

Non-Pauli-Fierz Massive Gravitons

Gia Dvali,^{1,2} Oriol Pujolàs,² and Michele Redi³

¹*CERN, Theory Division, CH-1211 Geneva 23, Switzerland*

²*Center for Cosmology and Particle Physics, New York University, New York, New York 10003, USA*

³*Institut de Théorie des Phénomènes Physiques, EPFL, CH-1015, Lausanne, Switzerland*

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We study general Lorentz invariant theories of massive gravitons. We show that, contrary to the standard lore, there exist consistent theories where the graviton mass term violates Pauli-Fierz structure. For theories where the graviton is a resonance, this does not imply the existence of a scalar ghost if the deviation from a Pauli-Fierz structure becomes sufficiently small at high energies. These types of mass terms are required by any consistent realization of the Dvali-Gabadadze-Porrati model in higher dimension.

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Introduction.—The current accelerated expansion of the Universe is arguably the most relevant observation in modern cosmology. The fact that it might be signaling a failure of General Relativity (GR) at large distances is a compelling idea that motivates the investigation of large distance modifications of gravity. Since GR is the unique consistent theory of massless spin 2 field, whose low energy limit is fixed by the principle of invariance under general coordinate transformations, any infrared modification of GR requires some sort of mass for the graviton.

The subject of massive spin-2 fields traces back to the classical work of Pauli and Fierz [1]. At quadratic order around a flat background, there are two possible mass terms compatible with Lorentz invariance,

$$V = \frac{M_4^2}{4} [m_{\text{PF}}^2 h^{\mu\nu} (h_{\mu\nu} - \eta_{\mu\nu} h) + m^2 h^2], \quad (1)$$

where $h_{\mu\nu}$ is the metric fluctuation and $h = h^\mu{}_\mu$. The celebrated Pauli-Fierz (PF) mass term [1] corresponds to $m = 0$. Because of the mass, $h_{\mu\nu}$ propagates more degrees of freedom than in the massless case (2). In the PF case, the graviton has five polarizations while when also $m \neq 0$, there is an extra scalar. This can be seen as follows. The scalar longitudinal degree of freedom of the graviton can be isolated performing the diffeomorphism [2],

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{2\partial_\mu \partial_\nu \phi}{m_{\text{PF}}^2}. \quad (2)$$

The mass term (1) is not invariant and generates a kinetic term for the scalar,

$$-\delta \mathcal{L} = M_4^2 \left[\frac{m^2}{m_{\text{PF}}^2} (\square \phi)^2 + \phi (\partial_\mu \partial_\nu \hat{h}^{\mu\nu} - \square \hat{h}) + \dots \right]. \quad (3)$$

For $m = 0$, the scalar obtains a (healthy) kinetic term by mixing with the transverse polarizations of the graviton [2]. For $m \neq 0$, ϕ acquires a four derivatives kinetic term signaling the presence of a second scalar. The higher derivative structure in fact implies that this extra degree

of freedom is a ghost. For this reason, graviton mass with structure different from PF is normally discarded.

The purpose of this Letter will be to show that this conclusion does not hold in general and that non-PF massive gravity theories can be consistent if the graviton is a resonance. This corresponds to the case where the mass parameters depend on the energy scale which is natural from the point of view of extra dimensions. In fact, the only Lorentz invariant nonlinear theories of massive gravity known to date resort to extra dimensions. The prototype of these types of theories is provided by the Dvali-Gabadadze-Porrati (DGP) model [3] where five dimensional gravity is localized on a codimension one defect by means of a brane kinetic term for the graviton. An important feature of these geometrical models is that the tensor structure depends on the number of extra dimensions. For instance, one can show that the DGP model describes a graviton resonance with a mass term of the PF form [1]. The higher dimensional generalizations of DGP [4] require a different mass term, and essentially for this reason, they were thought to be inconsistent. Recently, however, a counterexample was presented where a ghost-free model in 6 dimensions was constructed [5]. As we will discuss, this can be understood as the first realization of a consistent non-PF massive gravity theory.

Consistent massive gravitons.—The basic reason why it might be possible to find a ghost-free theory of non-PF massive gravitons is due to the fact that such a theory is equivalent to a scalar-tensor theory. From the point of view of the irreducible representation of the Lorentz group, the spectrum can be decomposed into scalars and PF massive gravitons. Since they belong to different representations, it is consistent with Lorentz symmetry to change the UV behavior in Eq. (3) that is responsible for the presence of ghosts. To show how this works, let us consider mass terms of the form (1) where we assume in general that m_{PF} and m are scale-dependent functions. Adopting a parameterization similar to [6], by Lorentz invariance this implies that m_{PF} and m must be functions of the d'Alembertian, \square . The

connection with scalar-tensor theories can be realized integrating in a scalar field in the action,

$$\mathcal{L} = \frac{M_4^2}{4} [h^{\mu\nu}(\mathcal{E}h)_{\mu\nu} - m_{\text{PF}}^2 h^{\mu\nu}(h_{\mu\nu} - h\eta_{\mu\nu})] + \alpha\phi h + \beta\phi^2 + h^{\mu\nu}T_{\mu\nu} \quad (4)$$

[here, $(\mathcal{E}h)_{\mu\nu} = \square h_{\mu\nu} + \dots$ denotes the linearized Einstein tensor]. This is equivalent to the non-PF massive graviton (1) if $\alpha^2 = m^2 M_4^2 \beta$. We now perform the diffeomorphism (2) followed by a rescaling of the metric,

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \phi\eta_{\mu\nu} + \frac{2\partial_\mu\partial_\nu}{m_{\text{PF}}^2}\phi, \quad (5)$$

where the shift of the metric is chosen in order to remove the kinetic mixing term $\phi(\mathcal{E}h)$ in Eq. (3). One obtains

$$\begin{aligned} \mathcal{L} = & \frac{M_4^2}{4} [\tilde{h}^{\mu\nu}(\mathcal{E}\tilde{h})_{\mu\nu} - \tilde{h}^{\mu\nu}m_{\text{PF}}^2(\tilde{h}_{\mu\nu} - \tilde{h}\eta_{\mu\nu})] \\ & + \alpha\phi\left(\tilde{h} + 4\phi + \frac{2\square\phi}{m_{\text{PF}}^2}\right) + \beta\phi^2 + \frac{3M_4^2}{2}m_{\text{PF}}^2\phi\tilde{h} \\ & + \left(\tilde{h}_{\mu\nu} + \phi\eta_{\mu\nu} + \frac{2\partial_\mu\partial_\nu}{m_{\text{PF}}^2}\phi\right)T^{\mu\nu}. \end{aligned} \quad (6)$$

By choosing $\alpha = -\frac{3}{2}M_4^2m_{\text{PF}}^2$, we cancel the mixing between ϕ and h , and we arrive at

$$\begin{aligned} \mathcal{L} = & \frac{M_4^2}{4} [\tilde{h}^{\mu\nu}(\mathcal{E}\tilde{h})_{\mu\nu} - \tilde{h}^{\mu\nu}m_{\text{PF}}^2(\tilde{h}_{\mu\nu} - \tilde{h}\eta_{\mu\nu})] + \frac{M_4^2}{2}\phi\mathcal{O}\phi \\ & + \left(\tilde{h}_{\mu\nu} + \phi\eta_{\mu\nu} + \frac{2\partial_\mu\partial_\nu}{m_{\text{PF}}^2}\phi\right)T^{\mu\nu}, \end{aligned} \quad (7)$$

with $\mathcal{O} \equiv \frac{9}{2}\frac{m_{\text{PF}}^4}{m^2} - 3(\square + 2m_{\text{PF}}^2).$ (8)

Therefore, the non-PF graviton can be rewritten in terms of a PF graviton and a scalar decoupled from each other, in agreement with [6]. Note that an analogous decomposition is impossible for the scalar longitudinal component of a PF graviton because this polarization belongs to the same representation of the Lorentz group. The propagator of non-PF graviton can be reconstructed using Eq. (5) in terms of PF and scalar propagators. We can also read off from (7) the amplitude for the exchange between two conserved sources,

$$\mathcal{A} = \frac{1}{M_4^2} \left(\frac{T_{\mu\nu}T'^{\mu\nu} - (1/3)TT'}{\square - m_{\text{PF}}^2} + \frac{1}{2} \frac{TT'}{\mathcal{O}} \right) \quad (9)$$

where the first contribution corresponds to the massive PF graviton contribution and the second to the scalar exchange.

More in general, for massive gravitons, gauge invariance does not demand conservation of $T_{\mu\nu}$. However, the non-conserved part is still constrained and has to vanish in the limit of zero graviton mass. At the quantum level, this brings the following subtlety. For a nonconserved source, the coupling of the form $\frac{\phi\partial_\mu\partial_\nu}{\bar{m}^2}T^{\mu\nu}$ can generate a higher derivative kinetic term for ϕ ,

$$\frac{\partial_\mu\partial_\nu\phi}{\bar{m}^2} \langle T^{\mu\nu}T^{\alpha\beta} \rangle \frac{\partial_\alpha\partial_\beta\phi}{\bar{m}^2}. \quad (10)$$

For the conserved $T_{\mu\nu}$, the correlator $\langle T^{\mu\nu}T^{\alpha\beta} \rangle$ is proportional to the transverse projector, and the higher derivative kinetic term (10) vanishes. For nonconserved sources, the structure can be nonzero but, as we mentioned, the non-conserved part of the source must go to zero at least as \bar{m}^2 itself. This means that the higher derivative kinetic term will be suppressed by the cutoff of the theory, and the resulting ghost pole is automatically at the cutoff. Notice that the same correlator will generate exactly the same type of higher derivative mass term for the helicity zero polarization of the PF graviton, with $\bar{m}^2 = m_{\text{PF}}^2$. So the absence of any ghost pole below the cutoff requires that the divergence of $T_{\mu\nu}$ is controlled by m_{PF}^2 . Since our scalar couples to $T_{\mu\nu}$ with $\bar{m}^2 = m_{\text{PF}}^2$, the absence of ghosts at the quantum level in the PF graviton automatically implies the absence of similar ghosts in ϕ .

Coming back to Eq. (8), in any theory with constant m_{PF} and m , the scalar propagator has a pole at $p^2 = -\frac{3m_{\text{PF}}^4}{2m^2} + 2m_{\text{PF}}^2$. From the sign of the kinetic term in Eq. (8), this pole is always a ghost, with positive or negative mass squared depending on the sign of m^2 . One interesting feature is that no matter how small the deviation from PF structure is, the amplitude (9) has the same tensor structure of massless gravity in the UV. This is because the ghost couples to T_μ^μ with the same strength as the longitudinal polarizations of the massive graviton, and therefore it exactly cancels its contribution at high energy. For $m \rightarrow 0$, the ghost becomes heavy and decouples from the theory.

When m and m_{PF} are scale dependent and different, however, then one does not necessarily have ghosts in the spectrum due to the positive contributions in Eq. (8). Let us now discuss how the consistency of the theory actually constrains the masses. In general, absence of ghosts demands the spectral decomposition of the scalar and spin-2 amplitudes in Eq. (9) to be positive definite. A necessary condition is that the coefficient of the scalar contribution is strictly positive since a negative value (corresponding to repulsion) could only be provided by ghosts. This strongly restricts the form that m_{PF} and m can take.

We can obtain different constraints considering the infrared (IR) and ultraviolet (UV) behavior of the amplitude. In the UV, the positivity of the scalar amplitude requires

$$m^2(\square) < \frac{3}{2} \frac{m_{\text{PF}}^4(\square)}{\square} \quad (11)$$

in the limit $\square \rightarrow \infty$. If this condition is violated, then a ghost appears. Note that in this case, \mathcal{O} scales as \square in the UV so this ghost corresponds a 4D scalar bound state. The condition above can also be understood as the fact that non-PF term does not generate ghosts as long as it is sufficiently subleading at high energies. This also agrees with the effective field theory intuition that small deviations from PF are acceptable [2].

In the IR, we can parameterize the behavior of the tensor and the scalar as

$$m_{\text{PF}}^2 \approx A_2(-\square)^{a_2}, \quad -\mathcal{O} \approx A_0(-\square)^{a_0}. \quad (12)$$

Since we are interested in large distance modifications of gravity, we consider $a_{0,2} < 1$ so that these terms become dominant with respect to the kinetic term at large distances. As discussed in [6], the spectral decomposition requires that the scalar and spin 2 amplitude should not vanish at zero momenta. This implies $a_{0,2} \geq 0$ and $A_{0,2} > 0$. Note that if the positivity constraint is violated in the IR, the amplitude has a branch cut corresponding to a continuum of ghosts. From this and (8), one finds that the healthy forms of m^2 in the IR are

$$m^2(\square) = \frac{3}{4} \frac{A_2(-\square)^{a_2}}{1 - A_0/(6A_2)(-\square)^{a_0 - a_2}}. \quad (13)$$

This automatically classifies all possible cases in 3 families. (i) $a_0 > a_2$, corresponding to the scalar being “lighter” than the tensor ($-\mathcal{O} \ll m_{\text{PF}}^2$) in the IR. These cases have $m^2/m_{\text{PF}}^2 \rightarrow 3/4$. (ii) $a_0 = a_2$, which includes the geometrical models discussed below. In this case, m^2/m_{PF}^2 approaches a constant with a value outside the range between 0 and 3/4. (iii) $a_0 < a_2$. In this case, $m^2 \rightarrow (9/2)(m_{\text{PF}}^4/\mathcal{O})$, so one has $-m^2 \ll m_{\text{PF}}^2 \ll -\mathcal{O}$.

Geometrical realizations.—We now turn to the geometrical realization of gravitons with non-PF structure. As mentioned earlier, a natural arena for theories of massive gravitons is higher dimensional theories of gravity. Following [3,4], the addition of a kinetic term for the graviton on the lower dimensional defect insures the existence of a 4D regime,

$$S = \frac{M_4^2}{2} \int R_4 + \frac{M_{4+n}^2}{2} \int R_{4+n}. \quad (14)$$

The connection with massive gravity theories can be made manifest computing the boundary effective action obtained by integrating out the bulk degrees of freedom. Expanding the action (14) around flat space, one can choose a gauge where the brane is located at $\vec{y} = 0$ in the n -dimensional transverse space. The induced metric fluctuation perceived by brane observers then reduces to the 4-dimensional components evaluated on the brane, $h_{\mu\nu}(\vec{y} = 0)$. One can further fix the gauge so that the graviton propagator takes the form

$$G_{MNPQ} = \frac{1}{M_{4+n}^2} \frac{1}{p^2 + \vec{q}^2} \left[\frac{1}{2} \eta_{MP} \eta_{NQ} + \frac{1}{2} \eta_{MQ} \eta_{NP} - \frac{1}{2+n} \eta_{MN} \eta_{PQ} \right] \quad (15)$$

where M, N, \dots denote $(4+n)$ -dimensional indices and p (\vec{q}) the 4- (n -) dimensional components of the momentum. Neglecting the 4D term in (14), the amplitude between two brane localized sources is

$$\int d^4 p G(p) \left(\tilde{T}_{\mu\nu}(p) \tilde{T}'^{\mu\nu}(-p) - \frac{\tilde{T}(p) \tilde{T}'(-p)}{2+n} \right) \quad (16)$$

where $\tilde{T}_{\mu\nu}(p)$ is the Fourier transform of the source and

$$G(p) = \frac{1}{M_{4+n}^2} \int \frac{d^n q}{p^2 + \vec{q}^2}. \quad (17)$$

This integral is divergent for $n \geq 2$ and therefore requires some regularization. Introducing a momentum cutoff Λ , [7]

$$G(p) \approx \frac{\Omega_n}{M_{4+n}^2} \left[\frac{\Lambda^{n-2}}{n-2} + \dots + (-1)^{(n/2)} p^{n-2} \log \frac{p}{\Lambda} + \dots \right], \quad (18)$$

where $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$ and local terms of the form $(p/\Lambda)^{2k} \Lambda^{n-2}$ with $k = 1, 2, \dots$ that are present for $n > 4$ have been omitted. The displayed cutoff-independent non-local term generates the higher dimensional Newtonian potential $\sim 1/r^{n+1}$.

The amplitude (16) can be derived from the boundary effective action,

$$\int d^4 x h^{\mu\nu} G^{-1}(\square) \left[h_{\mu\nu} - \frac{1}{n-2} \eta_{\mu\nu} h \right] + h_{\mu\nu} T^{\mu\nu} \quad (19)$$

where $h_{\mu\nu}$ is the metric measured by the brane observer. The addition of the brane localized kinetic term $\int R_4$ then leads to a massive gravity action where the bulk provides a scale-dependent mass term for the graviton. For the case $n = 1$, the 5D DGP model, this corresponds to a resonance with Pauli-Fierz mass. For $n > 1$, the tensor structure of the higher dimensional theory is encoded in a non-PF mass term.

From (18) and (19), we can identify

$$m_{\text{PF}}^2 = \frac{1}{M_4^2 G(\square)}, \quad m^2 = \frac{n-1}{n-2} m_{\text{PF}}^2. \quad (20)$$

The necessity of the non-PF mass for $n > 1$ can also be understood from the Kaluza-Klein decomposition of these theories. For $n > 1$, aside from the tower of gravitons, there is a tower of spin-0 states, which are encoded in the scalar ϕ that we integrated in.

We are now in the position to see why the $n > 1$ theories propagate ghosts, as first shown in [8,9]. This is just a consequence of m not obeying (11) in the UV. More precisely, from (9), one derives that there is a light ghost pole with a mass

$$m_{\text{ghost}}^2 \approx -\frac{1}{2\Omega_n} \frac{n+2}{n-1} \frac{n-2}{\Lambda^{n-2}} \frac{M_{4+n}^2}{M_4^2}. \quad (21)$$

The most important feature of this formula is the dependence on the inverse cutoff (for codimension 2 this becomes logarithmic). While a heavy ghost can be consistent within an effective field theory approach, this formula shows that the ghost is light. For this reason, higher dimensional generalizations of DGP were believed to be inconsistent.

Following our general analysis, this conclusion is a manifestation of the fact that m_{PF} and m have the same

momentum dependence so that m never becomes subleading. As we have seen, this is not mandatory because the extra scalars can, consistently with 4D Lorentz invariance, couple differently than the massive spin 2 states. In order to possibly avoid the ghost, m and m_{PF} should scale differently in the UV. This was explicitly realized in the “cascading DGP model” [5]. In the 6 dimensional case considered in that Letter, a 3-brane is embedded within a co-dimension 1 brane, each with its own induced gravity term

$$S = \frac{M_4^2}{2} \int R_4 + \frac{M_5^3}{2} \int R_5 + \frac{M_6^4}{2} \int R_6$$

In a certain limit of the model [12,5], with the addition of a tension λ on the 3-brane, it was found that the 4D boundary effective action reduces to

$$L_4 = -\frac{M_5^3}{2} h^{\mu\nu} \sqrt{-\square} (h_{\mu\nu} - h \eta_{\mu\nu}) - 3M_5^3 \pi \sqrt{-\square} h + \frac{M_4^2}{4} h^{\mu\nu} (\mathcal{E}h)_{\mu\nu} + \frac{9\lambda}{4m_6^2} \pi \square \pi + h^{\mu\nu} T_{\mu\nu}, \quad (22)$$

where $m_6 = M_6^4/M_5^3$. The scalar π corresponds to the brane bending mode of the 4-brane whose 4D kinetic term arises from the tension. To see the relationship with our general analysis, one can integrate out π using its equation of motion, $\pi = -(2M_5^3 m_6^2)/(3\lambda)(-\square)^{-1/2} h$. Substituting into the action generates a non-PF mass,

$$m_{\text{PF}}^2 = \frac{2M_5^3}{M_4^2} \sqrt{-\square}, \quad m^2 = -4 \frac{M_5^6 m_6^2}{M_4^2 \lambda}. \quad (23)$$

Since m_{PF} grows linearly with energy, m becomes subleading in the UV, and the condition (11) is satisfied for $3\lambda > 2M_4^2 m_6^2$. This reproduces the bound of Ref. [5].

Outlook.—Before concluding, we wish to speculate on other realizations of graviton resonances with non-PF structure. One possible direction is the generalization of the cascading DGP model to higher codimensions. To realize this, we need to consider a tower of DGP kinetic terms embedded into each other,

$$S = \frac{M_4^2}{2} \int R_4 + \frac{M_5^3}{2} \int R_5 + \dots + \frac{M_{4+n}^{2+n}}{2} \int R_{4+n}. \quad (24)$$

At very large distances, physics is dominated by the $4+n$ -dimensional kinetic term and asymptotically can be described by a non-PF resonance with parameters given by Eq. (20). As in the codimension 2 case, the lower dimensional kinetic terms generate brane localized ghosts which can be studied using the method of the boundary effective action. This can be derived integrating out the bulk degrees of freedom step by step starting from the highest codimension. To obtain a consistent theory, one should insure that at each step, one obtains a consistent massive gravity theory on the lower dimensional defect. This should be achieved by introducing sources, such as the tension in codimension 2, which render the non-PF mass term appropriately subleading in the UV.

A different possibility is to consider theories with a noninteger number of extra-dimensions, which corresponds to a fractal extra space. Formally for a brane observer, these theories can be obtained by analytic continuation of Eqs. (16) and (17). In the range $0 < n < 2$, (17) is finite and one obtains $m_{\text{PF}}^2 = p_c^n p^{2-n}$ with

$$p_c^n = \frac{\Gamma(n/2) \sin(n\pi/2)}{\pi^{1+(n/2)}} \frac{M_{4+n}^{n+2}}{M_4^2}$$

and $m^2 = (n-1)/(n-2)m_{\text{PF}}^2$. For the case $n < 1$, the scalar amplitude corresponds to the propagation of a continuum of light ghosts. This can be readily seen because the scalar exchange in Eq. (9) is negative. This rules out the range $n < 1$. For $1 < n < 2$, the masses above satisfy the constraints in the IR (13) but not in the UV (11). The ghost pole is at $p^2 = p_c^2 [(1+n/2)/(n-1)]^{1/n}$ so that for n close to 1, it is heavy and hence could be accepted within an effective field theory approach. From a geometrical point of view, these theories can be realized through a bulk space which is fractal. At quadratic order, they can be defined using a lattice as in [10] (see, however, [11] for difficulties at the interacting level). It is appealing that this construction only generates $n > 1$. Theories with $1 < n \ll 2$ are also the most interesting from a phenomenological point of view as they could be tested by future lunar ranging experiments [6]. We leave the detailed construction of these generalizations and their possible implications for the acceleration of the Universe to future work.

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