_p, d_f, \beta)} + \mathcal{P}\left[R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,r}(d_p, d_f)\right],
\end{aligned}
\tag{6.2}
$$

where the last step follows from (6.1) and the second term in the last step could be characterized as follows,

$$
\begin{aligned}
\mathcal{P}\left[R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,r}(d_p, d_f)\right] &= \mathcal{P}\left[\sum_{i=1,2} \log\left(1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2\right) < (d_f - d_p)\log\mathsf{SNR}\right] \\
&= \mathcal{P}\left[\prod_{i=1,2}\left(1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2\right) < \mathsf{SNR}^{d_f - d_p}\right].
\end{aligned}
$$

Let $|\mathbf{h}_i|^2 = \mathsf{SNR}^{-\alpha_i}$, then at high SNR, we have $1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2 \doteq \mathsf{SNR}^{(1-\beta-\alpha_i)^+}$. The above expression can be written as follows,

$$
\begin{aligned}
\mathcal{P}\left[R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,r}(d_p, d_f)\right] &= \mathcal{P}\left[\prod_{i=1,2}\left(1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2\right) < \mathsf{SNR}^{d_f - d_p}\right] \\
&\doteq \mathcal{P}\left[\prod_{i=1,2}\mathsf{SNR}^{(1-\beta-\alpha_i)^+} < \mathsf{SNR}^{d_f - d_p}\right] \\
&\doteq \mathcal{P}\left[\sum_{i=1,2}(1-\beta-\alpha_i)^+ < d_f - d_p\right] \\
&\doteq \mathsf{SNR}^{-\delta_f'(d_p, d_f, \beta)},
\end{aligned}
\tag{6.3}
$$

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where we have again applied the Laplace principle and

$$\delta'_f(d_p, d_f, \beta) \triangleq \min_{\alpha_i \geq 0, \sum_{i=1,2}(1-\beta-\alpha_i)^+ < d_f - d_p} \sum_{i=1,2} \alpha_i \,.$$

Following from (6.2), the full outage probability is

$$
\begin{aligned}
\mathcal{P}_f^{out}(d_p, d_f, \beta) &\doteq \mathsf{SNR}^{-\delta_p(d_p, d_f, \beta)} + \mathcal{P}\left[R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \beta) < R_{s,r}(d_p, d_f)\right] \\
&\doteq \mathsf{SNR}^{-\delta_p(d_p, d_f, \beta)} + \mathsf{SNR}^{-\delta'_f(d_p, d_f, \beta)} \qquad\qquad (6.4) \\
&\doteq \mathsf{SNR}^{-\delta_f(d_p, d_f, \beta)} \,,
\end{aligned}
$$

where Step (6.4) follows from Equation (6.3) and the full diversity order is

$$\delta_f(d_p, d_f, \beta) \triangleq \min\left\{\delta_p(d_p, d_f, \beta), \delta'_f(d_p, d_f, \beta)\right\} \,.$$

Lemma 6.1.1. *The MR-based source-channel scheme achieves two levels of distortion exponents d_p, d_f with the following diversity orders,*

$$
\begin{aligned}
\delta_p &= (2 - d_p) + \lceil -d_p/\beta \rceil (1 - \beta) \,, &(6.5) \\
\delta_f &= \min\left\{\delta_p, (2(1 - \beta) - (d_f - d_p))\right\} \,, &(6.6)
\end{aligned}
$$

where $0 < d_p < 2\beta$ and $0 < d_f - d_p < 2(1 - \beta)$.

Remark 6.1.2. *Note that β needs to be within $(0, 1)$ for δ_f to be positive.*

Proof. The detailed proof is in Appendix B. □

From Lemma 6.1.1, we get the following distortion-diversity tradeoff achieved by the MR-based source-channel scheme,

$$
\left.
\begin{aligned}
0 &< d_p < 2\beta \,, \\
0 &< d_f - d_p < 2(1 - \beta) \,, \\
\delta_p &= (2 - d_p) + \lceil -d_p/\beta \rceil (1 - \beta) \,, \\
\delta_f &= \min\left\{\delta_p, 2(1 - \beta) - (d_f - d_p)\right\} \,,
\end{aligned}
\right\} \quad 0 < \beta < 1 \,. \qquad (6.7)
$$

The distortion-diversity tradeoff is a 4 dimensional region. For the sake of illustration, we take a cut along the direction of $d_p = d$, $d_f = \alpha d$. In figure 6-1, we illustrate the distortion-diversity tradeoff among (δ_p, δ_f, d) for the case of $\alpha = 1.5$.
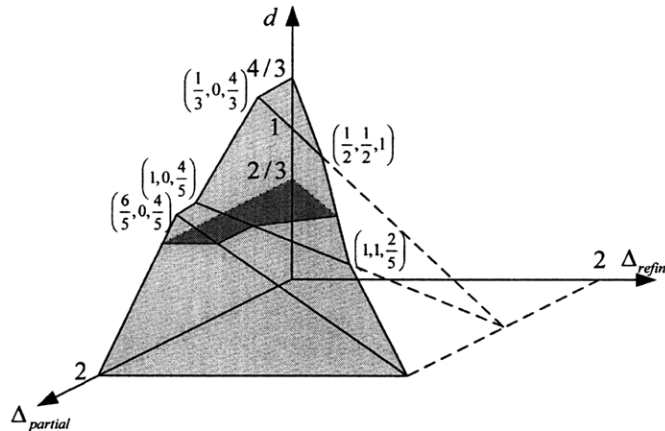
Figure 6-1: Double-Level MR-Based Source-Channel Scheme, Distortion-Diversity Tradeoff, $\alpha = 1.5$

6.2 Symmetric MD-Based Source-Channel Scheme

The symmetric MD-based source-channel scheme is composed of the simplified symmetric El-Gamal-Cover source code (reviewed in Theorem 4.3.1 and the remark after that), the V-BLAST channel code structure and a joint source-channel decoder [25]. Since we are dealing with parallel fading channel, the V-BLAST channel code structure degenerates to separate channel code for each subchannel. Suppose that the two distortion exponents requested by the two classes of receivers are d_p and d_f ($d_p < d_f$), respectively. For each SNR, the source-channel scheme achieves distortion levels of $D_p = \mathsf{SNR}^{-d_p}$ and $D_f = \mathsf{SNR}^{-d_f}$. The symmetric MD-based source-channel scheme proceeds as follows,

1. The simplified symmetric El-Gamal-Cover source code converts the length-N source sequence $\mathbf{s}$ to two length-NR_s bit sequences (description i_1 and i_2). According to the remark following Theorem 4.3.1, to achieve distortion levels of

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D_p and D_f, the parameters of the simplified symmetric El-Gamal-Cover source code need to be configured as follows,

$$\sigma^2 = \frac{D_p}{1 - D_p},$$

$$\rho = -1 + 2 \times \frac{D_f(1 - D_p)}{D_p(1 - D_f)}.$$

Therefore, the source code rate needs to satisfy the following condition,

$$R_s > \frac{1}{2} \log \frac{(1 - D_f)^2}{4D_f(D_p - D_f)(1 - D_p)}. \tag{6.8}$$

Moreover, the mutual information between $\mathbf{s}_1$ and $\mathbf{s}_2$ is

$$I(\mathbf{s}_1; \mathbf{s}_2) = \log \frac{(1 - D_f)^2}{4(1 - D_p)(D_p - D_f)}. \tag{6.9}$$

2. The sender encodes and modulates i_1, i_2 separately with power SNR into length-N channel input sequences $\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2$, which are transmitted over the parallel fading channel, respectively;

3. At the receiver, the following joint source-channel decoder is used,

- By comparing the channel output sequences $\underline{\mathbf{y}}_1$ and $\underline{\mathbf{y}}_2$ with the channel codewords in their own channel codebook according to the joint typicality criterion, the receiver forms two lists, $\mathcal{L}_1$ and $\mathcal{L}_2$, of candidate source descriptions. To differentiate them from the true source descriptions i_1 and i_2, we use $\hat{i}_1$ and $\hat{i}_2$ to denote the constituents of lists $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. The size of each list is approximately $e^{N(R_s - R_{c,i})}$, where $R_{c,i}$ is defined as follows (assuming Gaussian input distribution with covariance matrix of SNR for the channel code of each subchannel),

$$R_{c,i}(\mathbf{h}_i) = I(\mathbf{x}_i; \mathbf{y}_i \,|\, \mathbf{h}_i) = \log(1 + \mathsf{SNR}|\mathbf{h}_i|^2), \quad i = 1, 2; \tag{6.10}$$

- For each $(\hat{i}_1, \hat{i}_2) \in \mathcal{L}_1 \times \mathcal{L}_2$, the receiver compares the $\hat{i}_1$-th codeword

from side codebook 1, $(\underline{s}_1(\hat{i}_1)$, with the $\hat{i}_2$ codeword from side codebook 2, $\underline{s}_2(\hat{i}_2))$, according to the joint typicality criterion with regard to the joint distribution of (s_1, s_2), which is specified in the remark following Theorem 4.3.1. If only one pair satisfies the joint typicality criterion, then the receiver outputs the reconstruction sequence $g_c(\underline{s}_1(\hat{i}_1), \underline{s}_2(\hat{i}_2))$. Otherwise, the receiver declares a "full outage" and proceeds to the next step;

- If $|\mathcal{L}_1| = 1$, then the receiver outputs the reconstruction sequence $g_1(\underline{s}_1(\hat{i}_1))$. If $|\mathcal{L}_2| = 1$, then the receiver outputs the reconstruction sequence $g_2(\underline{s}_2(\hat{i}_2))$. Otherwise, the receiver declares a "partial outage".

Since the receiver declares "full outage" when the size of the product list $\mathcal{L}_1 \times \mathcal{L}_2$ is too large, we can express express the full outage event as follows,

$$
\begin{aligned}
\mathcal{O}_f(d_p, d_f) &= \left\{ \sum_{i=1,2} (R_s - R_{c,i}(\mathbf{h}_i))^+ > I(s_1; s_2) \right\} \\
&= \left\{ \sum_{i=1,2} (R_s - \log(1 + \mathsf{SNR}|\mathbf{h}_i|^2))^+ > I(s_1; s_2) \right\} \\
&\doteq \left\{ \sum_{i=1,2} (R_s - (1 - \alpha_i)^+ \log \mathsf{SNR})^+ > I(s_1; s_2) \right\} \\
&\doteq \left\{ \sum_{i=1,2} \left(\lim_{\mathsf{SNR} \to \infty} \frac{R_s}{\log \mathsf{SNR}} - (1 - \alpha_i)^+ \right)^+ > \lim_{\mathsf{SNR} \to \infty} \frac{I(s_1; s_2)}{\log \mathsf{SNR}} \right\} \\
&\doteq \left\{ \sum_{i=1,2} \left(\frac{d_p + d_f}{2} - (1 - \alpha_i)^+ \right)^+ > d_p \right\},
\end{aligned}
$$

where in the last step we have applied the following asymptotic properties of R_s and $I(s_1; s_2)$ $(d_p < d_f)$,

$$
\lim_{\mathsf{SNR} \to \infty} \frac{R_s}{\log \mathsf{SNR}} \geq \lim_{\mathsf{SNR} \to \infty} \frac{\frac{1}{2} \log \frac{(1 - D_f)^2}{4D_f(D_p - D_f)(1 - D_p)}}{\log \mathsf{SNR}} = \frac{d_p + d_f}{2}, \tag{6.11}
$$

$$
\lim_{\mathsf{SNR} \to \infty} \frac{I(s_1; s_2)}{\log \mathsf{SNR}} = \lim_{\mathsf{SNR} \to \infty} \frac{\log \frac{(1 - D_f)^2}{4(1 - D_p)(D_p - D_f)}}{\log \mathsf{SNR}} = d_p, \tag{6.12}
$$

which follows from (8.18) and (8.19) respectively. Therefore, the full outage probability is expressed as follows,

$$\mathcal{P}_f^{out}(d_p, d_f) = \mathcal{P}\left[\mathcal{O}_f(d_p, d_f)\right] \doteq \mathsf{SNR}^{-\delta_f(d_p, d_f)},$$

where the full diversity order is

$$\delta_f(d_p, d_f) = \min_{\alpha_i \geq 0, \sum_{i=1,2}\left(\frac{d_p+d_f}{2}-(1-\alpha_i)^+\right)^+ > d_p} \sum_{i=1,2} \alpha_i.$$

Note that the receiver declares a "partial outage" when not only the size of the product list $\mathcal{L}_1 \times \mathcal{L}_2$ is too large but the size of each individual list $\mathcal{L}_1$ and $\mathcal{L}_2$ is also too large. We can then express the partial outage event as follows,

$$
\begin{aligned}
\mathcal{O}_p(D_p, D_f) &= \mathcal{O}_f(D_p, D_f) \cap \{R_{c,1}(\mathbf{h}_1) < R_s\} \cap \{R_{c,2}(\mathbf{h}_2) < R_s\} \\
&= \mathcal{O}_f(D_p, D_f) \cap \left\{\log(1 + \mathsf{SNR}|\mathbf{h}_1|^2) < R_s\right\} \cap \left\{\log(1 + \mathsf{SNR}|\mathbf{h}_2|^2) < R_s\right\} \\
&= \mathcal{O}_f(D_p, D_f) \cap \cap_{i=1,2} \left\{1 + \mathsf{SNR}|\mathbf{h}_i|^2 < R_s\right\} \\
&\doteq \mathcal{O}_f(D_p, D_f) \cap \cap_{i=1,2} \left\{(1 - \alpha_i)^+ \log \mathsf{SNR} < R_s\right\} \\
&\doteq \mathcal{O}_f(D_p, D_f) \cap \cap_{i=1,2} \left\{(1 - \alpha_i)^+ < \lim_{\mathsf{SNR} \to \infty} \frac{R_s}{\log \mathsf{SNR}}\right\} \\
&= \mathcal{O}_f(D_p, D_f) \cap \cap_{i=1,2} \left\{(1 - \alpha_i)^+ < \frac{d_p + d_f}{2}\right\},
\end{aligned}
\tag{6.13}
$$

where the last step follows from (6.11). Therefore, the partial outage probability is expressed as follows,

$$\mathcal{P}_p^{out}(d_p, d_f) = \mathcal{P}\left[\mathcal{O}_p(d_p, d_f)\right] \doteq \mathsf{SNR}^{-\delta_p(d_p, d_f)},$$

where the partial diversity order is

$$\delta_p(d_p, d_f) = \min_{\alpha_i \geq 0, (1-\alpha_1)^+ < \frac{d_p+d_f}{2}, (1-\alpha_2)^+ < \frac{d_p+d_f}{2}, \sum_{i=1,2}\left(\frac{d_p+d_f}{2}-(1-\alpha_i)^+\right)^+ > d_p} \sum_{i=1,2} \alpha_i,$$

Lemma 6.2.1. *The symmetric MD-based source-channel scheme achieves two levels*

of distortion exponents d_p, d_f with the following diversity orders,

$$\delta_p = 2 - d_f, \tag{6.14}$$

$$\delta_f = \min\left\{2 - d_f,\ 1 - \frac{d_f - d_p}{2}\right\}, \tag{6.15}$$

where $0 < d_p < d_f < 2$.

Proof. The detailed proof is in Appendix C. □

From Lemma 6.2.1, we get the following distortion-diversity tradeoff achieved by the symmetric MD-based source-channel scheme,

$$\left\{\begin{array}{c} 0 < d_p < 2\beta, \\ 0 < d_f - d_p < 2(1 - \beta), \\ \delta_p = 2 - d_f, \\ \delta_f = \min\left\{2 - d_f,\ 1 - \frac{d_f - d_p}{2}\right\}, \end{array} \quad 0 < \beta < 1, \right\}. \tag{6.16}$$

The distortion-diversity tradeoff is a 4 dimensional region. For the sake of illustration, we take a cut along the direction of $d_p = d$, $d_f = \alpha d$. In figure 6-2, we illustrate the distortion-diversity tradeoff among (δ_p, δ_f, d) for the case of $\alpha = 1.5$.
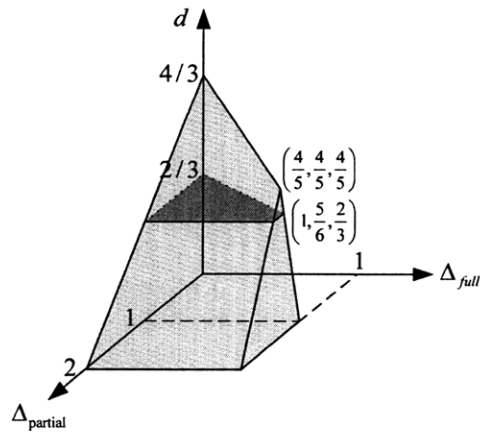
Figure 6-2: Double-Level Symmetric MD-Based Source-Channel Scheme, Distortion-Diversity Tradeoff, $\alpha = 1.5$

6.3 Performance Comparison

In Figure 6-1 and Figure 6-2, we have illustrated the distortion-diversity regions achieved by the double-level MR-based source-channel scheme and the double-level MD-based source-channel scheme along the direction of $d_p = d$, $d_f = 1.5d$. For the sake of illustration, we further cut these 3 dimensional regions along the direction of $d = 2/3$, and overlap the cuts in Figure 6-3 for comparison. Note that there
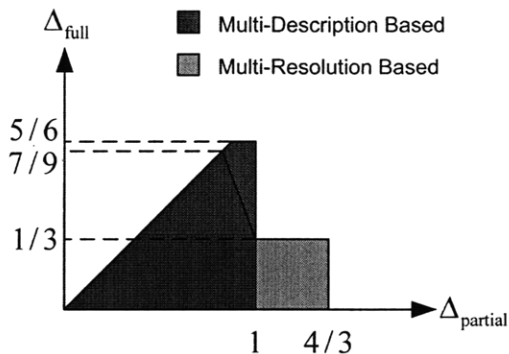
Figure 6-3: Double-Level Source-Channel Schemes, Distortion-Diversity Tradeoff Comparison, $d_p = 2/3$, $d_f = 1$

is no universal winner in the performance comparison between the two double-level source-channel schemes. From Figure 6-3, we observe that the MD-based scheme outperforms the MR-based scheme when achieving diversity orders that are relatively close to each other. However, the MR-based scheme outperforms the MD-based scheme when achieving diversity orders that are relatively disparate from each other. The performance comparison in our proposed distortion-diversity framework reveals several observations and intuitions that were not available in previous approaches [25]. Moreover, the superior performance of MD-based scheme demonstrates that specifically designed channel codes, such as the superposition code, may limit our choice of source coding techniques and thus do not necessarily lead to optimal end-to-end performance.

The significantly different performance regions in Figures 6-1 and 6-2 prompts us to wonder if a unifying scheme exists that encompass both the MR-based and the MD-based schemes. In the following section, we would show that this is indeed the case

by proposing a triple-level source-channel scheme that unifies the two double-level schemes from the perspective of distortion-diversity tradeoff.

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Chapter 7

Triple-Level Source-Channel Scheme

In this section, we propose a three-level source-channel scheme, which is composed of the symmetric MD with common refinement source code (reviewed in Section 4.4), a mixture of V-Blast channel code with superposition channel code, and a joint source-channel decoder. Suppose that three requested distortion exponents are d_p, d_f and d_r, resp ($d_p \leq d_f \leq d_r$). For each SNR, the source-channel scheme achieves distortion levels of $D_p = \mathsf{SNR}^{-d_p}$, D_f (the setting of D_f will be explained later) and $D_r = \mathsf{SNR}^{-d_r}$. The symmetric MD with common refinement source-channel scheme proceeds as follows,

1. The symmetric El-Gamal-Cover source code converts the length-N source sequence $\mathbf{s}$ to three bit sequences: two base descriptions $i_{b,1}$ and $i_{b,2}$, each of length $NR_{s,b}$ and a refinement description i_r of length NR_r. According to the remark following Theorem 4.4.1, the parameters of the symmetric El-Gamal-Cover source code need to be configured as follows to achieve distortion levels

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of D_p, D_f and D_r,

$$\sigma_b^2 = \frac{D_p - D_r}{(1 - D_p)(1 - D_r)},$$

$$\sigma_r^2 = \frac{D_r}{1 - D_r},$$

$$\rho = -1 + 2 \times \frac{(D_f - D_r)(1 - D_p)}{(D_p - D_r)(1 - D_f)}.$$

The source code rates need to satisfy the following condition,

$$R_{s,b} > \frac{1}{2} \log \frac{(1 - D_f)^2}{4 D_f (D_p - D_f)(1 - D_p)},$$

$$R_{s,r} > \log \frac{D_f}{D_r}.$$

Moreover, the mutual information between $s_{b,1}$ and $s_{b,2}$ is

$$I(s_{b,1}; s_{b,2}) = \log \frac{(1 - D_f)^2}{4(D_p - D_f)(1 - D_p)}.$$

2. The sender encodes and modulates $i_{b,1}, i_{b,2}$ separately with power $\mathsf{SNR} - \mathsf{SNR}^{1-\beta} \doteq \mathsf{SNR}$ into length-N base channel codewords $\underline{x}_{b,1}, \underline{x}_{b,2}$. The sender encodes and modulates i_r with power $\mathsf{SNR}^{1-\beta}$ into refinement channel codewords $\underline{x}_{r,1}, \underline{x}_{r,2}$. The sender then sends $\underline{x}_{b,1} + \underline{x}_{r,1}$ and $\underline{x}_{b,2} + \underline{x}_{r,2}$ over the parallel fading channel;

3. At the receiver, the following joint source-channel decoder is used,

   - By comparing the channel output sequence with the corresponding channel codewords (treating the refinement channel codewords as noise) according to the joint typicality criterion, the receiver forms two lists, $\mathcal{L}_1$ and $\mathcal{L}_2$, of candidates $\hat{i}_1$ and $\hat{i}_2$, respectively. The size of each list is approximately $e^{N(R_s - R_{c,b,i})}$, where $R_{c,b,i}$ is defined as follows,

$$R_{c,b,i}(\mathbf{h}_i) = I(\mathbf{x}_{b,i}; \mathbf{y}_i \mid \mathbf{h}_i) = \log \frac{1 + \mathsf{SNR}|\mathbf{h}_i|^2}{1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2}, \quad i = 1, 2;$$

   - For each $(\hat{i}_{b,1}, \hat{i}_{b,2}) \in \mathcal{L}_1 \times \mathcal{L}_2$, the receiver compare $(\underline{s}_{b,1}(\hat{i}_{b,1})$ with $\underline{s}_{b,2}(\hat{i}_{b,2}))$

according to the joint typicality criterion with regard to the joint distribution of $(\mathbf{s}_{b,1}, \mathbf{s}_{b,2})$, which is specified in the remark following Theorem 4.4.1. If more than one pair satisfies the joint typicality criterion, proceed to the next step. Otherwise, if only one pair satisfies the joint typicality criterion,

– The receiver encodes and modulates $\hat{i}_{b,1}, \hat{i}_{b,2}$ and subtract the base channel codewords from the channel output sequences. The receiver then decodes for $\hat{i}_r$ using the joint typicality criterion. Assuming a Gaussian input distribution with covariance matrix of SNR I, the refinement mutual information is

$$R_{c,r} = I((\mathbf{x}_{r,1}, \mathbf{x}_{r,2}); (\mathbf{y}_1, \mathbf{y}_2) \mid \mathbf{x}_{b,1}, \mathbf{x}_{b,2}, \mathbf{h}_1, \mathbf{h}_2) = \sum_{i=1,2} \log(1 + \mathsf{SNR}^{1-\beta} |\mathbf{h}_i|^2) \, ;$$

– If the receiver fails to decode the refinement description $\hat{i}_r$, he declares "refinement outage" and outputs $g_c(\mathbf{s}_{b,1}(\hat{i}_{b,1}), \mathbf{s}_{b,2}(\hat{i}_{b,2}))$ as the source reconstruction sequence;

• If $|\mathcal{L}_1| = 1$, then the receiver outputs the reconstruction sequence $g_1(\mathbf{s}_1(\hat{i}_1))$. If $|\mathcal{L}_2| = 1$, then the receiver outputs the reconstruction sequence $g_2(\mathbf{s}_2(\hat{i}_2))$. Otherwise, the receiver declares a "partial outage".

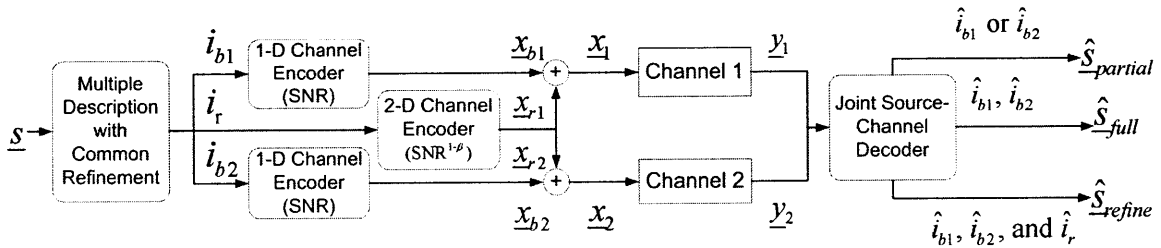The structure of the above triple-level source-channel scheme is illustrated in Figure 7-1

Figure 7-1: Triple-Level Source-Channel Scheme

The full outage event is characterized as follows,

$$
\mathcal{O}_f(d_p, d_f, d_r, \beta)
$$

$$
= \left\{ \sum_{i=1,2} \left(R_{s,b} - R_{c,b,i} \right)^+ > I(\mathbf{s}_{b,1}; \mathbf{s}_{b,2}) \right\}
$$

$$
= \left\{ \sum_{i=1,2} \left(R_{s,b} - \log \frac{1 + \mathsf{SNR}|\mathbf{h}_i|^2}{1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2} \right)^+ > I(\mathbf{s}_{b,1}; \mathbf{s}_{b,2}) \right\}
$$

$$
\doteq \left\{ \sum_{i=1,2} \left(R_{s,b} - \left((1-\alpha_i)^+ - (1-\beta-\alpha_i)^+ \right) \log \mathsf{SNR} \right)^+ > I(\mathbf{s}_{b,1}; \mathbf{s}_{b,2}) \right\}
$$

$$
\doteq \left\{ \sum_{i=1,2} \left(\lim_{\mathsf{SNR} \to \infty} \frac{R_{s,b}}{\log \mathsf{SNR}} - \left((1-\alpha_i)^+ - (1-\beta-\alpha_i)^+ \right) \right)^+ > \lim_{\mathsf{SNR} \to \infty} \frac{I(\mathbf{s}_{b,1}; \mathbf{s}_{b,2})}{\log \mathsf{SNR}} \right\} .
$$

The partial outage event is characterized as follows,

$$
\mathcal{O}_p(d_p, d_f, d_r, \beta)
$$

$$
= \mathcal{O}_f(d_p, d_f, d_r, \beta) \cap \{ R_{c,b,1} < R_{s,b} \} \cap \{ R_{c,b,2} < R_{s,b} \}
$$

$$
= \mathcal{O}_f(d_p, d_f, d_r, \beta) \cap \left\{ \log \frac{1 + \mathsf{SNR}|\mathbf{h}_1|^2}{1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_1|^2} < R_{s,b} \right\} \cap \left\{ \log \frac{1 + \mathsf{SNR}|\mathbf{h}_2|^2}{1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_2|^2} < R_{s,b} \right\}
$$

$$
\doteq \mathcal{O}_f(d_p, d_f, d_r, \beta) \cap \left(\cap_{i=1,2} \left\{ \left((1-\alpha_i)^+ - (1-\beta-\alpha_i)^+ \right) \log \mathsf{SNR} < R_{s,b} \right\} \right)
$$

$$
\doteq \mathcal{O}_f(d_p, d_f, d_r, \beta) \cap \left(\cap_{i=1,2} \left\{ (1-\alpha_i)^+ - (1-\beta-\alpha_i)^+ < \lim_{\mathsf{SNR} \to \infty} \frac{R_{s,b}}{\log \mathsf{SNR}} \right\} \right) .
$$

The refinement outage event is characterized as follows,

$$
\mathcal{O}_r(d_p, d_f, d_r, \beta) = \mathcal{O}_f(d_p, d_f, d_r, \beta) \cup \{ R_{c,r} < R_{s,r} \}
$$

$$
= \mathcal{O}_f(d_p, d_f, d_r, \beta) \cup \left\{ \sum_{i=1,2} \log(1 + \mathsf{SNR}^{1-\beta}|\mathbf{h}_i|^2) < \log \frac{D_f}{D_r} \right\}
$$

$$
\doteq \mathcal{O}_f(d_p, d_f, d_r, \beta) \cup \left\{ \prod_{i=1,2} \mathsf{SNR}^{(1-\beta-\alpha_i)^+} < \frac{D_f}{D_r} \right\}
$$

$$
\doteq \mathcal{O}_f(d_p, d_f, d_r, \beta) \cup \left\{ \sum_{i=1,2} (1-\beta-\alpha_i)^+ < \lim_{\mathsf{SNR} \to \infty} \frac{\log \frac{D_f}{D_r}}{\log \mathsf{SNR}} \right\} .
$$

Lemma 7.0.1. *The triple-level source-channel scheme achieves three levels of distortion exponents* d_p, d_f *and* d_r *($d_p \leq d_f \leq d_r$) with the following diversity orders,*

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- *Partial diversity order is*

$$
\delta_p = \begin{cases} 0, & \text{if } \beta < \frac{d_f}{2}, \\ 1 + \beta - d_f, & \text{if } \frac{d_f}{2} < \beta < \frac{d_p + d_f}{2}, \\ 2 - d_f, & \text{if } \beta > \frac{d_p + d_f}{2}; \end{cases}
$$

- *Full diversity order is*

  - *If $d_p < d_f \le d_r$, then*

$$
\delta_f = \begin{cases} 0, & \text{if } \beta < \frac{d_f}{2}, \\ \min\left\{1 - \frac{d_f - d_p}{2}, 1 + \beta - d_f\right\}, & \text{if } \frac{d_f}{2} < \beta < d_f, \\ \min\left\{1 - \frac{d_f - d_p}{2}, 2 - d_f\right\}, & \text{if } \beta > d_f; \end{cases}
$$

  - *If $d_p = d_f \le d_r$, then*

$$
\delta_f = \begin{cases} 0, & \text{if } \beta < \frac{d_f}{2}, \\ 1 + \beta - d_f, & \text{if } \frac{d_f}{2} < \beta < d_f, \\ 2 - d_f, & \text{if } \beta > d_f; \end{cases}
$$

- *Refinement diversity order is*

$$
\delta_r = \min\left\{\delta_f, 2(1 - \beta) - (d_r - d_f)\right\}^+.
$$

Proof. The detailed proof can be found in Appendix D. □

Now, we are ready to compare our triple-level source-channel scheme with the two double-level schemes. For fair comparison, we let $d_p = 2/3$, $d_r = 1$ and allow d_f to vary between 2/3 and 1. Under this setting, the distortion-diversity region achieved by the triple-level source-channel scheme is characterized as a 3 dimensional region of $(\delta_p, \delta_f, \delta_r)$. We illustrate the two extreme cases of $d_f = 2/3$ and $d_f = 1$ in the following.

- If we set $d_f = 2/3$, the achievable region of $(\delta_p, \delta_f, \delta_r)$ is shown in Figure 7-2. In this case, d_p and d_f are identical, so the partial reconstruction and the full
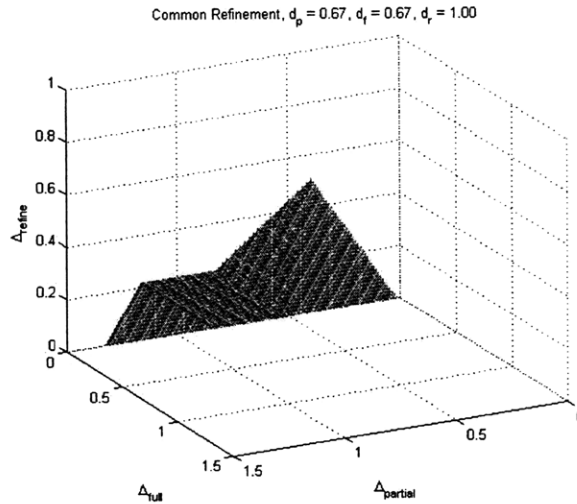
Figure 7-2: Triple-Level Source-Channel Scheme, Distortion-Diversity Tradeoff, $d_p = 2/3$, $d_f = 2/3$, $d_r = 1$

reconstruction are considered as one level of reconstruction. With the refinement reconstruction as the second level, we have specialized our triple-level scheme to a double-level scheme. Accordingly, we project the 3-dimensional region in Figure 7-2 onto the (δ_p, δ_r) plane, which is illustrated in Figure 7-3. Note that the 2-dimensional region in Figure 7-3 is the same as the red region in Figure 6-3, which is achieved by the MR-based double-level scheme. Therefore, our triple-level scheme includes the MR-based scheme as a special case;

- If we set $d_f = 1$, the achievable region of $(\delta_p, \delta_f, \delta_r)$ is shown in Figure 7-4. In this case, d_f and d_r are identical, so the full reconstruction and the refinement reconstruction are considered as one level of reconstruction. With the partial reconstruction as the other level, we have again specialized our triple-level scheme to a double-level scheme. Accordingly, we project the 3-dimensional region in Figure 7-4 onto the (δ_p, δ_f) plane, which is illustrated in Figure 7-5. Note that the 2-dimensional region in Figure 7-5 is the same as the blue region in Figure 6-3, which is achieved by the MD-based double-level scheme. Therefore, our
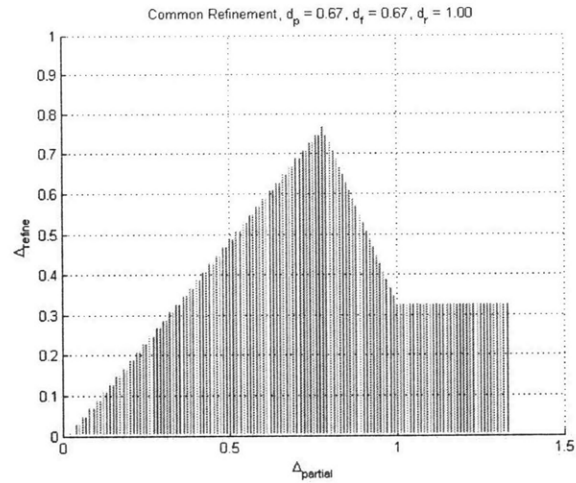
Figure 7-3: Triple-Level Source-Channel Scheme, Distortion-Diversity Tradeoff Projection on (δ_p, δ_r) plane, $d_p = 2/3$, $d_f = 2/3$, $d_r = 1$

triple-level scheme also includes the MD-based scheme as a special case.
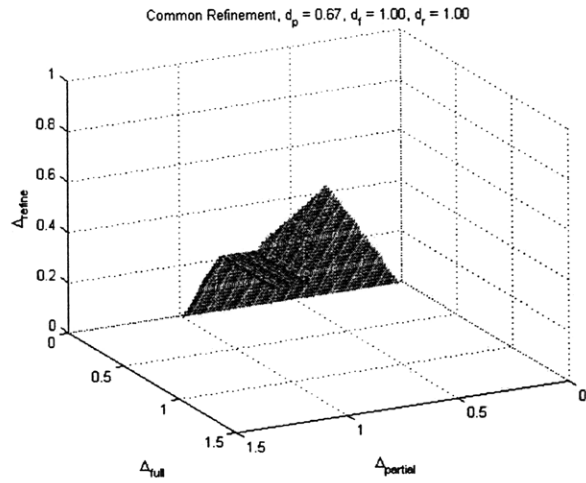
Figure 7-4: Triple-Level Source-Channel Scheme Distortion-Diversity Tradeoff, $d_p = 2/3$, $d_f = 1$, $d_f = 1$
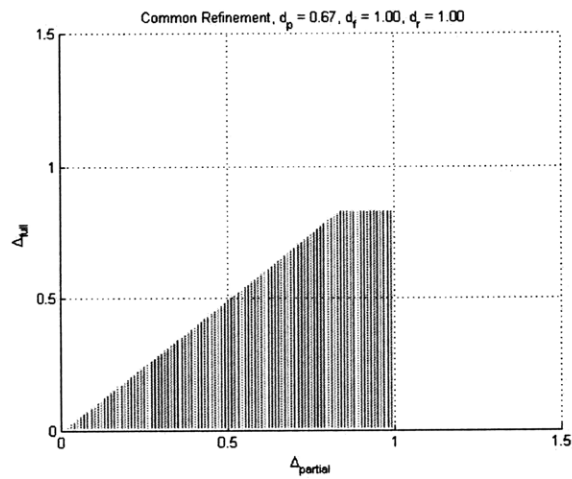
Figure 7-5: Triple-Level Source-Channel Scheme, Distortion-Diversity Tradeoff Projection on (δ_p, δ_f) plane, $d_p = 2/3$, $d_f = 1$, $d_r = 1$

Chapter 8

Low SNR Scenario

In previous sections, we focused on the high SNR scenario, which is a common set-
ting where source codes and MIMO channel model are studied. We propose to use
the distortion-diversity tradeoff as our new performance metric to characterize and
compare source-channel schemes. We also demonstrate that certain good channel
codes, such as superposition code, may limit the choice of source codes and do not
necessarily provide the optimal end-to-end performance. In particular, we show that
a source-channel scheme based on MD provided better performance in certain oper-
ating regions. However, the MD-based scheme uses a joint source-channel decoder
which breaks the interface between source code and channel code. In this section, we
shall extend the distortion-diversity tradeoff to the low SNR regime and characterize
the performance of several source-channel schemes. In particular, we demonstrate
that the MD-based scheme, even without using a joint source-channel decoder, could
still outperform the MR-based scheme. This finding confirms the distortion-diversity
tradeoff as a more appropriate source-channel interface than the traditional bit rate
interface.

8.1 MR-Based Source-Channel Scheme

The MR-based source-channel scheme is composed of a double-description MR source
code (reviewed in Section 4.2) and a superposition channel code. Suppose that the

distortion levels requested by the two classes of users are D_p an D_f, respectively, where $0 < D_f < D_p < 1$. The double-level MR-based source-channel scheme proceeds as follows,

1. The double-description MR source encoder converts the length-N source sequence $\underline{s}$ into a length-$NR_{s,b}$ bit sequence (base description i_b) and a length-$NR_{s,r}$ bit sequence (refinement description i_r). According to Theorem 4.2.1, the source code rates $(R_{s,b}, R_{s,r})$ required to achieve the distortion levels of D_p, D_f are

$$R_{s,b} = -\ln D_p, \tag{8.1}$$

$$R_{s,r} = -\ln D_f + \ln D_p; \tag{8.2}$$

2. Superposition channel encoder with power split of $(1 - \gamma)\mathsf{SNR}$ and $\gamma\mathsf{SNR}$ ($0 < \gamma < 1$): the base description i_b is encoded and modulated with power $(1-\gamma)\mathsf{SNR}$ into $(\underline{x}_{b,1}, \underline{x}_{b,2})$; the refinement description i_r is encoded and modulated with power $\gamma\mathsf{SNR}$ into $(\underline{x}_{r,1}, \underline{x}_{r,2})$; the channel input sequences $(\underline{x}_1, \underline{x}_2)$ are the sum of $(\underline{x}_{b,1}, \underline{x}_{b,2})$ and $(\underline{x}_{r,1}, \underline{x}_{r,2})$, respectively;

3. Successive interference cancelation decoder: from the received sequence $\underline{y}$, the decoder first decodes for i_b treating $(\underline{x}_{r,1}, \underline{x}_{r,2})$ as noise; if i_b is decoded successfully, the decoder re-encodes and re-modulates the decoded i_b into $(\underline{x}_{b,1}, \underline{x}_{b,2})$ and subtracts their effect from the channel output sequences $\underline{y}$; the decoder then decodes for i_r;

4. Assuming a Gaussian input distribution of covariance matrix $(1 - \gamma)\mathsf{SNR} \cdot I$ for the base description channel code and a Gaussian input distribution of covari-

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ance matrix $\gamma\mathsf{SNR} \cdot I$ for the refinement description channel code, we obtain

$$
\begin{aligned}
R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \gamma) &= I((\mathbf{x}_{b,1}, \mathbf{x}_{b,2}); \mathbf{y} \mid \mathbf{h}_1, \mathbf{h}_2) \\
&= \ln\left(\frac{1 + \mathsf{SNR}\,(|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2)}{1 + \gamma\mathsf{SNR}\,(|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2)}\right), \qquad (8.3) \\
R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \gamma) &= I((\mathbf{x}_{r,1}, \mathbf{x}_{r,2}); \mathbf{y} \mid (\mathbf{x}_{b,1}, \mathbf{x}_{b,2}), \mathbf{h}_1, \mathbf{h}_2) \\
&= \ln\left(1 + \gamma\mathsf{SNR}\,(|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2)\right); \qquad (8.4)
\end{aligned}
$$

5. The receiver reconstructs the source sequence as follows,

- If $R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \gamma) > R_{s,b}$ and $R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \gamma) > R_{s,r}$, the receiver outputs the reconstruction sequence corresponding to (i_b, i_r);

- If $R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \gamma) > R_{s,b}$ and $R_{c,r}(\mathbf{h}_1, \mathbf{h}_2, \gamma) < R_{s,r}$, the receiver declares *full outage* and outputs the reconstruction sequence corresponding to i_b;

- If $R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \gamma) < R_{s,b}$, the receiver declares *partial outage* and outputs the length-N all zero sequence as the source reconstruction sequence.

The partial outage probability is

$$
\begin{aligned}
\mathcal{P}\left[\mathcal{O}_p\right] &= \mathcal{P}\left[R_{c,b}(\mathbf{h}_1, \mathbf{h}_2, \gamma) < R_{s,b}\right] \\
&= \mathcal{P}\left[\ln\left(\frac{1 + \mathsf{SNR}(|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2)}{1 + \gamma\mathsf{SNR}(|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2)}\right) < R_{s,b}\right] \\
&= \mathcal{P}\left[|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2 < \frac{e^{R_{s,b}} - 1}{\mathsf{SNR} - \gamma e^{R_{s,b}}\mathsf{SNR}}\right] \\
&= F\left(\frac{4\left(e^{R_{s,b}} - 1\right)}{(1 - \gamma e^{R_{s,b}})\,\mathsf{SNR}}; 4\right), \qquad (8.5)
\end{aligned}
$$

where $\gamma e^{R_{s,b}} < 1$ and $F(\cdot\,;n)$ is the cumulative probability function of χ_n^2 random variables. The full outage probability is

$$
\begin{aligned}
\mathcal{P}[\mathcal{O}_f] &= \mathcal{P}\left[\left\{\ln\left(1 + \gamma\mathsf{SNR}(|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2)\right) < R_{s,r} - R_{s,b}\right\} \cup \left\{|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2 < \frac{e^{R_{s,b}} - 1}{(1 - \gamma e^{R_{s,b}})\,\mathsf{SNR}}\right\}\right] \\
&= \mathcal{P}\left[|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2 < \max\left\{\frac{e^{R_{s,b}} - 1}{(1 - \gamma e^{R_{s,b}})\,\mathsf{SNR}}, \frac{e^{R_{s,r} - R_{s,b}} - 1}{\gamma\mathsf{SNR}}\right\}\right] \\
&= F\left(\max\left\{\frac{e^{R_{s,b}} - 1}{(1 - \gamma e^{R_{s,b}})\,\mathsf{SNR}}, \frac{e^{R_{s,r} - R_{s,b}} - 1}{\gamma\mathsf{SNR}}\right\};4\right).
\end{aligned}
\tag{8.6}
$$

In the following, we would first digress a bit connect a widely adopted channel code, Alamouti code, to the MR-based scheme. After that, we would analyze the distortion-diversity tradeoff achieved by the MR-based scheme.

8.1.1 Alamouti Scheme

For the 2×1 MIMO channel model, a popular channel code widely adopted in practice is the Alamouti scheme [26]. Denote the two bit streams from source encoder as i_b and i_r, respectively. The Alamouti scheme proceeds by encoding the two bit streams separately into the base layer channel codeword $(\tilde{\mathbf{x}}_b[1], \cdots, \tilde{\mathbf{x}}_b[N])$ and the refinement layer channel codeword $(\tilde{\mathbf{x}}_r[1], \cdots, \tilde{\mathbf{x}}_r[N])$, where $\tilde{\mathbf{x}}_b[n]$ and $\tilde{\mathbf{x}}_r[n]$ are i.i.d. $\mathcal{CN}(0,1)$. The base layer and refinement layer channel codewords are modulated and combined as follows,

$$
\tilde{\mathbf{x}}[n] = \sqrt{(1 - \gamma)\mathsf{SNR}} \cdot \tilde{\mathbf{x}}_b[n] + \sqrt{\gamma\mathsf{SNR}} \cdot \tilde{\mathbf{x}}_r[n]
$$

where $(1 - \gamma)\mathsf{SNR}$ and $\gamma\mathsf{SNR}$ $(0 < \gamma < 1)$ are the power allocated to the base layer and the refinement layer, respectively. The Alamouti scheme arranges the channel inputs as follows,

$$
\begin{pmatrix} \mathbf{x}_1[2n] \\ \mathbf{x}_2[2n] \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{x}}[2n] \\ \tilde{\mathbf{x}}[2n+1] \end{pmatrix}, \quad \begin{pmatrix} \mathbf{x}_1[2n+1] \\ \mathbf{x}_2[2n+1] \end{pmatrix} = \begin{pmatrix} -\tilde{\mathbf{x}}^*[2n+1] \\ \tilde{\mathbf{x}}^*[2n] \end{pmatrix}, \quad n = 0, \cdots, N/2.
$$

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The receiver first re-organizes the channel outputs as follows,

$$\tilde{\mathbf{y}}[2n] = \frac{1}{\sqrt{|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2}} (\mathbf{h}_1 \mathbf{y}[2n] + \mathbf{h}_2 \mathbf{y}^\star[2n+1]) ,$$

$$\tilde{\mathbf{y}}[2n+1] = \frac{1}{\sqrt{|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2}} (\mathbf{h}_2 \mathbf{y}[2n] - \mathbf{h}_1 \mathbf{y}^\star[2n+1]) ,$$

to obtain the following effective channel,

$$\tilde{\mathbf{y}}[n] = \sqrt{|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2} \tilde{\mathbf{x}}[n] + \tilde{\mathbf{w}}[n]$$

$$= \sqrt{|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2} \left(\sqrt{(1-\gamma)\mathsf{SNR}} \cdot \tilde{\mathbf{x}}_b[n] + \sqrt{\gamma\mathsf{SNR}} \cdot \tilde{\mathbf{x}}_r[n] \right) + \tilde{\mathbf{w}}[n] , \quad (8.7)$$

where $n = 1, \cdots, N$ and $\tilde{\mathbf{w}}[n] \sim \mathcal{CN}(0,1)$. The receiver then proceeds with a successive interference cancelation decoder to decode for i_b and i_r. The supportable rates for i_b and i_r are

$$R_{c,b} = I\left(\tilde{\mathbf{x}}_b[n]; \tilde{\mathbf{y}}[n] \mid \mathbf{h}_1, \mathbf{h}_2\right) = \ln\left(\frac{1 + \mathsf{SNR}\left(|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2\right)}{1 + \gamma\mathsf{SNR}\left(|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2\right)}\right) ,$$

$$R_{c,r} = I\left(\tilde{\mathbf{x}}_r[n]; \tilde{\mathbf{y}}[n] \mid \mathbf{h}_1, \mathbf{h}_2, \tilde{\mathbf{x}}_b[n]\right) = \ln\left(1 + \gamma\mathsf{SNR}\left(|\mathbf{h}_1|^2 + |\mathbf{h}_2|^2\right)\right) ,$$

which coincides with the supportable rates of the superposition code (8.3) and (8.4). Therefore, we claim that the Alamouti scheme achieves the same performance as the superposition code for the 2×1 MIMO channel model we are considering in this section.

In the following, we would characterize the distortion-diversity performance achieved by the MR-based scheme for three different cases. Firstly, we characterize the distortion performance of the MR-based scheme for certain fixed level of outage probabilities. Secondly, we tune the MR-based scheme to achieve outage probabilities that decrease with SNR. Finally, we consider the case where the outage probabilities increase with SNR. The distortion-diversity tradeoff for these three different cases were defined in Section 3.2.

8.1.2 Constant Outage Probability Case

We use δ_p and δ_f to denote the partial diversity order and the full diversity order, respectively, where $\delta_f > \delta_p > 0$. According to the definition of diversity order for the constant outage probability case (3.6), the partial outage probability and the full outage probability should be bounded as follows,

$$\mathcal{P}\left[\mathcal{O}_p\right] \le \delta_p, \quad \mathcal{P}\left[\mathcal{O}_f\right] \le \delta_f.$$

Using (8.5), (8.6) and the following explicit expression for $F(x; 4)$,

$$F(x; 4) = 1 - e^{-x/2} - \frac{x}{2}e^{-x/2}, \tag{8.8}$$

we obtain the following inequalities on $R_{s,b}$ and $R_{s,r}$,

$$1 - \exp\left(-\frac{2\left(e^{R_{s,b}} - 1\right)}{\left(1 - \gamma e^{R_{s,b}}\right)\mathsf{SNR}}\right) - \frac{2\left(e^{R_{s,b}} - 1\right)}{\left(1 - \gamma e^{R_{s,b}}\right)\mathsf{SNR}} \exp\left(-\frac{2\left(e^{R_{s,b}} - 1\right)}{\left(1 - \gamma e^{R_{s,b}}\right)\mathsf{SNR}}\right) \le \delta_p,$$

$$1 - \exp\left(-\frac{2\left(e^{R_{s,r} - R_{s,b}} - 1\right)}{P_r}\right) - \frac{2\left(e^{R_{s,r} - R_{s,b}} - 1\right)}{P_r} \exp\left(-\frac{2\left(e^{R_{s,r} - R_{s,b}} - 1\right)}{P_r}\right) \le \delta_f.$$

where $0 < \gamma < \frac{e^{R_{s,r} - R_{s,b}} - 1}{e^{R_{s,r}} - 1}$. We are able to simplify the above inequalities to be

$$\frac{4\left(e^{R_{s,b}} - 1\right)}{\left(1 - \gamma e^{R_{s,b}}\right)\mathsf{SNR}} \le -2 - 2W_{-1}\left(\frac{\delta_p - 1}{e}\right),$$

$$\frac{4\left(e^{R_{s,r} - R_{s,b}} - 1\right)}{\gamma\mathsf{SNR}} \le -2 - 2W_{-1}\left(\frac{\delta_f - 1}{e}\right).$$

Note that, in the low SNR regime, allocating a fixed fraction of SNR to the refinement layer does not affect the asymptotic interference power level when decoding for the base layer, since unit power of the additive noise dominates the power of the refinement layer when SNR is low. Therefore, by solving the above inequalities, we obtain

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the following bounds on $R_{s,b}$ and $R_{s,r}$,

$$\begin{aligned}
R_{s,b} &\leq \frac{-1 - W_{-1}\left(\frac{\delta_p - 1}{e}\right)}{2}(1-\gamma)\,\text{SNR} + O\left(\text{SNR}^2\right) \\
&= c_b\text{SNR} + O\left(\text{SNR}^2\right), \\
R_{s,r} &\leq R_{s,b} + \frac{-1 - W_{-1}\left(\frac{\delta_f - 1}{e}\right)}{2}\gamma\text{SNR} + O\left(\text{SNR}^2\right) \\
&= \left(\gamma\frac{-1 - W_{-1}\left(\frac{\delta_f - 1}{e}\right)}{2} + (1-\gamma)\frac{-1 - W_{-1}\left(\frac{\delta_p - 1}{e}\right)}{2}\right)\text{SNR} + O\left(\text{SNR}^2\right) \\
&= c_r\text{SNR} + O\left(\text{SNR}^2\right),
\end{aligned}$$

where

$$c_b = (1-\gamma)\frac{-1 - W_{-1}\left(\frac{\delta_p - 1}{e}\right)}{2}, \quad c_r = \gamma\frac{-1 - W_{-1}\left(\frac{\delta_f - 1}{e}\right)}{2} + (1-\gamma)\frac{-1 - W_{-1}\left(\frac{\delta_p - 1}{e}\right)}{2}. \tag{8.9}$$

According to (8.1) and (8.2), we are able to bound the partial and full distortion from below as follows,

$$D_p \geq 1 - c_b\text{SNR} + O\left(\text{SNR}^2\right), \tag{8.10}$$

$$D_f \geq 1 - c_r\text{SNR} + O\left(\text{SNR}^2\right), \tag{8.11}$$

where $\gamma \in (0,1)$. Applying the distortion coefficient definition (3.5), we are able to bound the partial and full distortion coefficients from above as follows,

$$d_p \leq c_b, \tag{8.12}$$

$$d_f \leq c_r. \tag{8.13}$$

Equations (8.9), (8.12) and (8.13) characterize the distortion coefficients achieved by the MR-based scheme for the constant outage probability case in the low SNR regime.

8.1.3 Low Outage Probability Case

We use δ_p and δ_f to denote the partial diversity order and the full diversity order, respectively, where $\delta_p > \delta_f > 0$. According to the diversity order definition for the low outage probability case (3.2), it is sufficient to set the constraints on the partial and full outage probabilities as $\theta_p \mathsf{SNR}^{\delta_p}$ and $\theta_f \mathsf{SNR}^{\delta_f}$, respectively, where $\theta_p > 0$ and $\theta_f > 0$. Using (8.5), (8.6) and the explicit expression for $F(x; 4)$ (8.8), we obtain the following inequalities on $R_{s,b}$ and $R_{s,r}$,

$$1 - \exp\left(-\frac{2\left(e^{R_{s,b}} - 1\right)}{\left(1 - \gamma e^{R_{s,b}}\right)\mathsf{SNR}}\right) - \frac{2\left(e^{R_{s,b}} - 1\right)}{\left(1 - \gamma e^{R_{s,b}}\right)\mathsf{SNR}}\exp\left(-\frac{2\left(e^{R_{s,b}} - 1\right)}{\left(1 - \gamma e^{R_{s,b}}\right)\mathsf{SNR}}\right) \le \theta_p\mathsf{SNR}^{\delta_p},$$

$$1 - \exp\left(-\frac{2\left(e^{R_{s,r} - R_{s,b}} - 1\right)}{\gamma\mathsf{SNR}}\right) - \frac{2\left(e^{R_{s,r} - R_{s,b}} - 1\right)}{\gamma\mathsf{SNR}}\exp\left(-\frac{2\left(e^{R_{s,r} - R_{s,b}} - 1\right)}{\gamma\mathsf{SNR}}\right) \le \theta_f\mathsf{SNR}^{\delta_f},$$

where $\gamma \in (0, 1)$. We are to simplify the above inequalities as follows,

$$\frac{4\left(e^{R_{s,b}} - 1\right)}{\left(1 - \gamma e^{R_{s,b}}\right)\mathsf{SNR}} \le -2 - 2W_{-1}\left(\frac{\theta_p\mathsf{SNR}^{\delta_p} - 1}{e}\right),$$

$$\frac{4\left(e^{R_{s,r} - R_{s,b}} - 1\right)}{\gamma\mathsf{SNR}} \le -2 - 2W_{-1}\left(\frac{\theta_f\mathsf{SNR}^{\delta_f} - 1}{e}\right).$$

By applying the following approximation for $W_{-1}(-1/e + x)$,

$$W_{-1}(-1/e + x) = -1 - \sqrt{2ex} - 2ex/3 + O\left(x^{3/2}\right),$$

we are able to bound $R_{s,b}$ and $R_{s,r}$ from above as follows,

$$R_{s,b} \le (1 - \gamma)\sqrt{\frac{\theta_p}{2}}\mathsf{SNR}^{1 + \frac{1}{2}\delta_p} + O\left(\mathsf{SNR}^{1 + \delta_p}\right),$$

$$R_{s,r} \le R_{s,b} + \gamma\sqrt{\frac{\theta_f}{2}}\mathsf{SNR}^{1 + \frac{1}{2}\delta_f} + O\left(\mathsf{SNR}^{1 + \delta_f}\right)$$

$$= \frac{\gamma}{\sqrt{2}}\mathsf{SNR}^{1 + \frac{1}{2}\delta_f} + O\left(\mathsf{SNR}^{1 + \min\left\{\frac{1}{2}\delta_p, \delta_f\right\}}\right).$$

According to (8.1) and (8.2), we are able to bound the partial and full distortion from below as follows,

$$D_p \geq 1 - (1 - \gamma)\sqrt{\frac{\theta_p}{2}}\mathsf{SNR}^{1+\frac{1}{2}\delta_p} + O\left(\mathsf{SNR}^{1+\delta_p}\right),$$

$$D_f \geq 1 - \frac{\gamma}{\sqrt{2}}\mathsf{SNR}^{1+\frac{1}{2}\delta_f} + O\left(\mathsf{SNR}^{1+\min\left\{\frac{1}{2}\delta_p, \delta_f\right\}}\right).$$

Applying the definition of distortion exponent for the low outage probability case (3.1), we are able to bound the distortion exponent from above as follows,

$$d_p \leq 1 + \frac{1}{2}\delta_p, \tag{8.14}$$

$$d_f \leq 1 + \frac{1}{2}\delta_f, \tag{8.15}$$

which characterize the distortion exponents achieved by the MR-based scheme for the low outage probability case in the low SNR regime.

8.1.4 High Outage Probability Case

We use δ_p and δ_f to denote the partial diversity order and the full diversity order, respectively, where $\delta_f > \delta_p > 0$. According to the diversity order definition for the high outage probability case (3.4), it is sufficient to set the constraints on the partial and full outage probabilities as $1 - \theta_p\mathsf{SNR}^{\delta_p}$ and $1 - \theta_f\mathsf{SNR}^{\delta_f}$, respectively, where $\theta_p > 0$ and $\theta_f > 0$. Using (8.5), (8.6) and the explicit expression for $F(x; 4)$ (8.8), we obtain the following inequalities on $R_{s,b}$ and $R_{s,r}$,

$$1 - \exp\left(-\frac{2\left(e^{R_{s,b}} - 1\right)}{(1 - \gamma e^{R_{s,b}})\mathsf{SNR}}\right) - \frac{2\left(e^{R_{s,b}} - 1\right)}{(1 - \gamma e^{R_{s,b}})\mathsf{SNR}}\exp\left(-\frac{2\left(e^{R_{s,b}} - 1\right)}{(1 - \gamma e^{R_{s,b}})\mathsf{SNR}}\right) \leq 1 - \theta_p\mathsf{SNR}^{\delta_p},$$

$$1 - \exp\left(-\frac{2\left(e^{R_{s,r}-R_{s,b}} - 1\right)}{\gamma\mathsf{SNR}}\right) - \frac{2\left(e^{R_{s,r}-R_{s,b}} - 1\right)}{\gamma\mathsf{SNR}}\exp\left(-\frac{2\left(e^{R_{s,r}-R_{s,b}} - 1\right)}{\gamma\mathsf{SNR}}\right) \leq 1 - \theta_f\mathsf{SNR}^{\delta_f}.$$

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where $\gamma \in (0, 1)$. We are able to simplify the above inequalities as follows,

$$\frac{4\left(e^{R_{s,b}} - 1\right)}{(1 - \gamma e^{R_{s,b}})\mathsf{SNR}} \leq -2 - 2W_{-1}\left(-\frac{\theta_p \mathsf{SNR}^{\delta_p}}{e}\right),$$

$$\frac{4\left(e^{R_{s,r} - R_{s,b}} - 1\right)}{\gamma \mathsf{SNR}} \leq -2 - 2W_{-1}\left(-\frac{\theta_f \mathsf{SNR}^{\delta_f}}{e}\right).$$

By applying the following approximation for $W_{-1}(x)$ when x is close to 0^-,

$$W_{-1}(x) = \ln(-x) - \ln\left(-\ln(-x)\right) + \sum_{l=0}^{+\infty}\sum_{m=1}^{+\infty} c_{lm} \frac{\ln^m\left(-\ln(-x)\right)}{\ln^{l+m}(-x)},$$

where $c_{lm} = (-1)^l \frac{S(l+m, l+1)}{m!}$ and $S(l + m, l + 1)$ is a non-negative Stirling number of the first kind. Let $P_r = \gamma \mathsf{SNR}$ with $\gamma \in (0, 1)$, we obtain the following upper bounds on the source code rates,

$$R_{s,b} \leq \frac{1-\gamma}{2}\delta_p \mathsf{SNR} \ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\ln\ln\frac{1}{\mathsf{SNR}}\right),$$

$$R_{s,r} \leq R_b + \frac{\gamma}{2}\delta_f \mathsf{SNR}\ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\ln\ln\frac{1}{\mathsf{SNR}}\right)$$

$$= \left(\frac{1-\gamma}{2}\delta_p + \frac{\gamma}{2}\delta_f\right)\mathsf{SNR}\ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\ln\ln\frac{1}{\mathsf{SNR}}\right).$$

According to (8.1) and (8.2), we are able to bound the partial and full distortion from below as follows,

$$D_p \geq 1 - \frac{1-\gamma}{2}\delta_p \mathsf{SNR}\ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\ln\ln\frac{1}{\mathsf{SNR}}\right),$$

$$D_f \geq 1 - \left(\frac{1-\gamma}{2}\delta_p + \frac{\gamma}{2}\delta_f\right)\mathsf{SNR}\ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\ln\ln\frac{1}{\mathsf{SNR}}\right).$$

Applying the definition of distortion exponent for the low outage probability case (3.3), we are able to bound the distortion coefficient from above as follows,

$$d_p \leq \frac{1-\gamma}{2}\delta_p, \tag{8.16}$$

$$d_f \leq \left(\frac{1-\gamma}{2}\delta_p + \frac{\gamma}{2}\delta_f\right), \tag{8.17}$$

which characterize the distortion coefficients achieved by the MR-based scheme for the high outage probability case in the low SNR regime.

8.2 Symmetric MD-Based Source-Channel Scheme with Separate Decoding

The symmetric MD-based source-channel scheme with separate decoding is composed of the simplified symmetric El-Gamal-Cover source code (reviewed in Theorem 4.3.1 and the remark after that), the V-BLAST channel code structure and a separate decoder. Since we are dealing with parallel fading channel, the V-BLAST channel code structure degenerates to separate channel code for each subchannel. Suppose that the distortion levels requested by the two classes of receivers are D_p and D_f ($D_f < D_p$), respectively. The symmetric MD-based source-channel scheme with separate decoding proceeds as follows,

1. The simplified symmetric El-Gamal-Cover source code converts the length-N source sequence s to two length-NR_s bit sequences (description i_1 and i_2). According to the remark following Theorem 4.3.1, to achieve distortion levels of D_p and D_f, the parameters of the simplified symmetric El-Gamal-Cover source code need to be configured as follows,

$$\sigma^2 = \frac{D_p}{1 - D_p},$$
$$\rho = -1 + 2 \times \frac{D_f(1 - D_p)}{D_p(1 - D_f)}.$$

Therefore, the source code rate needs to satisfy the following condition,

$$R_s > \frac{1}{2} \log \frac{(1 - D_f)^2}{4D_f(D_p - D_f)(1 - D_p)}. \tag{8.18}$$

Moreover, the mutual information between s_1 and s_2 is

$$I(s_1; s_2) = \log \frac{(1 - D_f)^2}{4(1 - D_p)(D_p - D_f)} . \tag{8.19}$$

2. The two bit streams i_1 and i_2 are channel coded and modulated separately and transmitted on the two input antennas, respectively. To avoid interference at the receiver, the slots of channel uses are alternated between the two bit streams (when the first antenna is active, the second antenna is silent and vice versa). Since only one antenna is active in each time slot, the bit streams are modulated to the power level of 2SNR. The mutual information between the channel input and the channel output is

$$I\left(x_1[n]; y[n] \,|\, h_1\right) = \ln\left(1 + 2|h_1|^2 \mathsf{SNR}\right) ;$$

3. At the receiver, the following separate decoder is used,

- By comparing the channel output sequence $\underline{y}$ at time slots when antenna 1 is active with the channel codewords in channel codebook for bit stream 1 according to the joint typicality criterion, the receiver decodes bit stream 1 as $\hat{i}_1$. Since, the two bit streams share the time slots equally, the supportable rate for the first bit stream is

$$R_{c,1} = \frac{1}{2} I\left(x_1[n]; y[n] \,|\, h_1\right) = \frac{1}{2} \ln\left(1 + 2|h_1|^2 \mathsf{SNR}\right) .$$

Similarly, the receiver decodes bit stream 2 as $\hat{i}_2$. The supportable rate for the second bit stream is

$$R_{c,2} = \frac{1}{2} I\left(x_2[n]; y[n] \,|\, h_2\right) = \frac{1}{2} \ln\left(1 + 2|h_2|^2 \mathsf{SNR}\right) . \tag{8.20}$$

Since h_1 and h_2 are independent, the supportable rates for the two bit streams $R_{c,1}$ and $R_{c,2}$ are also independent of each other;

- If $R_s < R_{c,1}$ and $R_s < R_{c,1}$, the receiver outputs the reconstruction sequence $g_c(\underline{s}_1(\hat{i}_1), \underline{s}_2(\hat{i}_2))$. Otherwise, the receiver declares a "full outage" and proceeds to the next step;

- If $R_s < R_{c,1}$, the receiver outputs the reconstruction sequence $g_1(\underline{s}_1(\hat{i}_1))$; If $R_s < R_{c,2}$, the receiver outputs the reconstruction sequence $g_2(\underline{s}_2(\hat{i}_2))$; Otherwise, the receiver declares a "partial outage" and outputs the all-zero sequence.

The partial and full outage event is characterized as follows,

$$\mathcal{O}_p = \{R_{c,1} < R_s\} \cap \{R_{c,2} < R_s\}\,, \quad \mathcal{O}_f = \{R_{c,1} < R_s\} \cup \{R_{c,2} < R_s\}\,.$$

The partial outage probability is

$$
\begin{aligned}
\mathcal{P}\left[\mathcal{O}_p\right] &= \mathbf{P}\left(R_{c,1} < R_s\right)\mathbf{P}\left(R_{c,s} < R_s\right) \\
&= \mathbf{P}\left(|\mathbf{h}_1|^2 < \frac{e^{2R_s}-1}{2\mathsf{SNR}}\right)\mathbf{P}\left(|\mathbf{h}_2|^2 < \frac{e^{2R_s}-1}{2\mathsf{SNR}}\right) \\
&= F^2\left(\frac{2\left(e^{2R_s}-1\right)}{\mathsf{SNR}}; 2\right)\,,
\end{aligned}
$$

where $F\left(\cdot; n\right)$ is the cumulative function of χ_n^2 random variable. In particular, for $n = 2$, we have that

$$F(x; 2) = 1 - e^{-x/2}\,, \tag{8.21}$$

so the partial outage probability can be expressed as

$$\mathcal{P}\left[\mathcal{O}_p\right] = F^2\left(\frac{2\left(e^{2R_s}-1\right)}{\mathsf{SNR}}; 2\right) = \left(1 - \exp\left(-\frac{\left(e^{2R_s}-1\right)}{\mathsf{SNR}}\right)\right)^2\,. \tag{8.22}$$

The full outage probability is characterized as follows,

$$
\begin{aligned}
\mathcal{P}\left[\mathcal{O}_f\right] &= 1 - \left(1 - \mathbf{P}\left(R_{c,1} < R_s\right)\right)\left(1 - \mathbf{P}\left(R_{c,2} < R_s\right)\right) \\
&= 1 - \exp\left(-\frac{2\left(e^{2R_s}-1\right)}{\mathsf{SNR}}\right)\,.
\end{aligned} \tag{8.23}
$$

As in the previous section, we would characterize the distortion-diversity performance achieved by the MD-based scheme for three different cases, namely the constant outage probability case, the low outage probability case and the high outage probability case. The distortion-diversity tradeoff for these three different cases were defined in Section 3.2.

8.2.1 Constant Outage Probability Case

We use δ_p and δ_f to denote the partial diversity order and the full diversity order, respectively, where $\delta_f > \delta_p > 0$. According to the definition of diversity order for the constant outage probability case (3.6), the partial outage probability and the full outage probability should be bounded as follows,

$$\mathcal{P}\left[\mathcal{O}_p\right] \leq \delta_p, \quad \mathcal{P}\left[\mathcal{O}_f\right] \leq \delta_f.$$

Plugging in (8.22) and (8.23), we have the following inequalities on R_s,

$$\left(1 - \exp\left(-\frac{\left(e^{2R_s} - 1\right)}{\mathsf{SNR}}\right)\right)^2 \leq \delta_p,$$

$$1 - \exp\left(-\frac{2\left(e^{2R_s} - 1\right)}{\mathsf{SNR}}\right) \leq \delta_f.$$

which can be simplified as follows,

$$
\begin{aligned}
R_s &\leq \min\left\{\frac{1}{2}\ln\left(1 - \mathsf{SNR}\ln\left(1 - \sqrt{\delta_p}\right)\right), \frac{1}{2}\ln\left(1 - \frac{\mathsf{SNR}}{2}\ln\left(1 - \delta_f\right)\right)\right\} \\
&= \min\left\{\frac{1}{2}\ln\frac{1}{1 - \sqrt{\delta_p}}\cdot\mathsf{SNR} + O\left(\mathsf{SNR}^2\right), \frac{1}{4}\ln\frac{1}{1 - \delta_f}\cdot\mathsf{SNR} + O\left(\mathsf{SNR}^2\right)\right\} \\
&= \min\left\{\frac{1}{2}\ln\frac{1}{1 - \sqrt{\delta_p}}, \frac{1}{4}\ln\frac{1}{1 - \delta_f}\right\}\mathsf{SNR} + O\left(\mathsf{SNR}^2\right) \\
&= c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right),
\end{aligned}
$$

(8.24)

where $c = \min\left\{\frac{1}{2}\ln\frac{1}{1-\sqrt{\delta_p}}, \frac{1}{4}\ln\frac{1}{1-\delta_f}\right\}$. The achievable partial and full distortion levels are characterized by applying Lemma 4.3.3, which we summarize here,

$$D_f \geq \max\left\{a, e^{-2R_s}\right\}, \quad D_p \geq \max\left\{e^{-R_s}, \frac{1+a}{2} - \frac{1-a}{2}\sqrt{1 - \frac{e^{-2R_s}}{a}}\right\}, \quad (8.25)$$

where $a \in \left[e^{-2R_s}, \frac{e^{-R_s}}{2-e^{-R_s}}\right]$. Plugging in the upper bound on R_s (8.24), we are able to simplify the lower bound on a as follows,

$$
\begin{aligned}
e^{-2R_s} &= \exp\left(-2c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right)\right) \\
&= 1 - 2c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right), \quad (8.26)
\end{aligned}
$$

and also the upper bound as follows,

$$
\begin{aligned}
\frac{e^{-R_s}}{2-e^{-R_s}} &= \frac{\exp\left(-c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right)\right)}{2 - \exp\left(-c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right)\right)} \\
&= \frac{1 - c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right)}{1 + c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right)} \\
&= 1 - 2c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right). \quad (8.27)
\end{aligned}
$$

Therefore, it is safe to set $a = 1 - 2c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right)$. According to (8.25), we are able to bound the full distortion from below by $1 - 2c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right)$, while the partial distortion is lower bounded as follows,

$$
\begin{aligned}
D_p &\geq \max\left\{e^{-R_s}, \frac{1+a}{2} - \frac{1-a}{2}\sqrt{1 - \frac{e^{-2R_s}}{a}}\right\} \\
&= \max\left\{1 - c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right), 1 - c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right)\right\} \\
&= 1 - c\mathsf{SNR} + O\left(\mathsf{SNR}^2\right). \quad (8.28)
\end{aligned}
$$

Applying the distortion coefficient definition (3.5) for the constant outage probability case, we obtain the following achievable partial and full distortion coefficients

$$d_p \leq c, \tag{8.29}$$

$$d_f \leq 2c, \tag{8.30}$$

where

$$c = \min \left\{ \frac{1}{2} \ln \frac{1}{1 - \sqrt{\delta_p}}, \frac{1}{4} \ln \frac{1}{1 - \delta_f} \right\}.$$

Note that (8.29) and (8.30) characterize the distortion-diversity tradeoff achieved by the MD-based scheme for the constant outage probability case in the low SNR regime. We leave the performance comparison with the MR-based scheme to the next section.

8.2.2 Low Outage Probability Case

We use δ_p and δ_f to denote the partial diversity order and the full diversity order, respectively, where $\delta_p > \delta_f > 0$. According to the diversity order definition for the low outage probability case (3.2), it is sufficient to set the constraints on the partial and full outage probabilities as $\theta_p \text{SNR}^{\delta_p}$ and $\theta_f \text{SNR}^{\delta_f}$, respectively, where $\theta_p > 0$ and $\theta_f > 0$. Using (8.22), (8.23), we obtain the following inequalities on R_s,

$$\theta_p \text{SNR}^{\delta_p} \geq \left(1 - \exp\left(-\frac{\left(e^{2R_s} - 1 \right)}{\text{SNR}} \right) \right)^2,$$

$$\theta_f \text{SNR}^{\delta_f} \geq 1 - \exp\left(-\frac{2 \left(e^{2R_s} - 1 \right)}{\text{SNR}} \right).$$

Simplifying the above inequalities, we obtain the following upper bound on R_s,

$$
\begin{aligned}
R_s &\leq \min\left\{\frac{1}{2}\ln\left(1 - \mathsf{SNR}\ln\left(1 - \sqrt{\theta_p}\mathsf{SNR}^{\frac{1}{2}\delta_p}\right)\right), \frac{1}{2}\ln\left(1 - \frac{\mathsf{SNR}}{2}\ln\left(1 - \theta_f\mathsf{SNR}^{\delta_f}\right)\right)\right\} \\
&= \min\left\{\frac{\sqrt{\theta_p}}{2}\mathsf{SNR}^{1+\frac{1}{2}\delta_p} + O\left(\mathsf{SNR}^{1+\delta_p}\right), \frac{\theta_f}{4}\mathsf{SNR}^{1+\delta_f} + O\left(\mathsf{SNR}^{1+2\delta_f}\right)\right\} \\
&= \begin{cases} \frac{\sqrt{\theta_p}}{2}\mathsf{SNR}^{1+\frac{1}{2}\delta_p} + O\left(\mathsf{SNR}^{1+\delta_p}\right), & \text{if } \delta_p > 2\delta_f, \\ \frac{\theta_f}{4}\mathsf{SNR}^{1+\delta_f} + O\left(\mathsf{SNR}^{1+2\delta_f}\right), & \text{if } \delta_p < 2\delta_f. \end{cases}
\end{aligned}
\tag{8.31}
$$

The above upper bound on R_s can be summarized as

$$
\begin{aligned}
R_s &\leq c\mathsf{SNR}^{1+\max\left\{\frac{1}{2}\delta_p, \delta_f\right\}} + O\left(\mathsf{SNR}^{1+\max\left\{\delta_p, 2\delta_f\right\}}\right) \\
&= c\mathsf{SNR}^{1+\delta} + O\left(\mathsf{SNR}^{1+2\delta}\right).
\end{aligned}
\tag{8.32}
$$

where $\delta = \max\left\{\frac{1}{2}\delta_p, \delta_f\right\}$, and the constant coefficient $c = \sqrt{\theta_p}/2$ if $\delta_p > 2\delta_f$ and $c = \theta_f/4$ if $\delta_p < 2\delta_f$. In the following, we would apply Lemma 4.3.3, which is summarized here,

$$
D_f \geq a, \quad D_p \geq \max\left\{e^{-R_s}, \frac{1+a}{2} - \frac{1-a}{2}\sqrt{1 - \frac{e^{-2R_s}}{a}}\right\},
\tag{8.33}
$$

where $a \in \left[e^{-2R_s}, \frac{e^{-R_s}}{2-e^{-R_s}}\right]$. Setting R_s to its upper bound (8.32), we can simplify the lower bound on a as follows,

$$
\begin{aligned}
e^{-2R_s} &= \exp\left(-2c\mathsf{SNR}^{1+\delta} + O\left(\mathsf{SNR}^{1+2\delta}\right)\right) \\
&= 1 - 2c\mathsf{SNR}^{1+\delta} + O\left(\mathsf{SNR}^{1+2\delta}\right),
\end{aligned}
$$

and also the upper bound as follows,

$$
\begin{aligned}
\frac{e^{-R_s}}{2-e^{-R_s}} &= \frac{\exp\left(-c\mathsf{SNR}^{1+\delta} + O\left(\mathsf{SNR}^{1+2\delta}\right)\right)}{2 - \exp\left(-c\mathsf{SNR}^{1+\delta} + O\left(\mathsf{SNR}^{1+2\delta}\right)\right)} \\
&= \frac{1 - c\mathsf{SNR}^{1+\delta} + O\left(\mathsf{SNR}^{1+2\delta}\right)}{1 + c\mathsf{SNR}^{1+\delta} + O\left(\mathsf{SNR}^{1+2\beta}\right)} \\
&= 1 - 2c\mathsf{SNR}^{1+\delta} + O\left(\mathsf{SNR}^{1+2\delta}\right).
\end{aligned}
$$

Therefore, it is safe set $a = 1 - 2c\text{SNR}^{1+\delta} + O\left(\text{SNR}^{1+2\delta}\right)$. Applying Lemma 4.3.3, we are able to bound the full distortion from below by $1 - 2c\text{SNR}^{1+\delta} + O\left(\text{SNR}^{1+2\delta}\right)$, while the partial distortion is lower bounded as follows,

$$
\begin{aligned}
D_p \;\geq\; & \max\left\{ e^{-R_s}, \; \frac{1+a}{2} - \frac{1-a}{2}\sqrt{1 - \frac{e^{-2R_s}}{a}} \right\} \\
=\; & \max\left\{ 1 - c\text{SNR}^{1+\delta} + O\left(\text{SNR}^{1+2\delta}\right), \; 1 - c\text{SNR}^{1+\delta} + O\left(\text{SNR}^{1+2\delta}\right) \right\} \\
=\; & 1 - c\text{SNR}^{1+\delta} + O\left(\text{SNR}^{1+2\delta}\right).
\end{aligned}
$$

According to the distortion exponent definition (3.1), we obtain the following achievable distortion exponents by the MD-based scheme,

$$
d_p \;\leq\; 1 + \delta, \tag{8.34}
$$

$$
d_f \;\leq\; 1 + \delta, \tag{8.35}
$$

where

$$
\delta = \max\left\{ \frac{1}{2}\delta_p, \; \delta_f \right\}, \quad c = \begin{cases} \frac{\sqrt{\theta_p}}{2}, & \text{if } \delta_p > 2\delta_f, \\[2mm] \frac{\theta_f}{4}, & \text{if } \delta_p < 2\delta_f. \end{cases}
$$

Note that (8.34) and (8.35) characterize the distortion-diversity tradeoff achieved by the MD-based scheme for the low outage probability case in the low SNR regime.

8.2.3 High Outage Probability Case

We use δ_p and δ_f to denote the partial diversity order and the full diversity order, respectively, where $\delta_f > \delta_p > 0$. According to the diversity order definition for the high outage probability case (3.4), it is sufficient to set the constraints on the partial and full outage probabilities as $1 - \theta_p\text{SNR}^{\delta_p}$ and $1 - \theta_f\text{SNR}^{\delta_f}$, respectively, where $\theta_p > 0$ and $\theta_f > 0$. Using (8.5), (8.6) and the explicit expression for $F(x; 4)$ (8.8), we

obtain the following inequalities on R_s,

$$1 - \theta_p \mathsf{SNR}^{\delta_p} \geq \left(1 - \exp\left(-\frac{\left(e^{2R_s} - 1\right)}{\mathsf{SNR}}\right)\right)^2,$$

$$1 - \theta_f \mathsf{SNR}^{\delta_f} \geq 1 - \exp\left(-\frac{2\left(e^{2R_s} - 1\right)}{\mathsf{SNR}}\right),$$

Simplifying the above inequalities, we obtain the following upper bound on R_s,

$$
\begin{aligned}
R_s &\leq \min\left\{\frac{1}{2}\ln\left(1 - \mathsf{SNR}\ln\left(1 - \sqrt{1 - \theta_p \mathsf{SNR}^{\delta_p}}\right)\right), \frac{1}{2}\ln\left(1 - \frac{\mathsf{SNR}}{2}\ln\left(\theta_f \mathsf{SNR}^{\delta_f}\right)\right)\right\} \\
&= \min\left\{\frac{1}{2}\delta_p \mathsf{SNR}\ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\right), \frac{1}{4}\delta_f \mathsf{SNR}\ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\right)\right\} \\
&= \min\left\{\frac{1}{2}\delta_p, \frac{1}{4}\beta_f\right\} \mathsf{SNR}\ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\right) \\
&= c\mathsf{SNR}\ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\right),
\end{aligned}
\tag{8.36}
$$

where $c = \min\left\{\frac{1}{2}\delta_p, \frac{1}{4}\delta_f\right\}$. To characterize the achievable distortion levels, we would apply Lemma 4.3.3, which is summarized here,

$$D_f \geq a, \quad D_p \geq \max\left\{e^{-R_s}, \frac{1+a}{2} - \frac{1-a}{2}\sqrt{1 - \frac{e^{-2R_s}}{a}}\right\},$$

where $a \in \left[e^{-2R_s}, \frac{e^{-R_s}}{2-e^{-R_s}}\right]$. Setting R_s to its upper bound (8.36), we can simplify the lower bound on a as follows,

$$
\begin{aligned}
e^{-2R_s} &= \exp\left(-2c\mathsf{SNR}\ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\right)\right) \\
&= 1 - 2c\mathsf{SNR}\ln\frac{1}{\mathsf{SNR}} + O\left(\mathsf{SNR}\right),
\end{aligned}
$$

and also the upper bound as follows,

$$
\begin{aligned}
\frac{e^{-R_s}}{2 - e^{-R_s}} &= \frac{\exp\left(-c\mathrm{SNR}\ln\frac{1}{\mathrm{SNR}} + O\left(\mathrm{SNR}\right)\right)}{2 - \exp\left(-c\mathrm{SNR}\ln\frac{1}{\mathrm{SNR}} + O\left(\mathrm{SNR}\right)\right)} \\
&= \frac{1 - c\mathrm{SNR}\ln\frac{1}{\mathrm{SNR}} + O\left(\mathrm{SNR}\right)}{1 + c\mathrm{SNR}\ln\frac{1}{\mathrm{SNR}} + O\left(\mathrm{SNR}\right)} \\
&= 1 - 2c\mathrm{SNR}\ln\frac{1}{\mathrm{SNR}} + O\left(\mathrm{SNR}\right).
\end{aligned}
\tag{8.37}
$$

Therefore, it is safe to set $a = 1 - 2c\mathrm{SNR}\ln\frac{1}{\mathrm{SNR}} + O\left(\mathrm{SNR}\right)$. Applying Lemma 4.3.3, we are able to bound the full distortion from below by $1 - 2c\mathrm{SNR}\ln\frac{1}{\mathrm{SNR}} + O\left(\mathrm{SNR}\right)$, while the partial distortion is lower bounded as follows,

$$
\begin{aligned}
D_p &\geq \max\left\{e^{-R_s}, \frac{1+a}{2} - \frac{1-a}{2}\sqrt{1 - \frac{e^{-2R_s}}{a}}\right\} \\
&= \max\left\{1 - c\mathrm{SNR}\ln\frac{1}{\mathrm{SNR}} + O\left(\mathrm{SNR}\right), 1 - c\mathrm{SNR}\ln\frac{1}{\mathrm{SNR}} + O\left(\mathrm{SNR}\right)\right\} \\
&= 1 - c\mathrm{SNR}\ln\frac{1}{\mathrm{SNR}} + O\left(\mathrm{SNR}\right).
\end{aligned}
$$

According to the definition of distortion coefficients (3.3), we characterize the achievable distortion coefficients of the MD-based scheme as follows,

$$
\begin{aligned}
d_p &\leq c, \tag{8.38} \\
d_f &\leq 2c, \tag{8.39}
\end{aligned}
$$

where $c = \min\left\{\frac{1}{2}\delta_p, \frac{1}{4}\delta_f\right\}$.

8.3 Performance Comparison

8.3.1 Constant Outage Probability Case

For given partial and full diversity orders (δ_p, δ_f), the distortion coefficients achieved by the MR-based scheme are $(d_p, d_f) = (c_b, c_r)$ (8.12) (8.13), where

$$c_b = (1 - \gamma)\frac{-1 - W_{-1}\left(\frac{\delta_p - 1}{e}\right)}{2}, \quad c_r = \gamma\frac{-1 - W_{-1}\left(\frac{\delta_f - 1}{e}\right)}{2} + (1 - \gamma)\frac{-1 - W_{-1}\left(\frac{\delta_p - 1}{e}\right)}{2},$$

and $\gamma \in (0,1)$. The distortion coefficients achieved by the MD-based scheme is $(d_p, d_f) = (c, 2c)$ (8.29) (8.30) where

$$c = \min\left\{\frac{1}{2}\ln\frac{1}{1 - \sqrt{\delta_p}}, \ \frac{1}{4}\ln\frac{1}{1 - \delta_f}\right\}.$$

The performance comparison between the MR-based and the MD-based schemes are mixed. When the outage probability is small, the MR-based scheme achieves a better performance, as illustrated in Figure 8-1. However, when the outage probability is large, the MD-based scheme wins out, as illustrated in Figure 8-2. In order to
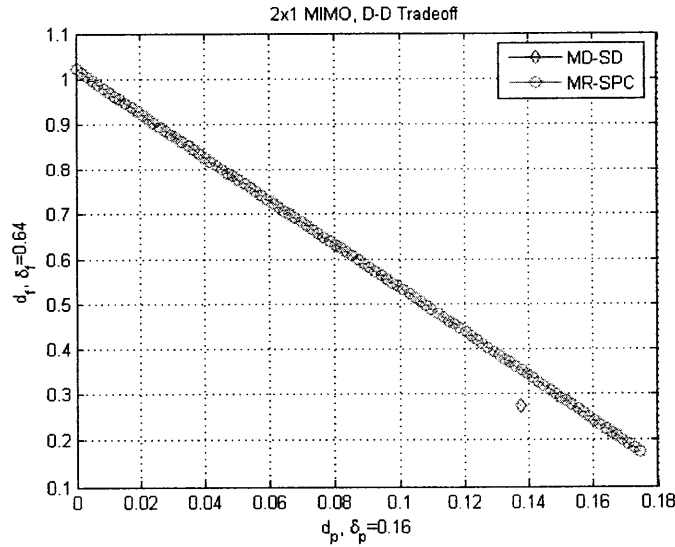

Figure 8-1: MR-Based Scheme v.s. MD-Based Scheme, Constant Outage Probability Case, $\delta_p = 0.16$ and $\delta_f = 0.64$
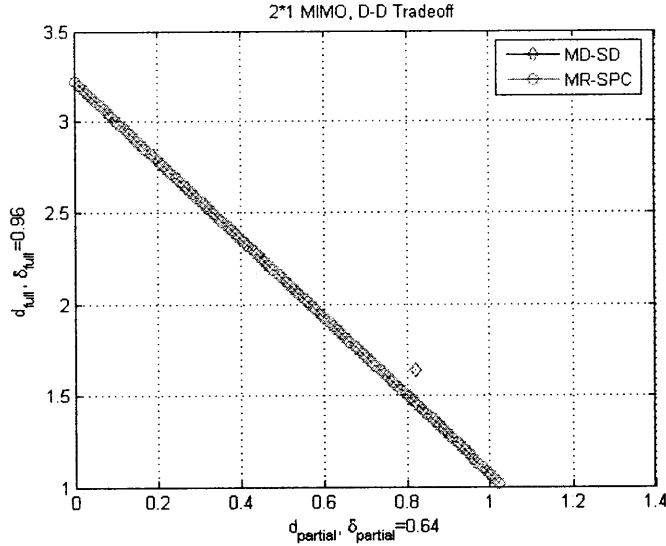
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Figure 8-2: MR-Based Scheme v.s. MD-Based Scheme, Constant Outage Probability Case, $\delta_p = 0.64$ and $\delta_f = 0.96$

confirm this observation, we carry out performance comparison in the two extreme cases where the outage probability either decreases to 0 or increases to 1 as SNR decreases. We expect the MR-based scheme to wins out in the former case while the MD-based scheme to outperform in the latter case.

8.3.2 Low Outage Probability Case

For given partial and full diversity orders (δ_p, δ_f), the distortion exponents achieved by the MR-based scheme are $(d_p, d_f) = \left(1 + \frac{1}{2}\delta_p, 1 + \frac{1}{2}\delta_f\right)$ (8.14) (8.15), while the MD-based scheme achieves distortion exponents of $(d_p, d_f) = (1 + \delta, 1 + \delta)$ (8.34) (8.35), where $\delta = \max\left\{\frac{1}{2}\delta_p, \delta_f\right\}$. Since

$$\frac{1}{2}\delta_p \leq \max\left\{\frac{1}{2}\delta_p, \delta_f\right\} , \quad \frac{1}{2}\delta_f \leq \delta_f \leq \max\left\{\frac{1}{2}\delta_p, \delta_f\right\} ,$$

we conclude that the MR-based scheme outperforms the MD-based scheme in the low outage probability case. This extreme case confirms the observation we made in the constant outage probability case (Figure 8-1).

8.3.3 High Outage Probability Case

For given partial and full diversity orders (δ_p, δ_f), the distortion coefficients achieved by the MR-based scheme are (8.16) (8.17)

$$(d_p, d_f) = \left(\frac{1-\gamma}{2} \delta_p, \; \frac{1-\gamma}{2} \delta_p + \frac{\gamma}{2} \delta_f \right),$$

where $\gamma \in (0, 1)$. The MD-based scheme achieves the following distortion coefficients (8.38) (8.39),

$$(d_p, \, d_f) = \min \left\{ \left(\frac{1}{2} \delta_p, \; \delta_p \right), \; \left(\frac{1}{4} \delta_f, \; \frac{1}{2} \delta_f \right) \right\}. \tag{8.40}$$

Lemma 8.3.1 (High Outage Probability Case). *For the high outage probability case in the low SNR regime, the MD-based scheme outperforms the MR-based scheme from the perspective of distortion-diversity tradeoff.*

Proof. To show that the MD-based scheme outperforms the MR-based scheme, we only need to show that both $\left(\frac{1}{2} \delta_p, \; \delta_p \right)$ and $\left(\frac{1}{4} \delta_f, \; \frac{1}{2} \delta_f \right)$ lie above the achievable (d_p, d_f) boundary of the MR-based scheme. Note that the (d_p, d_f) boundary of the MR-based scheme is a line segment connecting $\left(\frac{1}{2} \beta_p, \; \frac{1}{2} \beta_p \right)$ to $\left(0, \; \frac{1}{2} \beta_f \right)$. Clearly, the point $\left(\frac{1}{2} \beta_p, \; \beta_p \right)$ lies above the line segment, since $\left(\frac{1}{2} \beta_p, \; \beta_p \right) \geq \left(\frac{1}{2} \beta_p, \; \frac{1}{2} \beta_p \right)$. To show that the point $\left(\frac{1}{4} \beta_f, \; \frac{1}{2} \beta_f \right)$ also lies above the line segment, we find out the corresponding point on the line segment with the same d_p, which turns out to be

$$\frac{\delta_f}{2\delta_p} \left(\frac{1}{2} \beta_p, \; \frac{1}{2} \beta_p \right) + \left(1 - \frac{\beta_f}{2\beta_p} \right) \left(0, \; \frac{1}{2} \beta_f \right) = \left(\frac{1}{4} \beta_f, \; \frac{1}{2} \beta_f + \frac{\beta_f}{4\beta_p} (\beta_p - \beta_f) \right) \leq \left(\frac{1}{4} \beta_f, \; \frac{1}{2} \beta_f \right).$$

Since both $\left(\frac{1}{2} \delta_p, \; \delta_p \right)$ and $\left(\frac{1}{4} \delta_f, \; \frac{1}{2} \delta_f \right)$ lie above the achievable (d_p, d_f) boundary of the MR-based scheme, the (d_p, d_f) point achieved by the MD-based scheme also lies above the the achievable (d_p, d_f) boundary of the MR-based scheme. Therefore, we conclude that, for the high outage probability case, the MD-based scheme outperforms the MR-based scheme. $\square$

Chapter 9

Conclusions and Further Directions

The new performance metric we have proposed in this paper, the distortion-diversity tradeoff, provides more detailed characterization of source-channel schemes. In essence, the distortion-diversity tradeoff is a re-parameterized version of the cumulative distribution function of the end-to-end distortion level achieved by source-channel schemes. This performance metric is in significant contrast to the traditional average distortion performance metric, which is essentially the statistical mean of the end-to-end distortion level. We have demonstrated the effectiveness of this distortion-diversity tradeoff framework by revisiting the two source-channel coding schemes that have been studied in [25], the MR-based and the MD-based source-channel schemes. In [25], the authors conclude that the average distortion exponent achieved by both schemes are identical. However, our results in Section 6 provides more operational intuitions. Therefore, the distortion-diversity tradeoff could be used as a powerful tool to study source-channel schemes.

Aided with the distortion-diversity tradeoff framework, we propose a triple-level source-channel scheme to unify the MR-based and the MD-based schemes. Though the triple-level scheme seems to include both the MR-based and the MD-based schemes structurally, it is not true that the triple-level scheme includes the two double-level schemes as special cases, due to the rate explosion issue. However, within the distortion-diversity framework, we are able to show that the triple-level scheme does indeed include both double-level schemes as special cases. Therefore, we conclude

that the triple-level scheme unifies the MR-based and the MD-based schemes from the distortion-diversity perspective.

Another interesting observation is that the specifically designed channel codes do not necessarily provide good end-to-end performance. In our study of the double-class broadcast network, we found that the superposition channel codes, usually regarded as good channel codes, limits our choice in source code and do not universally outperform the MD-based scheme. This performance comparison is demonstrated in Figure 6-3. One drawback in this comparison is that a joint source-channel decoder has to be implemented at the receiver for the MD-based scheme. However, we provide one example showing that a joint-source channel decoder is not always necessary for MD-based scheme to outperform MR-based scheme. In particular, we consider a 2×1 MIMO channel in the low SNR scenario and demonstrate that the MD-based scheme could outperform the MR-based scheme without breaking the source-channel interface.

Here are some future directions that could emerge from the results reported in this paper,

- Extend the distortion-diversity tradeoff to general MIMO channel models. How would the performance of the MR-based scheme compare with that of the MD-based scheme?

- Extend our single-block analysis to multiple-block scenario and compare different source-channel scheme within the distortion-diversity framework.

Appendix A

Two Useful Lemmas

The outage events in the high SNR regime often reduce to the following form,

$$\left\{ \sum_{s \in \mathcal{S}} (1 - \alpha_s)^+ < \theta \right\}, \quad \mathcal{S} \subset \mathbb{Z}^+. \tag{A.1}$$

A typical example of $\mathcal{S}$ in this paper is a subset of $\{(i, l), i = 1, 2, l = 1, \cdots, L\}$. In this paper, we only need to deal with the $|\mathcal{S}| = 1$ and $|\mathcal{S}| = 2$ cases, as we focus on a single fading block, which only has two fading coefficients $\mathbf{h}_1$ and $\mathbf{h}_2$. However, the general form (A.1) is useful in dealing with multiple fading blocks, which have more than two fading coefficients. In this section, we provide two lemmas to characterize outage events in (A.1). The first lemma reduces (A.1) to a simpler format, from which one could easily read off the boundary of the outage event. The second lemma characterizes the minimal value of $\sum_{s \in \mathcal{S}} \alpha_s$ in (A.1), which is essentially the diversity order of the outage event (A.1).

Lemma A.0.2 (Boundary of $\{\sum_{s \in \mathcal{S}} (1 - \alpha_s)^+ < \theta\}$). *Let $\mathcal{S}$ be a set of finite size. Let $\mathcal{S}_k$ denote the size-k subset of $\mathcal{S}$ ($k = 0, \cdots, |\mathcal{S}|$). For positive α_s ($s \in \mathcal{S}$) and $0 < \theta < |\mathcal{S}|$, the outage event in (A.1) can be expressed as follows,*

$$\left\{ \sum_{s \in \mathcal{S}} (1 - \alpha_s)^+ < \theta \right\} = \left\{ \sum_{s \in \mathcal{S}} (1 - \alpha_s)^+ < \theta \right\} = \bigcap_{k = \lceil \theta \rceil}^{|\mathcal{S}|} \bigcap_{\mathcal{S}_k} \left\{ \sum_{s \in \mathcal{S}_k} \alpha_s > k - \theta \right\}. \tag{A.2}$$

Remark A.0.3. *For the sake of clarity, we illustrate the special case of $\mathcal{S} = \{1, 2\}$ in Figure A-1.*
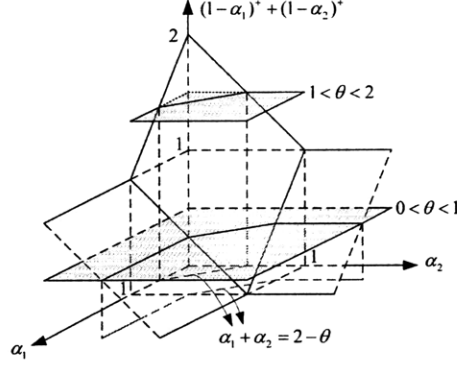

Figure A-1: Illustration of $(1 - \alpha_1)^+ + (1 - \alpha_2)^+ < \theta$

Proof. Without loss of generality, we let $\mathcal{S}$ be the index set $\mathcal{I}(n) = \{1, \cdots, n\}$, where the set size $n \in \mathbb{N}^+$. The proof proceeds by mathematical induction in the set size n. The induction step is detailed in the following.

Suppose that (A.2) holds for set $\mathcal{S}$ of size up to n. We need to show that (A.2) also holds for $\mathcal{I}(n + 1) = \mathcal{I}(n) \cup \{n + 1\}$. We abbreviate $\mathcal{I}(n)$ and its size-k subsets as $\mathcal{I}$ and $\mathcal{I}_k$ respectively, where $k = 0, \cdots, n$. We also $\mathcal{I}(n + 1)$ and its size-k subsets as $\mathcal{I}'$ and $\mathcal{I}'_k$ respectively, where $k = 0, \cdots, n + 1$. Since

$$
(1 - \alpha_{n+1})^+ = \begin{cases} 1 - \alpha_{n+1}, & \text{if } \alpha_{n+1} \leq 1, \\ 0, & \text{if } \alpha_{n+1} > 1, \end{cases}
$$

we are able to express $\left\{ \sum_{i \in \mathcal{I}'} (1 - \alpha_i)^+ < \theta \right\}$ as the following union of two sets,

$$
\left\{ \alpha_{n+1} > 1, \ \sum_{i \in \mathcal{I}} (1 - \alpha_i)^+ < \theta \right\} \cup \left\{ \alpha_{n+1} \leq 1, \ \sum_{i \in \mathcal{I}} (1 - \alpha_i)^+ < \theta - (1 - \alpha_{n+1}) \right\}.
$$

$$(A.3)$$

Since (A.2) holds for all sets of size up to n, it should apply to $\mathcal{I}$. Therefore, we can

express the first term on the right hand side of (A.3) as follows,

$$
\left\{ \alpha_{n+1} > 1, \ \sum_{i \in \mathcal{I}} (1 - \alpha_i)^+ < \theta \right\}
$$

$$
= \left\{ \alpha_{n+1} > 1 \right\} \cap \left\{ \bigcap_{k=\lceil \theta \rceil}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i \in \mathcal{I}_k} \alpha_i > k - \theta \right\} \right\}, \tag{A.4}
$$

$$
= \left\{ \alpha_{n+1} > 1, \ \bigcap_{\mathcal{I}_{\lceil \theta \rceil - 1}} \left\{ \sum_{i \in \mathcal{I}_{\lceil \theta \rceil - 1} \cup \{n+1\}} \alpha_i > \lceil \theta \rceil - \theta \right\} \right\}
$$

$$
\cap \left\{ \bigcap_{k=\lceil \theta \rceil}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i \in \mathcal{I}_k} \alpha_i > k - \theta, \ \sum_{i \in \mathcal{I}_k \cup \{n+1\}} \alpha_i > k + 1 - \theta \right\} \right\} \tag{A.5}
$$

$$
= \left\{ \alpha_{n+1} > 1, \ \bigcap_{k=\lceil \theta \rceil}^{|\mathcal{I}'|} \bigcap_{\mathcal{I}'_k} \left\{ \sum_{i \in \mathcal{I}'_k} \alpha_i > k - \theta \right\} \right\}. \tag{A.6}
$$

Step (A.4) follows from the induction assumption that (A.2) holds for all sets of size up to n, which include $\mathcal{I}$. Step (A.5) can be shown in two steps. On the one hand, (A.4) clearly includes (A.5) as a subset, since every set that shows up in the intersection of (A.4) also shows up in the intersection of (A.5). On the other hand, we need to show that (A.4) is also a subset of (A.5). Firstly, since $\alpha_{n+1} > 1$ implies $\alpha_{n+1} > \lceil \theta \rceil - \theta$, which together with $\alpha_i > 0$, implies that $\sum_{i \in \mathcal{I}_{\lceil \theta \rceil - 1} \cup \{n+1\}} \alpha_i > \lceil \theta \rceil - \theta$ for any $\mathcal{I}_{\lceil \theta \rceil - 1}$, we then have that

$$
\{ \alpha_{n+1} > 1 \} \subset \left\{ \alpha_{n+1} > 1, \ \bigcap_{\mathcal{I}_{\lceil \theta \rceil - 1}} \left\{ \sum_{i \in \mathcal{I}_{\lceil \theta \rceil - 1} \cup \{n+1\}} \alpha_i > \lceil \theta \rceil - \theta \right\} \right\}. \tag{A.7}
$$

Secondly, for each $\mathcal{I}_k$, $\sum_{i \in \mathcal{I}_k} \alpha_i > k - \theta$ and $\alpha_{n+1} > 1$ imply $\sum_{i \in \mathcal{I}_k \cup \{n+1\}} \alpha_i > k + 1 - \theta$, we then have that

$$
\left\{ \bigcap_{k=\lceil \theta \rceil}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i \in \mathcal{I}_k} \alpha_i > k - \theta \right\} \right\} \subset \left\{ \bigcap_{k=\lceil \theta \rceil}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i \in \mathcal{I}_k} \alpha_i > k - \theta, \ \sum_{i \in \mathcal{I}_k \cup \{n+1\}} \alpha_i > k + 1 - \theta \right\} \right\}.
$$
$$\tag{A.8}$$

Combining (A.7) and (A.8), we have that (A.4) is also a subset of (A.5). Therefore,

the equality in (A.5) follows. Step (A.6) follows by noting that

$$
\{\mathcal{I}_k'\} = \begin{cases} \{\mathcal{I}_k, \ \mathcal{I}_{k-1} \cup \{n+1\}\}, & k = \lceil\theta\rceil, \cdots, n, \\[2ex] \mathcal{I} \cup \{n+1\}, & k = n+1. \end{cases} \tag{A.9}
$$

The second term on the right hand side of (A.3) can be expressed as follows,

$$
\left\{ \alpha_{n+1} \le 1, \ \sum_{i \in \mathcal{I}} (1 - \alpha_i)^+ < \theta - (1 - \alpha_{n+1}) \right\}
$$

$$
= \left\{ \alpha_{n+1} \le 1, \ \bigcap_{k=\lceil\theta-(1-\alpha_{n+1})\rceil}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i \in \mathcal{I}_k} \alpha_i > k - \theta + (1 - \alpha_{n+1}) \right\} \right\} \tag{A.10}
$$

$$
= \left\{ \lceil\theta\rceil - \theta < \alpha_{n+1} \le 1, \ \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i \in \mathcal{I}_k \cup \{n+1\}} \alpha_i > k + 1 - \theta \right\} \right\}
$$

$$
\cup \left\{ \alpha_{n+1} < \lceil\theta\rceil - \theta, \ \bigcap_{k=\lceil\theta\rceil-1}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i \in \mathcal{I}_k \cup \{n+1\}} \alpha_i > k + 1 - \theta \right\} \right\}. \tag{A.11}
$$

Step (A.10) follows by applying (A.2) to $\{\sum_{i \in \mathcal{I}}(1 - \alpha_i)^+ < \theta - (1 - \alpha_{n+1})\}$. Step (A.11) follows by breaking $\alpha_{n+1} \le 1$ into two regions: if $\lceil\theta\rceil - \theta < \alpha_{n+1} \le 1$, then $\lceil\theta - (1 - \alpha_{n+1})\rceil = \lceil\theta\rceil$, while if $\alpha_{n+1} < \lceil\theta\rceil - \theta$, then $\lceil\theta - (1 - \alpha_{n+1})\rceil = \lceil\theta\rceil - 1$. Now, let us consider the first term of (A.11). Firstly, since $\alpha_{n+1} > \lceil\theta\rceil - \theta$ implies $\bigcap_{\mathcal{I}_{\lceil\theta\rceil-1}} \left\{ \sum_{i \in \mathcal{I}_{\lceil\theta\rceil-1} \cup \{n+1\}} \alpha_i > \lceil\theta\rceil - \theta \right\}$, we have that

$$
\{\lceil\theta\rceil - \theta < \alpha_{n+1} \le 1\} = \left\{ \lceil\theta\rceil - \theta < \alpha_{n+1} \le 1, \ \bigcap_{\mathcal{I}_{\lceil\theta\rceil-1}} \left\{ \sum_{i \in \mathcal{I}_{\lceil\theta\rceil-1} \cup \{n+1\}} \alpha_i > \lceil\theta\rceil - \theta \right\} \right\}. \tag{A.12}
$$

Secondly, since $\lceil\theta\rceil - \theta < \alpha_{n+1} \le 1$ and $\bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i \in \mathcal{I}_k \cup \{n+1\}} \alpha_i > k + 1 - \theta \right\}$

imply $\bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}|}\bigcap_{\mathcal{I}_k}\left\{\sum_{i\in\mathcal{I}_k}\alpha_i>k-\theta\right\}$, we then have that

$$\left\{\lceil\theta\rceil-\theta<\alpha_{n+1}<1,\ \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}|}\bigcap_{\mathcal{I}_k}\left\{\sum_{i\in\mathcal{I}_k\cup\{n+1\}}\alpha_i>k+1-\theta\right\}\right\}$$

$$=\left\{\lceil\theta\rceil-\theta<\alpha_{n+1}<1,\ \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}|}\bigcap_{\mathcal{I}_k}\left\{\sum_{i\in\mathcal{I}_k\cup\{n+1\}}\alpha_i>k+1-\theta,\sum_{i\in\mathcal{I}_k}\alpha_i>k-\theta\right\}\right\} \quad (A.13)$$

Combining (A.12) and (A.13), we can express the first term of (A.11) as follows,

$$\left\{\lceil\theta\rceil-\theta<\alpha_{n+1}<1,\ \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}|}\bigcap_{\mathcal{I}_k}\left\{\sum_{i\in\mathcal{I}_k\cup\{n+1\}}\alpha_i>k+1-\theta\right\}\right\}$$

$$=\left\{\lceil\theta\rceil-\theta<\alpha_{n+1}<1,\ \bigcap_{k=\lceil\theta\rceil-1}^{|\mathcal{I}|}\bigcap_{\mathcal{I}_k}\left\{\sum_{i\in\mathcal{I}_k\cup\{n+1\}}\alpha_i>\lceil\theta\rceil-\theta\right\},\right.$$

$$\left.\bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}|}\bigcap_{\mathcal{I}_k}\left\{\sum_{i\in\mathcal{I}_k\cup\{n+1\}}\alpha_i>k+1-\theta,\sum_{i\in\mathcal{I}_k}\alpha_i>k-\theta\right\}\right\}$$

$$=\left\{\lceil\theta\rceil-\theta<\alpha_{n+1}<1,\ \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}'|}\bigcap_{\mathcal{I}'_k}\left\{\sum_{i\in\mathcal{I}'_k}\alpha_i>k-\theta\right\}\right\}, \quad (A.14)$$

where the last step follows from (A.9). Now, let us consider the second term of (A.11). Since $\alpha_{n+1}<\lceil\theta\rceil-\theta$ implies $\alpha_{n+1}<1$, which, together with $\sum_{i\in\mathcal{I}_k\cup\{n+1\}}\alpha_i>k+1-\theta$, implies that $\sum_{i\in\mathcal{I}_k}\alpha_i>k-\theta$, we then have that

$$\left\{\alpha_{n+1}<\lceil\theta\rceil-\theta,\ \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}|}\bigcap_{\mathcal{I}_k}\left\{\sum_{i\in\mathcal{I}_k\cup\{n+1\}}\alpha_i>k+1-\theta\right\}\right\}$$

$$=\left\{\alpha_{n+1}<\lceil\theta\rceil-\theta,\ \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}|}\bigcap_{\mathcal{I}_k}\left\{\sum_{i\in\mathcal{I}_k\cup\{n+1\}}\alpha_i>k+1-\theta,\sum_{i\in\mathcal{I}_k}\alpha_i>k-\theta\right\}\right\},$$

Therefore, we can express the second term of (A.11) as follows,

$$
\left\{ \alpha_{n+1} < \lceil\theta\rceil - \theta, \bigcap_{k=\lceil\theta\rceil-1}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i\in\mathcal{I}_k\cup\{n+1\}} \alpha_i > k+1-\theta \right\} \right\}
$$

$$
= \left\{ \alpha_{n+1} < \lceil\theta\rceil - \theta, \bigcap_{k=\lceil\theta\rceil-1}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i\in\mathcal{I}_k\cup\{n+1\}} \alpha_i > k+1-\theta \right\}, \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}|} \bigcap_{\mathcal{I}_k} \left\{ \sum_{i\in\mathcal{I}_k} \alpha_i > k-\theta \right\} \right\}
$$

$$
= \left\{ \alpha_{n+1} < \lceil\theta\rceil - \theta, \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}'|} \bigcap_{\mathcal{I}'_k} \left\{ \sum_{i\in\mathcal{I}'_k} \alpha_i > k-\theta \right\} \right\}, \tag{A.15}
$$

where the last step follows from (A.9). Combining (A.6), (A.14) and (A.15), we have that

$$
\left\{ \sum_{i\in\mathcal{I}'} (1-\alpha_i)^+ < \theta \right\}
$$

$$
= \left\{ \alpha_{n+1} > 1, \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}'|} \bigcap_{\mathcal{I}'_k} \left\{ \sum_{i\in\mathcal{I}'_k} \alpha_i > k-\theta \right\} \right\} \cup \left\{ \lceil\theta\rceil - \theta < \alpha_{n+1} < 1, \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}'|} \bigcap_{\mathcal{I}'_k} \left\{ \sum_{i\in\mathcal{I}'_k} \alpha_i > k-\theta \right\} \right\}
$$

$$
\cup \left\{ \alpha_{n+1} < \lceil\theta\rceil - \theta, \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}'|} \bigcap_{\mathcal{I}'_k} \left\{ \sum_{i\in\mathcal{I}'_k} \alpha_i > k-\theta \right\} \right\}
$$

$$
= \left\{ \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}'|} \bigcap_{\mathcal{I}'_k} \left\{ \sum_{i\in\mathcal{I}'_k} \alpha_i > k-\theta \right\} \right\},
$$

where the last step follows by applying the distributive law of set algebra, $(\mathcal{A}\cap\mathcal{B})\cup(\mathcal{A}\cap\mathcal{C})\cup(\mathcal{A}\cap\mathcal{D}) = \mathcal{A}\cap(\mathcal{B}\cup\mathcal{C}\cup\mathcal{D})$, where

$$
\mathcal{A} = \bigcap_{k=\lceil\theta\rceil}^{|\mathcal{I}'|} \bigcap_{\mathcal{I}'_k} \left\{ \sum_{i\in\mathcal{I}'_k} \alpha_i > k-\theta \right\}
$$

and

$$
\mathcal{B} = \{\alpha_{n+1} > 1\}, \quad \mathcal{C} = \{\lceil\theta\rceil - \theta < \alpha_{n+1} < 1\}, \quad \mathcal{D} = \{\alpha_{n+1} < \lceil\theta\rceil - \theta\}.
$$

$\square$

Lemma A.0.4. *Let $\mathcal{S}$ be a finite set and $\alpha_s(s \in \mathcal{S})$ be positive. The diversity order of the outage event $\{\sum_{s \in \mathcal{S}}(1 - \alpha_s)^+ < \theta\}$ is,*

$$\min_{\sum_{s \in \mathcal{S}}(1-\alpha_s)^+ < \theta} \sum_{s \in \mathcal{S}} \alpha_s = |\mathcal{S}| - \theta, \quad 0 < \theta < |\mathcal{S}| \ .$$

Proof.

$$
\begin{aligned}
\min_{\sum_{s \in \mathcal{S}}(1-\alpha_s)^+ < \theta} \sum_{s \in \mathcal{S}} \alpha_s \quad &\overset{\text{Lemma } A.0.2}{=} \quad \min_{\cap_{k=\lceil \theta \rceil}^{|\mathcal{S}|} \cap_{\mathcal{S}_k} \{\sum_{s \in \mathcal{S}_k} \alpha_s > k-\theta\}} \sum_{s \in \mathcal{S}} \alpha_s \\
&\geq \quad \max_{\substack{\mathcal{S}_k \\ k=\lceil \theta \rceil, \cdots, |\mathcal{S}|}} \min_{\sum_{s \in \mathcal{S}_k} \alpha_s > k-\theta} \sum_{s \in \mathcal{S}} \alpha_s \\
&= \quad \max_{k=\lceil \theta \rceil, \cdots, |\mathcal{S}|} k - \theta \\
&= \quad |\mathcal{S}| - \theta,
\end{aligned}
$$

which can be achieved by $\alpha_s = 1 - \frac{\theta}{|\mathcal{S}|}$ $(\forall \ s \in \mathcal{S})$. $\qquad \square$

Appendix B

Proof of Lemma 6.1.1

Proof. The partial diversity order can be simplified as follows,

$$
\begin{aligned}
\delta_p(d_p, d_f, \beta) \quad &= \quad \min_{\alpha_i \geq 0, \sum_{i=1,2}((1-\alpha_i)^+ - (1-\beta-\alpha_i)^+) < d_p} \sum_{i=1,2} \alpha_i \\
&= \quad \min_{\alpha_i \geq 0, \sum_{i=1,2}\left(\beta - (\beta - (1-\alpha_i)^+)^+\right) < d_p} \sum_{i=1,2} \alpha_i \qquad \text{(B.1)} \\
&= \quad \min_{\alpha_i \geq 0, \sum_{i=1,2}(\beta-(1-\alpha_i)^+)^+ > 2\beta - d_p} \sum_{i=1,2} \alpha_i \,, \\
&\overset{\text{Lemma }A.0.2}{=} \quad \min_{\alpha_i \geq 0, \, \bigcup_{k=\lceil 2-d_p/\beta\rceil}^{2} \bigcup_{\mathcal{S}_k}\left\{\sum_{i\in\mathcal{S}_k}(1-\alpha_i)^+ < k\beta-(2\beta-d_p)\right\}} \sum_{i=1,2} \alpha_i \\
&= \quad \min_{k=\lceil 2-d_p/\beta\rceil,\cdots,2} \, \min_{\mathcal{S}_k} \, \min_{\alpha_i \geq 0, \sum_{i\in\mathcal{S}_k}(1-\alpha_i)^+ < k\beta-(2\beta-d_p)} \sum_{i=1,2} \alpha_i \\
&\overset{\text{Lemma }A.0.4}{=} \quad \min_{k=\lceil 2-d_p/\beta\rceil,\cdots,2} k(1-\beta) + (2\beta - d_p) \\
&= \quad (2 - d_p) + \lceil -d_p/\beta\rceil (1 - \beta) \,,
\end{aligned}
$$

where $0 < d_p < 2\beta$. Note that step (B.1) follows from the following equality,

$$
(1 - \alpha_{i,l})^+ - (1 - \alpha_{i,l} - \beta)^+ = \beta - \left(\beta - (1 - \alpha_{i,l})^+\right)^+ .
$$

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Similarly, the full diversity order can be simplified as follows,

$$
\begin{aligned}
\delta_f(d_p, d_f, \beta) \quad &= \quad \min\left\{ \delta_p(d_p, d_f, \beta),\ \delta_f'(d_p, d_f, \beta) \right\} \\[2mm]
&= \quad \min\left\{ \delta_p(d_p, d_f, \beta),\ \min_{\alpha_i \geq 0,\ \sum_{i=1,2}(1-\beta-\alpha_i)^+ < d_f - d_p}\ \sum_{i=1,2} \alpha_i \right\} \\[2mm]
&= \quad \min\left\{ \delta_p(d_p, d_f, \beta),\ \min_{\alpha_i \geq 0,\ \sum_{i=1,2}(1-\frac{\alpha_i}{1-\beta})^+ < \frac{d_f - d_p}{1-\beta}}\ \sum_{i=1,2} \alpha_i \right\} \\[2mm]
&\overset{\text{Lemma } A.0.4}{=} \quad \min\left\{ \delta_p(d_p, d_f, \beta),\ 2(1-\beta) - (d_f - d_p) \right\},
\end{aligned}
$$

where $0 < d_f - d_p < 2(1-\beta)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Appendix C

Proof of Lemma 6.2.1

We first repeat Lemma 6.2.1 as follows,

Lemma C.0.5. *The symmetric MD-based source-channel scheme achieves two levels of distortion exponents d_p, d_f with the following diversity orders,*

$$
\begin{aligned}
\delta_p &= 2 - d_f, \\
\delta_f &= \min\left\{ 2 - d_f, \, 1 - \frac{d_f - d_p}{2} \right\},
\end{aligned}
$$

where $0 < d_p < d_f < 2$.

Proof. By applying Lemma A.0.2, we simplify the full outage event as follows,

$$
\begin{aligned}
\mathcal{O}_f(d_p, d_f) &= \left\{ \sum_{i=1,2} \left(\frac{d_p + d_f}{2} - (1 - \alpha_i)^+ \right)^+ > d_p \right\} \\
&= \left\{ \sum_{i=1,2} \left(1 - \frac{2}{d_p + d_f}(1 - \alpha_i)^+ \right)^+ < \frac{2d_p}{d_p + d_f} \right\}^c \\
&= \bigcup_{i=1,2} \left\{ (1 - \alpha_i)^+ < \frac{d_f - d_p}{2} \right\} \bigcup \left\{ \sum_{i=1,2} (1 - \alpha_i)^+ < d_f \right\} \quad \text{(C.1)} \\
&= \mathcal{O}'_1 \cup \mathcal{O}'_2 \cup \mathcal{O}'_3,
\end{aligned}
$$

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where

$$\mathcal{O}'_i = \left\{ (1 - \alpha_i)^+ < \frac{d_f - d_p}{2} \right\}, \quad i = 1, 2,$$

$$\mathcal{O}'_3 = \left\{ \sum_{i=1,2} (1 - \alpha_i)^+ < d_f \right\}.$$

By applying Lemma A.0.4, we derive the full diversity order as follows,

$$
\begin{aligned}
\delta_f &= \min_{\mathcal{O}_f} \sum_i \alpha_i \\
&= \min \left\{ \min_{\mathcal{O}'_1} \sum_i \alpha_i, \ \min_{\mathcal{O}'_2} \sum_i \alpha_i, \ \min_{\mathcal{O}'_3} \sum_i \alpha_i \right\} \\
&= \min \left\{ 1 - \frac{d_f - d_p}{2}, 1 - \frac{d_f - d_p}{2}, 2 - d_f \right\} \\
&= \min \left\{ 1 - \frac{d_f - d_p}{2}, 2 - d_f \right\}.
\end{aligned}
$$

Similarly, with the following notation,

$$\mathcal{O}_i = \left\{ (1 - \alpha_i)^+ < \frac{d_p + d_f}{2} \right\},$$

we can simplify the partial outage event as follows,

$$
\begin{aligned}
\mathcal{O}_p(d_p, d_f) &= \cap_{i=1,2} \left\{ (1 - \alpha_i)^+ < \frac{d_p + d_f}{2} \right\} \cap \mathcal{O}_f(D_p, D_f) \\
&= \mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}_f \\
&= (\mathcal{O}_1 \cap \mathcal{O}_2) \cap (\mathcal{O}'_1 \cup \mathcal{O}'_2 \cup \mathcal{O}'_3) \\
&\overset{\text{distributive}}{=} (\mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}'_1) \cup (\mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}'_2) \cup (\mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}'_3) \\
&\overset{\substack{\mathcal{O}'_1 \subset \mathcal{O}_1 \\ \mathcal{O}'_2 \subset \mathcal{O}_2}}{=} (\mathcal{O}'_1 \cap \mathcal{O}_2) \cup (\mathcal{O}_1 \cap \mathcal{O}'_2) \cup (\mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}'_3) \\
&\overset{\substack{\mathcal{O}_1 \cap \mathcal{O}'_2 \subset \mathcal{O}'_3 \\ \mathcal{O}'_1 \cap \mathcal{O}_2 \subset \mathcal{O}'_3}}{=} (\mathcal{O}'_1 \cap \mathcal{O}_2 \cap \mathcal{O}'_3) \cup (\mathcal{O}_1 \cap \mathcal{O}'_2 \cap \mathcal{O}'_3) \cup (\mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}'_3) \\
&\overset{\text{distributive}}{=} ((\mathcal{O}'_1 \cap \mathcal{O}_2) \cup (\mathcal{O}_1 \cap \mathcal{O}'_2) \cup (\mathcal{O}_1 \cap \mathcal{O}_2)) \cap \mathcal{O}'_3 \\
&\overset{\substack{\mathcal{O}'_1 \subset \mathcal{O}_1 \\ \mathcal{O}'_2 \subset \mathcal{O}_2}}{=} \mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}'_3.
\end{aligned}
\tag{C.2}
$$

Therefore, the partial diversity order can be lower bounded as follows,

$$
\begin{aligned}
\delta_p \quad &= \quad \min_{\mathcal{O}_p} \sum_i \alpha_i \\
&= \quad \min_{\mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}_3'} \sum_i \alpha_i \\
&\geq \quad \max\left\{ \min_{\mathcal{O}_1} \sum_i \alpha_i, \ \min_{\mathcal{O}_2} \sum_i \alpha_i, \ \min_{\mathcal{O}_3'} \sum_i \alpha_i \right\} \\
&\overset{\text{Lemma } A.0.4}{=} \quad \max\left\{ 1 - \frac{d_f - d_p}{2}, \ 1 - \frac{d_f - d_p}{2}, \ 2 - d_f \right\} \\
&= \quad 2 - d_f .
\end{aligned}
$$

Moreover, this lower bound is achieved by $\alpha_1 = \alpha_2 = 1 - d_f/2$. Hence, we conclude that $\delta_p = 2 - d_f$. $\qquad\square$

Appendix D

Proof of Lemma 7.0.1

Proof. Preceding Lemma 7.0.1, we have obtained the following expressions for the partial, full and refinement outage events,

$$\mathcal{O}_f(d_p, d_f, d_r, \beta) \doteq \left\{ \sum_{i=1,2} \left(\lim_{\mathsf{SNR} \to \infty} \frac{R_{s,b}}{\log \mathsf{SNR}} - g(\alpha_i, \beta) \right)^+ > \lim_{\mathsf{SNR} \to \infty} \frac{I(\mathsf{s}_{b,1}; \mathsf{s}_{b,2})}{\log \mathsf{SNR}} \right\},$$

$$\mathcal{O}_p(d_p, d_f, d_r, \beta) \doteq \mathcal{O}_f(d_p, d_f, d_r, \beta) \cap \cap_{i=1,2} \left\{ g(\alpha_i, \beta) < \lim_{\mathsf{SNR} \to \infty} \frac{R_{s,b}}{\log \mathsf{SNR}} \right\},$$

$$\mathcal{O}_r(d_p, d_f, d_r, \beta) \doteq \mathcal{O}_f(d_p, d_f, d_r, \beta) \cup \left\{ \sum_{i=1,2} (1 - \beta - \alpha_i)^+ < \lim_{\mathsf{SNR} \to \infty} \frac{\log \frac{D_f}{D_r}}{\log \mathsf{SNR}} \right\},$$

where

$$g(\alpha_i, \beta) = (1 - \alpha_i)^+ - (1 - \beta - \alpha_i)^+.$$

In the following, we treat two cases, $d_p < d_f$ and $d_p = d_f$, separately. We first characterize the full outage event and diversity order.

- Case 1: $0 \le d_p < d_f \le d_r$. The full outage event is

$$\mathcal{O}_f(d_p, d_f, d_r, \beta) \doteq \left\{ \sum_{i=1,2} \left(\lim_{\mathsf{SNR} \to \infty} \frac{R_{s,b}}{\log \mathsf{SNR}} - g(\alpha_i, \beta) \right)^+ > \lim_{\mathsf{SNR} \to \infty} \frac{I(\mathsf{s}_{b,1}; \mathsf{s}_{b,2})}{\log \mathsf{SNR}} \right\}.$$

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Let $D_p = \mathsf{SNR}^{-d_p}$, $D_f = \mathsf{SNR}^{-d_f}$ and $D_R = \mathsf{SNR}^{-d_r}$. Since $0 \le d_p < d_f \le d_r$,

$$\lim_{\mathsf{SNR}\to\infty} \frac{R_{s,b}}{\log\mathsf{SNR}} = \frac{d_p + d_f}{2}, \qquad \lim_{\mathsf{SNR}\to\infty} \frac{I(\mathbf{s}_{b,1};\mathbf{s}_{b,2})}{\log\mathsf{SNR}} = d_p.$$

The full outage event is then simplified to be

$$
\begin{aligned}
\mathcal{O}_f(d_p, d_f, d_r, \beta) &= \left\{ \sum_{i=1,2} \left(\frac{d_p + d_f}{2} - g(\alpha_i, \beta) \right)^+ > d_p \right\} \\
&= \left\{ \sum_{i=1,2} \left(1 - \frac{2}{d_p + d_f} g(\alpha_i, \beta) \right)^+ > \frac{2d_p}{d_p + d_f} \right\} \\
&= \cup_{i=1,2} \left\{ g(\alpha_i, \beta) < \frac{d_f - d_p}{2} \right\} \cup \left\{ \sum_{i=1,2} g(\alpha_i, \beta) < d_f \right\},
\end{aligned}
$$

where, in the last step, we have applied Lemma A.0.2. The first term can be simplified as follows,

$$
\begin{aligned}
\left\{ g(\alpha_i, \beta) < \frac{d_f - d_p}{2} \right\} &= \left\{ \beta - \left(\beta - (1 - \alpha_i)^+ \right)^+ < \frac{d_f - d_p}{2} \right\} \\
&= \left\{ \left(\beta - (1 - \alpha_i)^+ \right)^+ > \beta - \frac{d_f - d_p}{2} \right\} \\
&= \left\{ \left(1 - \frac{1}{\beta}(1 - \alpha_i)^+ \right)^+ > 1 - \frac{d_f - d_p}{2\beta} \right\},
\end{aligned}
$$

while the second term can be simplified as follows,

$$
\begin{aligned}
\left\{ \sum_{i=1,2} g(\alpha_i, \beta) < d_f \right\} &= \left\{ \sum_{i=1,2} \left(\beta - \left(\beta - (1 - \alpha_i)^+ \right)^+ \right) < d_f \right\} \\
&= \left\{ \sum_{i=1,2} \left(\beta - (1 - \alpha_i)^+ \right)^+ > 2\beta - d_f \right\} \\
&= \left\{ \sum_{i=1,2} \left(1 - \frac{1}{\beta}(1 - \alpha_i)^+ \right)^+ > 2 - \frac{d_f}{\beta} \right\}.
\end{aligned}
$$

Therefore, the full outage event reduces to

$$
\mathcal{O}_f(d_p, d_f, d_r, \beta)
$$

$$
= \bigcup_{i=1,2} \left\{ \left(1 - \frac{1}{\beta}(1-\alpha_i)^+\right)^+ > 1 - \frac{d_f - d_p}{2\beta} \right\} \cup \left\{ \sum_{i=1,2} \left(1 - \frac{1}{\beta}(1-\alpha_i)^+\right)^+ > 2 - \frac{d_f}{\beta} \right\}
$$

$$
= \begin{cases}
\{\alpha_i \geq 0\}, & \text{if } \beta < \frac{d_f}{2}, \\
\bigcup_{i=1,2} \left\{(1-\alpha_i)^+ < \frac{d_f - d_p}{2}\right\} \cup \bigcup_{i=1,2} \left\{(1-\alpha_i)^+ < d_f - \beta\right\} \\
\quad \cup \left\{\sum_{i=1,2}(1-\alpha_i)^+ < d_f\right\}, & \text{if } \frac{d_f}{2} < \beta < d_f, \\
\bigcup_{i=1,2} \left\{(1-\alpha_i)^+ < \frac{d_f - d_p}{2}\right\} \cup \left\{\sum_{i=1,2}(1-\alpha_i)^+ < d_f\right\}, & \text{if } \beta > d_f.
\end{cases}
$$

Hence, the full diversity order is characterized as follows,

$$
\delta_f(d_p, d_f, d_r, \beta) = \begin{cases}
0, & \text{if } \beta < \frac{d_f}{2}, \\
\min\left\{1 - \frac{d_f - d_p}{2}, 1 + \beta - d_f, 2 - d_f\right\}, & \text{if } \frac{d_f}{2} < \beta < d_f, \\
\min\left\{1 - \frac{d_f - d_p}{2}, 2 - d_f\right\}, & \text{if } \beta > d_f.
\end{cases}
$$

$$
= \begin{cases}
0, & \text{if } \beta < \frac{d_f}{2}, \\
\min\left\{1 - \frac{d_f - d_p}{2}, 1 + \beta - d_f\right\}, & \text{if } \frac{d_f}{2} < \beta < d_f, \\
\min\left\{1 - \frac{d_f - d_p}{2}, 2 - d_f\right\}, & \text{if } \beta > d_f,
\end{cases}
$$

since $\beta < 1$ implies that $1 + \beta - d_f < 2 - d_f$;

- If $0 \leq d_p = d_f \leq d_r$, we let $d \triangleq d_p = d_f$, $D_p = \mathsf{SNR}^{-d}$, $D_r = \mathsf{SNR}^{-d_r}$ and

$$
- \lim_{\mathsf{SNR} \to \infty} \frac{\log(D_p - D_f)}{\log \mathsf{SNR}} = d + \theta, \quad \theta > 0,
$$

where we essentially characterize that the second smallest exponent of D_f is $d + \theta$. With this distortion assignment, we have the following rates characterization,

$$
R_{s,b} \doteq (d + \theta/2) \log \mathsf{SNR},
$$

$$
R_{s,r} \doteq (d_r - d_f) \log \mathsf{SNR},
$$

$$
I(\mathsf{s}_{b,1}; \mathsf{s}_{b,2}) \doteq (d + \theta) \log \mathsf{SNR}.
$$

The full outage event is then characterized as follows,

$$
\mathcal{O}_f(d_p, d_f, d_r, \beta)
$$

$$
\doteq \quad \left\{ \sum_{i=1,2} \left(\lim_{\mathsf{SNR} \to \infty} \frac{R_{s,b}}{\log \mathsf{SNR}} - g(\alpha_i, \beta) \right)^+ > \lim_{\mathsf{SNR} \to \infty} \frac{I(\mathsf{s}_{b,1}; \mathsf{s}_{b,2})}{\log \mathsf{SNR}} \right\}
$$

$$
= \quad \left\{ \sum_{i=1,2} \left((d + \frac{\theta}{2}) - g(\alpha_i, \beta) \right)^+ > d + \theta \right\}
$$

$$
\overset{\text{Lemma } A.0.2}{=} \quad \left\{ \sum_{i=1,2} g(\alpha_i, \beta) < d \right\}
$$

$$
= \quad \left\{ \sum_{i=1,2} \left(\beta - (1 - \alpha_i)^+ \right)^+ > 2\beta - d \right\}
$$

$$
= \quad \begin{cases} \{\alpha_i \geq 0\}, & \text{if } \beta < \frac{d}{2}, \\ \cup_{i=1,2} \{(1 - \alpha_i)^+ < d - \beta\} \cup \left\{ \sum_{i=1,2}(1 - \alpha_i)^+ < d \right\}, & \text{if } \frac{d}{2} < \beta < d, \\ \left\{ \sum_{i=1,2}(1 - \alpha_i)^+ < d \right\}, & \text{if } \beta > d. \end{cases}
$$

Therefore, if $0 \leq d_p = d_f (\overset{\triangleq}{=} d) \leq d_r$, the full diversity order is

$$
\delta_f = \begin{cases} 0, & \text{if } \beta < \frac{d}{2}, \\ 1 + \beta - d, & \text{if } \frac{d}{2} < \beta < d, \\ 2 - d, & \text{if } \beta > d. \end{cases}
$$

The partial outage event and the partial diversity order are characterized as follows,

- If $0 \leq d_p < d_f \leq d_r$, we let $D_p = \mathsf{SNR}^{-d_p}$, $D_f = \mathsf{SNR}^{-d_f}$ and $D_R = \mathsf{SNR}^{-d_r}$. The partial outage event is characterized as follows,

$$
\mathcal{O}_p(d_p, d_f, d_r, \beta)
$$

$$
= \quad \mathcal{O}_f(d_p, d_f, d_r, \beta) \cap \cap_{i=1,2} \left\{ g(\alpha_i, \beta) < \frac{d_p + d_f}{2} \right\}
$$

$$
= \quad \left(\cup_{i=1,2} \left\{ g(\alpha_i, \beta) < \frac{d_f - d_p}{2} \right\} \cup \left\{ \sum_{i=1,2} g(\alpha_i, \beta) < d_f \right\} \right) \cap \cap_{i=1,2} \left\{ g(\alpha_i, \beta) < \frac{d_p + d_f}{2} \right\}
$$

$$
= \quad \left\{ \sum_{i=1,2} g(\alpha_i, \beta) < d_f \right\} \cap \cap_{i=1,2} \left\{ g(\alpha_i, \beta) < \frac{d_p + d_f}{2} \right\},
$$

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where the last step follows since $\cup_i \left\{ g(\alpha_i, \beta) < \frac{d_f - d_p}{2} \right\}$ together with $\cap_i \left\{ g(\alpha_i, \beta) < \frac{d_p + d_f}{2} \right\}$ implies $\sum_{i=1,2} g(\alpha_i, \beta) < d_f$. The partial outage event can be further simplified as follows,

$$
\begin{aligned}
& \mathcal{O}_p(d_p, d_f, d_r, \beta) \\
=\ & \left\{ \sum_{i=1,2} g(\alpha_i, \beta) < d_f \right\} \cap \cap_{i=1,2} \left\{ g(\alpha_i, \beta) < \frac{d_p + d_f}{2} \right\} \\
=\ & \left\{ \sum_{i=1,2} \left(\beta - (1 - \alpha_i)^+ \right)^+ > 2\beta - d_f \right\} \cap \cap_{i=1,2} \left\{ \left(\beta - (1 - \alpha_i)^+ \right)^+ > \beta - \frac{d_p + d_f}{2} \right\} \\
=\ & \begin{cases}
\{\alpha_i \geq 0\}, & \text{if } \beta < \frac{d_f}{2}, \\[2mm]
\left\{ \sum_{i=1,2} \left(1 - \frac{1}{\beta}(1 - \alpha_i)^+ \right)^+ > 2 - \frac{d_f}{\beta} \right\}, & \text{if } \frac{d_f}{2} < \beta < \frac{d_p}{} \\[2mm]
\left\{ \sum_{i=1,2} \left(1 - \frac{1}{\beta}(1 - \alpha_i)^+ \right)^+ > 2 - \frac{d_f}{\beta} \right\} \cap \cap_{i=1,2} \left\{ (1 - \alpha_i)^+ < \frac{d_p + d_f}{2} \right\}, & \text{if } \beta > \frac{d_p + d_f}{2}.
\end{cases}
\end{aligned}
$$

which can be further categorized as follows,

$$
=\ \begin{cases}
\{\alpha_i \geq 0\}, & \text{if } \beta < \frac{d_f}{2}, \\[2mm]
\cup_{i=1,2} \left\{ \frac{1}{\beta}(1 - \alpha_i)^+ < \frac{d_f}{\beta} - 1 \right\} \cup \left\{ \sum_{i=1,2} \frac{1}{\beta}(1 - \alpha_i)^+ < \frac{d_f}{\beta} \right\}, & \text{if } \frac{d_f}{2} < \beta < \frac{d_p + d_f}{2}, \\[2mm]
\left(\cup_{i=1,2} \left\{ \frac{1}{\beta}(1 - \alpha_i)^+ < \frac{d_f}{\beta} - 1 \right\} \cup \left\{ \sum_{i=1,2} \frac{1}{\beta}(1 - \alpha_i)^+ < \frac{d_f}{\beta} \right\} \right) & \\
\quad \cap \cap_{i=1,2} \left\{ (1 - \alpha_i)^+ < \frac{d_p + d_f}{2} \right\}, & \text{if } \frac{d_p + d_f}{2} < \beta < d_f \\[2mm]
\left\{ \sum_{i=1,2} \frac{1}{\beta}(1 - \alpha_i)^+ < \frac{d_f}{\beta} \right\} \cap \cap_{i=1,2} \left\{ (1 - \alpha_i)^+ < \frac{d_p + d_f}{2} \right\}, & \text{if } \beta > d_f.
\end{cases}
$$

$$
=\ \begin{cases}
\{\alpha_i \geq 0\}, & \text{if } \beta < \frac{d_f}{2}, \\[2mm]
\cup_{i=1,2} \left\{ \frac{1}{\beta}(1 - \alpha_i)^+ < \frac{d_f}{\beta} - 1 \right\} \cup \left\{ \sum_{i=1,2} \frac{1}{\beta}(1 - \alpha_i)^+ < \frac{d_f}{\beta} \right\}, & \text{if } \frac{d_f}{2} < \beta < \frac{d_p + d_f}{2}, \\[2mm]
\left\{ \sum_{i=1,2} \frac{1}{\beta}(1 - \alpha_i)^+ < \frac{d_f}{\beta} \right\} \cap \cap_{i=1,2} \left\{ (1 - \alpha_i)^+ < \frac{d_p + d_f}{2} \right\}, & \text{if } \beta > \frac{d_p + d_f}{2}.
\end{cases}
$$

Therefore, the partial diversity order is

$$
\delta_p = \begin{cases} 0, & \text{if } \beta < \frac{d_f}{2}, \\ \min\left\{1 + \beta - d_f, 2 - d_f\right\}, & \text{if } \frac{d_f}{2} < \beta < \frac{d_p + d_f}{2}, \\ \max\left\{1 - \frac{d_p + d_f}{2}, 2 - d_f\right\}, & \text{if } \beta > \frac{d_p + d_f}{2}. \end{cases}
$$

$$
= \begin{cases} 0, & \text{if } \beta < \frac{d_f}{2}, \\ 1 + \beta - d_f, & \text{if } \frac{d_f}{2} < \beta < \frac{d_p + d_f}{2}, \\ 2 - d_f, & \text{if } \beta > \frac{d_p + d_f}{2}. \end{cases} \tag{D.1}
$$

- If $0 \le d_p = d_f \le d_r$, we let $d \triangleq d_p = d_f$, $D_p = \mathsf{SNR}^{-d}$, $D_r = \mathsf{SNR}^{-d_r}$ and

$$
- \lim_{\mathsf{SNR} \to \infty} \frac{\log(D_p - D_f)}{\log \mathsf{SNR}} = d + \theta, \quad \theta > 0,
$$

where we essentially characterize that the second smallest exponent of D_f is $d + \theta$. The partial outage event is then characterized as follows,

$$
\begin{aligned}
\mathcal{O}_p(d_p, d_f, d_r, \beta) &= \mathcal{O}_f(d_p, d_f, d_r, \beta) \cap \bigcap_{i=1,2} \left\{ g(\alpha_i, \beta) < d + \frac{\theta}{2} \right\} \\
&= \left\{ \sum_{i=1,2} \left((d + \frac{\theta}{2}) - g(\alpha_i, \beta) \right)^+ > d + \theta \right\} \cap \bigcap_{i=1,2} \left\{ g(\alpha_i, \beta) < d + \frac{\theta}{2} \right\} \\
&= \left\{ \sum_{i=1,2} g(\alpha_i, \beta) < d \right\} \cap \bigcap_{i=1,2} \left\{ g(\alpha_i, \beta) < d + \frac{\theta}{2} \right\} \\
&= \left\{ \sum_{i=1,2} g(\alpha_i, \beta) < d \right\} \\
&= \left\{ \sum_{i=1,2} \left(\beta - (1 - \alpha_i)^+ \right)^+ > 2\beta - d \right\} \\
&= \begin{cases} \{\alpha_i \ge 0\}, & \text{if } \beta < \frac{d}{2}, \\ \cup_{i=1,2} \{(1 - \alpha_i)^+ < d - \beta\} \cup \left\{ \sum_{i=1,2} (1 - \alpha_i)^+ < d \right\}, & \text{if } \frac{d}{2} < \beta < d, \\ \left\{ \sum_{i=1,2} (1 - \alpha_i)^+ < d \right\}, & \text{if } \beta > d. \end{cases}
\end{aligned}
$$

Therefore, the partial diversity order is

$$
\delta_p = \begin{cases} 0, & \text{if } \beta < \frac{d}{2}, \\ 1 + \beta - d, & \text{if } \frac{d}{2} < \beta < d, \\ 2 - d, & \text{if } \beta > d. \end{cases} \tag{D.2}
$$

Note that the partial diversity orders (D.1) and (D.2) are essentially the same. Therefore, for $d_p \le d_f \le d_r$, the partial diversity order is

$$
\delta_p = \begin{cases} 0, & \text{if } \beta < \frac{d_f}{2}, \\ 1 + \beta - d_f, & \text{if } \frac{d_f}{2} < \beta < \frac{d_p + d_f}{2}, \\ 2 - d_f, & \text{if } \beta > \frac{d_p + d_f}{2}. \end{cases}
$$

The refinement outage event is characterized as follows,

$$
\begin{aligned}
\mathcal{O}_r(d_p, d_f, d_r, \beta) &= \mathcal{O}_f(d_p, d_f, d_r, \beta) \cup \left\{ \sum_{i=1,2} (1 - \beta - \alpha_i)^+ < d_r - d_f \right\} \\
&= \mathcal{O}_f(d_p, d_f, d_r, \beta) \cup \left\{ \sum_{i=1,2} (1 - \frac{\alpha_i}{1 - \beta})^+ < \frac{d_r - d_f}{1 - \beta} \right\} \\
&= \begin{cases} \mathcal{O}_f(d_p, d_f, d_r, \beta) \cup \{\alpha_i \ge 0\}, & \text{if } 1 - \beta < d_r - d_f, \\ \mathcal{O}_f(d_p, d_f, d_r, \beta) \cup (\cup_{i=1,2} \{\alpha_i > (1 - \beta) - (d_r - d_f)\} & \\ \quad \cup \left\{ \sum_{i=1,2} \alpha_i > 2(1 - \beta) - (d_r - d_f) \right\}), & \text{if } 1 - \beta > d_r - d_f. \end{cases}
\end{aligned}
$$

Therefore, the refinement diversity order is

$$
\delta_r = \begin{cases} 0, & \text{if } 1 - \beta < d_r - d_f, \\ \min\{\delta_f, 2(1 - \beta) - (d_r - d_f)\}, & \text{if } 1 - \beta > d_r - d_f. \end{cases}
$$

$\square$

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