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$d = 4$ attractors, effective horizon radius, and fake supergravity

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We consider extremal black hole attractors [both Bogomol'nyi-Prasad-Sommerfield (BPS) and non-BPS] for $\mathcal{N} = 3$ and $\mathcal{N} = 5$ supergravity in $d = 4$ space-time dimensions. Attractors for *matter-coupled* $\mathcal{N} = 3$ theory are similar to attractors in $\mathcal{N} = 2$ supergravity *minimally coupled* to Abelian vector multiplets. On the other hand, $\mathcal{N} = 5$ attractors are similar to attractors in $\mathcal{N} = 4$ *pure* supergravity, and in such theories only $\frac{1}{\mathcal{N}}$-BPS nondegenerate solutions exist. All the above-mentioned theories have a simple interpretation in the *first order (fake supergravity)* formalism. Furthermore, such theories do not have a $d = 5$ uplift. Finally we comment on the *duality* relations among the attractor solutions of $\mathcal{N} \geq 2$ supergravities sharing the same full bosonic sector.

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I. INTRODUCTION

The *attractor mechanism* [1–4] is an important dynamical phenomenon in the theory of gravitational objects, which naturally appears in modern theories of gravity, such as supergravity, superstrings [5–8], or M theory [9,10].

Even if such a phenomenon was originally shown to occur for $\frac{1}{2}$-Bogomol'nyi-Prasad-Sommerfield (BPS) extremal black holes (BHs) in $\mathcal{N} = 2$, $d = 4$ ungauged supergravity coupled to Abelian vector multiplets, it has a more general validity, since it may take place also for non-BPS extremal BHs, irrespectively, whether the underlying gravitational theory is endowed with local supersymmetry or not [11–66] (for further developments, see also e.g. [67–70]). Moreover, such a phenomenon also exists in higher space-time dimensions for black p-branes coupled to scalar fields, provided certain constraints are met.

In theories with $\mathcal{N} > 2$ local supersymmetry, extremal BH attractors with regular horizon geometry and nonvanishing classical Bekenstein-Hawking entropy exhibit a new feature, namely, the Hessian matrix of a suitably defined effective BH potential $V_{\rm BH}$ may present, in contrast with the $\mathcal{N} = 2$ case, "flat" directions even for ($\frac{1}{\mathcal{N}}$-)BPS configurations. This is, for instance, the case when the scalar manifold is a locally symmetric space, as it holds for all $\mathcal{N} > 2$, $d = 4$ supergravities. A general analysis of extremal BH attractors in $\mathcal{N} = 2$ symmetric special Kähler geometry was performed in [25], and the related moduli spaces were discovered and classified in [41]. For $\mathcal{N} > 2$ supergravities similar results were obtained in [56,71]. In [38] it was further observed that flat directions for $\mathcal{N} > 2$ (both $\frac{1}{\mathcal{N}}$-BPS and non-BPS) attractors, as well as for $\mathcal{N} = 2$ non-BPS attractors, are closely related to the fact that in $\mathcal{N} = 2$ ungauged supergravity the hypermultiplets' scalars do not participate in the attractor mechanism. As a consequence, the moduli space of $\frac{1}{\mathcal{N}}$-BPS attractors in $\mathcal{N} > 2$ supergravities results in being a quaternionic manifold, spanned by the leftover would-be hypermultiplets' scalar degrees of freedom in the supersymmetry reduction of the original theory down to $\mathcal{N} = 2$.

The corresponding orbits of electric and magnetic BH charges, supporting the critical points of $V_{\rm BH}$ which determine the attractor scalar configurations on the BH event horizon, have also been classified in [25,56]. The noncompactness of the stabilizer of such orbits (with the only exception of $\mathcal{N} = 2$ BPS orbits) is responsible for the existence of flat directions of the BH potential at its corresponding critical points.

Most of the supergravities based on symmetric scalar manifolds have the property that the classical BH entropy, as given by the Bekenstein-Hawking entropy-area formula [72], is given in terms of the square root of the absolute value of a (*n unique*) invariant I_4 of the relevant representation of the U-duality group. Such an invariant is *quartic* in electric and magnetic BH charges:

$$S_{\rm BH} = \frac{A}{4} = \pi V_{\rm BH}|_{\partial V_{\rm BH}=0} = \pi\sqrt{|I_4|}. \tag{1.1}$$

I_4, which is *moduli-independent*, can also be written in terms of the *"dressed"* charges, i.e. in terms of the

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(moduli-dependent) *central charge matrix* and *matter charges*, in a (unique) combination such that the overall dependence on moduli drops out. However, a peculiar class of $d = 4$ supergravities exists such that the unique U-duality invariant (and thus moduli-independent) combination of dressed charges turns out to be a *perfect square* of a *quadratic* expression in the skew-eigenvalues of the relevant central charge matrix. Namely, this holds for *pure* $\mathcal{N} = 4$ [73] and $\mathcal{N} = 5$ [74] supergravity.

Furthermore, another class of $d = 4$ theories exists, such that the unique U-invariant I_2 is quadratic in BH charges, yielding:

$$S_{\mathrm{BH}} = \frac{A}{4} = \pi V_{\mathrm{BH}}|_{\partial V_{\mathrm{BH}}=0} = \pi|I_2|. \tag{1.2}$$

I_2 is also given by a quadratic expression in terms of the dressed charges. Such a class of theories is given by $\mathcal{N} = 2$ supergravity *minimally coupled* to Abelian vector multiplets [75], and by $\mathcal{N} = 3$ supergravity coupled to matter (Abelian vector) multiplets [76].

For both the peculiar class of theories admitting I_4 as a perfect square of the skew-eigenvalues of the central charge matrix and the supergravities admitting I_2, there is a very simple alternative expression for the classical Bekenstein-Hawking entropy in terms of a *(square) effective horizon radius* R_H, which turns out to be moduli independent, and dependent *only* on the set of magnetic and electric BH charges, shortly indicated as (p, q). The formula for the entropy of the extremal BH in these cases reads:

$$\begin{aligned} S_{\mathrm{BH}} &= \frac{A}{4} = \pi V_{\mathrm{BH}}|_{\partial V_{\mathrm{BH}}=0} \equiv \pi R_H^2(p, q) = \begin{cases} \pi\sqrt{|I_4|} \\ \text{or} \\ \pi|I_2| \end{cases} \\ &= \pi\bigg[r_H^2(\varphi_\infty, p, q) - \frac{1}{2} G_{ab}(\varphi_\infty)\Sigma^a(\varphi_\infty, p, q) \\ &\quad \times \Sigma^b(\varphi_\infty, p, q)\bigg], \end{aligned} \tag{1.3}$$

where r_H is the radius of the unique *(event) horizon* of the extremal BH, Σ^a denotes the set of *scalar charges* asymptotically associated to the scalar field φ^a, and G_{ab} is the covariant metric tensor of the scalar manifold (in the real parametrization). Notice that the first line of Eq. (1.3) only contains the definition of R_H^2 itself, whereas the second line of the very same equation expresses it through a moduli-independent combination of moduli-dependent quantities, holding only for the aforementioned $d = 4$ supergravity theories.

Actually, R_H^2 can be expressed as a suitable integral in terms of a *(square) effective radius* R, as follows:

$$R_H^2 = R^2(r, \varphi_\infty, p, q)|^{r=r_H(\varphi_\infty,p,q)}_{r=\sqrt{(1/2)G_{ab}(\varphi_\infty)\Sigma^a(\varphi_\infty,p,q)\Sigma^b(\varphi_\infty,p,q)}}; \tag{1.4}$$

$$\begin{aligned} R^2(r, \varphi_\infty, p, q) &\equiv r^2 - \tfrac{1}{2}G_{ab}(\varphi_\infty)\Sigma^a(\varphi_\infty, p, q)\Sigma^b(\varphi_\infty, p, q) \\ &\leq r^2, \end{aligned} \tag{1.5}$$

where r is the usual radial coordinate, and in the last inequality the positive definiteness of G_{ab} was exploited.

It is worth pointing out that the second line of Eqs. (1.3), (1.4), and (1.5) are generalizations of the formulae holding in the so-called Maxwell-Einstein-axion-dilaton system (actually also in the *nonextremal case*, see e.g. [77,78]; see also the treatment, and, in particular, Eqs. (2.7), (2.8), and (2.15), of [63]). Notice that in the first of Refs. [77] the variable R is named *physical radial coordinate* (see e.g. Eq. (72) therein). Clearly, as the second line of Eq. (1.3), also the definition (1.7) of $R^2(r, \varphi_\infty, p, q)$ holds only for the aforementioned $d = 4$ supergravity theories.

Within the first order (fake supergravity) formalism [79], recently used to describe non-BPS attractor flows of $d = 4$ extremal BHs [35,40], the quantities appearing in the second line of Eq. (1.3) can easily be expressed in terms of a *real "fake superpotential"* $\mathcal{W}(\varphi, p, q)$ as follows (see Eqs. (2.5) and (2.6) below, respectively):

$$r_H(\varphi_\infty, p, q) = \mathcal{W}(\varphi_\infty, p, q); \tag{1.6}$$

$$\Sigma^a(\varphi_\infty, p, q) = 2G^{ab}(\varphi_\infty)(\partial_b \mathcal{W})(\varphi_\infty, p, q). \tag{1.7}$$

An explicit expression for $\mathcal{W}$ can be given for the supergravity theories mentioned above [40]. It is here worth noticing that for $\frac{1}{\mathcal{N}}$-BPS *nondegenerate* attractor flows, simply $\mathcal{W}(\varphi, p, q) = |Z|(\varphi, p, q)$, where $|Z|$ is the biggest (absolute value of the) skew-eigenvalues of the central charge matrix Z_{AB}, saturating the BPS bound [80].

Equation (1.3) would seem to yield a moduli-dependent expression for R_H^2, but, as we prove explicitly in the present paper, for the class of $d = 4$ ungauged supergravities under consideration it just turns out that the dependence on moduli drops out in the combination $r_H^2 - \frac{1}{2}G_{ab}\Sigma^a\Sigma^b$, when Eqs. (1.6) and (1.7) are taken into account. Summarizing, such a phenomenon happens in the following theories:

(i) $\mathcal{N} = 2$ supergravity minimally coupled to Abelian vector multiplets [75], whose scalar manifold is endowed with a symmetric special Kähler geometry with the completely symmetric rank-3 tensor $C_{ijk} = 0$, and with U-invariant quadratic in BH charges. For such a theory, in [63] Eq. (1.3) has been proved to hold for both $\frac{1}{2}$-BPS and non-BPS ($Z = 0$) attractor flows;

(ii) $\mathcal{N} = 3$ supergravity coupled to matter (Abelian vector) multiplets [76], with U-invariant quadratic in BH charges;

(iii) $\mathcal{N} = 4$ pure supergravity [73], with U-invariant quartic in BH charges;

(iv) $\mathcal{N} = 5$ supergravity [74], with U-invariant quartic in BH charges.

It is worth pointing out that $\mathcal{N} = 2$ supergravity minimally coupled to one Abelian vector multiplet, corresponding to the $(U(1))^6 \to (U(1))^2$ gauge truncation of $\mathcal{N} = 4$ pure supergravity, is nothing but the so-called Maxwell-Einstein-axion-dilaton system, studied in [77,78] and recently discussed in [71] and in [63], for which, as stated above, the formula (1.3) indeed holds true (actually, with suitable changes, also in the nonextremal case).

Furthermore, it is interesting to notice that all the above-mentioned theories are all the $\mathcal{N} \geq 2$, $d = 4$ supergravities based on symmetric scalar manifolds which do *not* admit an uplift[1] to $d = 5$ space-time dimensions [81].

The present paper is organized as follows.

In Sec. II we briefly introduce the fundamentals of the first order (fake supergravity) formalism for the nondegenerate attractor flows (both BPS and non-BPS) of extremal BHs in $d = 4$ space-time dimensions.

Then, Sec. III is devoted to a detailed study of $\mathcal{N} = 2$, $d = 4$ supergravity minimally coupled to Abelian vector multiplets. In Subsection III A the related attractor equations are explicitly solved, for both the classes of nondegenerate critical points of V_{BH}: the $\frac{1}{2}$-BPS one (Subsubsection III A 1) and the non-BPS $Z = 0$ one, this latter with related moduli space (Subsubsection III A 2). By exploiting the first order (fake supergravity) formalism, in Subsections III B and III C the Arnowitt-Deser-Misner (ADM) mass M_{ADM} [82], covariant scalar charges Σ_i, and (square) effective horizon radius R_H^2 are explicitly computed, respectively, for $\frac{1}{2}$-BPS and non-BPS $Z = 0$ attractor flows, proving that (the second line of) Eq. (1.3) holds true. This latter result, already proved in [63], generalizes the findings of [77,78] (also holding in the nonextremal case).

Section IV deals with $\mathcal{N} = 3$, $d = 4$ supergravity coupled to matter (Abelian vector) multiplets. In Subsection IV A the related attractor equations are explicitly solved, for both the classes of nondegenerate critical points of V_{BH}: the $\frac{1}{3}$-BPS one (Subsubsection IV A 1) and the non-BPS $Z_{AB} = 0$ one (Subsubsection IV A 2), both with related moduli space. Once again, by using the first order (fake supergravity) formalism, in Subsections IV B and IV C the ADM mass M_{ADM}, covariant scalar charges Σ_i, and (square) effective horizon radius R_H^2, respectively, for $\frac{1}{3}$-BPS and non-BPS $Z_{AB} = 0$ attractor flows are explicitly computed, proving that (the second line of) Eq. (1.3) holds true also for such a theory which, as the minimally coupled $\mathcal{N} = 2$ supergravity, has a unique U-invariant quadratic in BH charges.

Comments on the invariance properties of BH entropy in minimally coupled $\mathcal{N} = 2$, as well as in matter coupled $\mathcal{N} = 3$, ungauged $d = 4$ supergravity are given in Sec. V.

Next, Sec. VI deals with $\mathcal{N} = 5$, $d = 4$ supergravity, which does not allow for matter coupling and whose field content thus only consists of the gravity multiplet (pure theory). In Subsection VI A the related attractor equations are explicitly solved for the unique class of nondegenerate critical points of V_{BH}, namely, the $\frac{1}{5}$-BPS one. In Subsubsection such a class is studied, along with the related moduli space and Bekenstein-Hawking classical BH entropy [72]. This latter is proportional to the unique U-invariant I_4 of $\mathcal{N} = 5$ supergravity, whose quartic expression in terms of the BH charges is explicitly derived, as well. Through the formal machinery presented in Sec. II, in Subsection VI B the ADM mass M_{ADM}, covariant scalar charges Σ_i, and (square) effective horizon radius R_H^2 for the $\frac{1}{5}$-BPS attractor flow are explicitly given, proving that (the second line of) Eq. (1.3) holds true also for such a theory. This is somewhat surprising because, as mentioned above, $\mathcal{N} = 5$ supergravity, in contrast to the minimally coupled $\mathcal{N} = 2$ and matter coupled $\mathcal{N} = 3$ cases, has a unique U-invariant quartic, rather than quadratic, in BH charges.

Then, in Sec. VII the extremal BH attractors in $\mathcal{N} = 4$, $d = 4$ pure supergravity are revisited. In Subsection VII A the resolution of the corresponding attractor equation [71] is reviewed for the unique class of nondegenerate critical points of V_{BH}, namely, the $\frac{1}{4}$-BPS one. Its corresponding Bekenstein-Hawking classical BH entropy [72] is given by the unique U-invariant I_4 of $\mathcal{N} = 4$ pure supergravity, which is also reported. By using the formulae of Sec. II, in Subsection VII B the ADM mass M_{ADM}, covariant axion-dilaton charge Σ_s, and (square) effective horizon radius R_H^2 for the $\frac{1}{4}$-BPS attractor flow are explicitly given, proving that (the second line of) Eq. (1.3) holds true also for such a theory. Also such a result is rather surprising, for the same reason mentioned above: pure $\mathcal{N} = 4$ supergravity, as $\mathcal{N} = 5$ theory and in contrast to the minimally coupled $\mathcal{N} = 2$ and matter coupled $\mathcal{N} = 3$ cases, has a unique U-invariant quartic, rather than quadratic, in BH charges.

However, as pointed out in the introduction, pure $\mathcal{N} = 4$ and $\mathcal{N} = 5$ supergravities are peculiar theories, because their unique (moduli-independent) U-duality invariant, quartic in BH charges, when expressed as a (unique) combination of dressed (moduli-dependent) charges, turns out to be a perfect square of a quadratic expression in the skew-eigenvalues Z_1 and Z_2 of the relevant central charge matrix. Such a key feature is studied in Sec. VIII.

In Sec. IX we consider all ungauged $\mathcal{N} \geq 2$, $d = 4$ supergravities sharing the same bosonic sector, and thus

[1] Throughout all the treatment of the present paper, by "uplift to $d = 5$" we mean the dimensional uplift to a $d = 5$ Poincaré supergravity theory, having the same massless degrees of freedom of the original $d = 4$ supergravity.

with the same number of fermion fields, but with different supersymmetric completions. Besides the well-known case of the duality between $\mathcal{N}=2$ $J_3^{\mathbb{H}}$ (matter coupled) and $\mathcal{N}=6$ (pure) supergravity (see [25] and references therein), other two cases exist, namely:

(i) the duality exhibited by $\mathcal{N}=2$ supergravity minimally coupled to 3 Abelian vector multiplets, and $\mathcal{N}=3$ supergravity coupled to 1 matter (Abelian vector) multiplet;

(ii) the duality between $\mathcal{N}=2$ supergravity coupled to 6 Abelian vector multiplets, with scalar manifold given by the symmetric reducible special Kähler manifold $\frac{SU(1,1)}{U(1)}\times\frac{SO(2,6)}{SO(2)\times SO(6)}$, and $\mathcal{N}=4$ supergravity coupled to 2 matter (Abelian vector) multiplets.

It is here worth commenting that such dualities are evidences against the conventional wisdom that bosonic interacting theories have a unique supersymmetric extension. The sharing of the same bosonic backgrounds with different supersymmetric completions implies its dual interpretation with respect to the supersymmetry-preserving properties. Consistently with (local) supersymmetry, the number of fermion fields is the same in both theories, but with different spin/field contents, simply related by the interchange among spin-$\frac{1}{2}$ (gaugino) and spin-$\frac{3}{2}$ (gravitino) fields.

Section X contains some comments, outlook, and directions for further developments.

Finally, the appendix concludes the paper. It presents $\mathcal{N}=4$, $d=4$ ungauged supergravity coupled to 1 matter (Abelian vector) multiplet (upliftable to the $\mathcal{N}=4$, $d=5$ pure theory) as a counterexample of a theory with unique (moduli-independent) U-duality invariant quartic in BH charges which, when expressed as a combination of dressed (moduli-dependent) charges, does not turn out to be a perfect square of a quadratic expression in the skew-eigenvalues of the central charge matrix and in the matter charge(s). As a consequence, the explicit expression of R_H^2 given by (the second line of) Eq. (1.3) does not hold for such a theory, as well as for all other $d=4$ (ungauged) supergravities not explicitly mentioned above and in the treatment given below.

II. FAKE SUPERGRAVITY FORMALISM AND EFFECTIVE HORIZON RADIUS FOR $d=4$ EXTREMAL BLACK HOLES

We recall some facts about the first order (fake supergravity) formalism [79] for static, spherically symmetric, asymptotically flat dyonic extremal (i.e. with $c=0$) BHs in $d=4$, introduced in [35,40] (see also [63]).

Let us start with the general formula for the (positive definite) BH effective potential of $d=4$ supergravities:

$$V_{\rm BH}=\tfrac{1}{2}Z_{AB}\bar{Z}^{AB}+Z_I\bar{Z}^I, \tag{2.1}$$

where $Z_{AB}=Z_{[AB]}$ $(A,\ B=1,\ldots,\mathcal{N})$ is the central charge matrix, and Z_I $(I=1,\ldots,n)$ are the matter charges, $n\in\mathbb{N}$ being the number of matter multiplets (if any) coupled to the gravity multiplet. Equivalently, in the first order formalism (see Eq. (2.23) of [35]):

$$\begin{aligned}V_{\rm BH}&=\mathcal{W}^2+4G^{i\bar{j}}(\partial_i\mathcal{W})\bar{\partial}_{\bar{j}}\mathcal{W}\\&=\mathcal{W}^2+4G^{i\bar{j}}(\nabla_i\mathcal{W})\bar{\nabla}_{\bar{j}}\mathcal{W},\end{aligned} \tag{2.2}$$

where $\mathcal{W}$ is the moduli-dependent so-called first order fake superpotential, and ∇ denotes the relevant covariant differential operator.

An alternative expression for $V_{\rm BH}$ can be given as follows (see Eq. (5.7) of [22]):

$$V_{\rm BH}=e^{\mathcal{G}}[1+G^{i\bar{j}}(\partial_i\mathcal{G})\bar{\partial}_{\bar{j}}\mathcal{G}]=e^{\mathcal{G}}[1+G^{i\bar{j}}(\nabla_i\mathcal{G})\bar{\nabla}_{\bar{j}}\mathcal{G}], \tag{2.3}$$

where now

$$\mathcal{W}\equiv e^{\mathcal{G}/2}. \tag{2.4}$$

By recalling Eq. (65) of [63] and Eqs. (84) and (114) of [63] (which in turn can be traced back to Eq. (29) of [40]), in the same framework the covariant scalar charges and the squared ADM mass [82] can, respectively, be written as follows[2]:

$$\Sigma_i=2\lim_{\tau\to0^-}\nabla_i\mathcal{W}=2\lim_{\tau\to0^-}\partial_i\mathcal{W}; \tag{2.5}$$

$$\begin{aligned}M_{\rm ADM}^2=r_H^2&=\lim_{\tau\to0^-}[V_{\rm BH}-4G^{i\bar{j}}(\partial_i\mathcal{W})\bar{\partial}_{\bar{j}}\mathcal{W}]\\&=\lim_{\tau\to0^-}\mathcal{W}^2,\end{aligned} \tag{2.6}$$

where $\tau\equiv(r_H-r)^{-1}$. Then, one can introduce the (square) effective horizon radius (recall the notation $R_{+,c=0}=R_{-,c=0}\equiv R_H$; see the treatment of [63]):

$$\begin{aligned}R_H^2&\equiv\lim_{\tau\to-\infty}V_{\rm BH}=V_{\rm BH}|_{\partial V_{\rm BH}=0,V_{\rm BH}\neq0}=\lim_{\tau\to-\infty}\mathcal{W}^2\\&=\mathcal{W}^2|_{\partial\mathcal{W}=0,\mathcal{W}\neq0}=\frac{A_{\rm eff}(p,q)}{4\pi}=\frac{S_{\rm BH}(p,q)}{\pi},\end{aligned} \tag{2.7}$$

where (p,q) denotes the set of magnetic and electric BH charges, $A_{\rm eff}$ (simply named A in the introduction) is the effective area of the BH (i.e. the area of the surface pertaining to R_H), $S_{\rm BH}$ is the classical BH entropy, and the Bekenstein-Hawking entropy-area formula [72] has been used.

Whenever allowed by the symmetric nature of the scalar manifold, R_H^2 can thus be expressed in terms of a suitable power of the (generally unique) invariant of the relevant

[2]Here and in all our analysis we assume all functions of moduli to be sufficiently regular, in order to allow one to perform smoothly the radial asymptotical ($\tau\to0^-$) and near horizon ($\tau\to-\infty$) limits.

representation of the U-duality group G, determining the symplectic embedding of the vector field strengths. In $d = 4$ $S_{\rm BH}$ is homogeneous of degree two in (p, q), and only two possibilities arise:

$$R_H^2 = |\mathcal{I}_2(p, q)|, \quad \text{or} \quad R_H^2 = \sqrt{|\mathcal{I}_4(p, q)|}, \tag{2.8}$$

where $\mathcal{I}_2$ and $\mathcal{I}_4$ respectively denote U-invariants quadratic and quartic in BH charges.

By exploiting the τ-monotonicity of $\mathcal{W}$ (which is indeed an example of C-function [17] for *extremal* BHs) [40]:

$$\frac{d\mathcal{W}(z(\tau), \bar{z}(\tau); p, q)}{d\tau} \geqslant 0, \tag{2.9}$$

the following inequality (holding for $c = 0$) can be obtained [63]:

$$\begin{aligned} M_{\rm ADM}^2(z_\infty, \bar{z}_\infty, p, q) &= \lim_{\tau\to 0^-}[V_{\rm BH} - 4G^{i\bar{j}}(\partial_i\mathcal{W})\bar{\partial}_{\bar{j}}\mathcal{W}] \\ &= \lim_{\tau\to 0^-}\mathcal{W}^2 \equiv r_H^2(z_\infty, \bar{z}_\infty, p, q) \\ &\geqslant R_H^2(p, q) = \lim_{\tau\to -\infty}\mathcal{W}^2 = \lim_{\tau\to -\infty} V_{\rm BH}, \end{aligned} \tag{2.10}$$

where the radius r_H of the BH event horizon was introduced. More concisely,

$$r_H^2(z_\infty, \bar{z}_\infty, p, q) \geq R_H^2(p, q), \quad \forall\ (z_\infty, \bar{z}_\infty) \in \mathcal{M}_\infty, \tag{2.11}$$

holding in the whole asymptotical scalar manifold $\mathcal{M}_\infty$.

In the minimally matter coupled $\mathcal{N} = 2$, $d = 4$ supergravity based on the sequence of symmetric special Kähler manifolds (*complex Grassmannians*) $\frac{SU(1,n)}{SU(n)\times U(1)}$ [75] (see also the treatment of [63]), as well as in $\mathcal{N} = 3$, pure $\mathcal{N} = 4$, and $\mathcal{N} = 5$, $d = 4$ supergravity, it is possible to specialize further the inequality (2.11). Indeed, for such theories it holds that [recall Eq. (1.3), as well as Eqs. (1.4) and (1.5)]

$$\begin{aligned} R_H^2(p, q) \equiv \frac{S_{\rm BH}(p, q)}{\pi} &= r_H^2(z_\infty, \bar{z}_\infty, p, q) - G_{i\bar{j}}\Sigma^i\bar{\Sigma}^{\bar{j}} \\ &= r_H^2(z_\infty, \bar{z}_\infty, p, q) - 4\lim_{\tau\to 0^-} G^{i\bar{j}}(\partial_i\mathcal{W})\bar{\partial}_{\bar{j}}\mathcal{W}, \end{aligned} \tag{2.12}$$

where in the last step Eq. (2.5) was used. Equation (2.12), clearly yielding the inequality (2.11) by the presence of nonvanishing scalar charges and the (strict) positive definiteness of $G_{i\bar{j}}$, is nothing but a *many-moduli generalization* of the formula holding for the so-called (axion-) dilaton extremal BH [77]. The crucial feature, expressed by Eq. (2.12) and shared by the aforementioned supergravities, is the *disappearance* of the dependence on the *asymptotical moduli* $(z_\infty, \bar{z}_\infty)$ in the combination of quantities $r_H^2 - G_{i\bar{j}}\Sigma^i\bar{\Sigma}^{\bar{j}}$, which separately do depend on moduli.[3]

As a generalization of the formula holding (also in the nonextremal case) in the Maxwell-axion-dilaton supergravity (see e.g. [77,78], and also [63]), in [63] Eq. (2.12) was proved to hold in the extremal case for the whole sequence of $\mathcal{N} = 2$, $d = 4$ supergravity minimally coupled to Abelian vector multiplets [75], in terms of the (unique) invariant $\mathcal{I}_2$ of the U-duality group $G = SU(1, n)$, which is quadratic in charges:

$$\begin{aligned} R_H^2(p, q) &= r_H^2(z_\infty, \bar{z}_\infty, p, q) - 4\lim_{\tau\to 0^-} G^{i\bar{j}}(\partial_i\mathcal{W})\bar{\partial}_{\bar{j}}\mathcal{W} \\ &= |\mathcal{I}_2(p, q)|. \end{aligned} \tag{2.13}$$

We will report such results in Subsections III B and III C.

Then, by exploiting the first order formalism for $d = 4$ extremal BHs outlined above, we will show that the same happens for the following $d = 4$ supergravities:

(i) $\mathcal{N} = 3$ (matter coupled) [76], as intuitively expected by the strict similarity with the so-called minimally coupled $\mathcal{N} = 2$ theory (Subsections IV B and IV C);

(ii) $\mathcal{N} = 5$ [74], with $|\mathcal{I}_2|$ replaced by $\sqrt{|\mathcal{I}_4|}$ (Subsection VI B);

(iii) pure $\mathcal{N} = 4$ [73], with $|\mathcal{I}_2|$ replaced by $\sqrt{|\mathcal{I}_4|}$ (Subsection VII B).

Let us here note that while $\mathcal{N} = 5$ theory cannot be coupled to matter, in the case $\mathcal{N} = 4$ matter coupling is allowed, but Eq. (2.12) holds only in $\mathcal{N} = 4$ pure supergravity. Having a (n unique) U-invariant $\mathcal{I}_4$ quartic in charges, the aforementioned $\mathcal{N} = 4$ and $\mathcal{N} = 5$ theories are pretty different from the minimally coupled $\mathcal{N} = 2$ and $\mathcal{N} = 3$, $d = 4$ supergravity, as we will point out in the treatment below.

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[3] $r_H(z_\infty, \bar{z}_\infty, p, q)$ is the radius of the BH event horizon, which is the unique geometrical horizon for extremal BHs (in which $c = 0 \Leftrightarrow r_- = r_+ \equiv r_H$; for a recent treatment, see e.g. [63]). It depends on the dyonic BH charges p^Λ and q_Λ, but, in the presence of nonvanishing scalar charges, also on the asymptotical scalar fields $(z_\infty, \bar{z}_\infty)$.
In order to make contact with the attractor mechanism, and thus to characterize r_H as the *fixed point* of the scalar radial dynamics (in the considered static, spherically symmetric, and asymptotically flat extremal BH background), one has to evaluate r_H at the peculiar *geometrical locus* in the (asymptotical) moduli space defined by the (nondegenerate) criticality condition of $V_{\rm BH}$ (i.e. by $\partial V_{\rm BH} = 0$, with $V_{\rm BH}|_{\partial V_{\rm BH}=0} \neq 0$). Equation (2.5) yields that

$$\Sigma^i(z_H(p, q), \bar{z}_H(p, q), p, q) = 0 \quad \forall\ i,$$

where $(z_H(p, q), \bar{z}_H(p, q))$ are defined by

$$[\partial V_{\rm BH}(z, \bar{z}, p, q)]_{(z,\bar{z})=(z_H(p,q),\bar{z}_H(p,q))} \equiv 0.$$

Thus, Eqs. (2.6) and (2.7) [or Eq. (2.12)] consistently yield that

$$r_H(z_H(p, q), \bar{z}_H(p, q), p, q) = R_H(p, q).$$

It is here worth pointing out that in the nonextremal case (i.e. $c \neq 0$) the expression generalizing Eq. (2.12), namely

$$R_+^2(z_\infty, \bar{z}_\infty, p, q) \equiv \frac{S_{\mathrm{BH},c\neq 0}(z_\infty, \bar{z}_\infty, p, q)}{\pi} \equiv R_+^2(z_\infty, \bar{z}_\infty, p, q) = r_+^2(z_\infty, \bar{z}_\infty, p, q) - G_{i\bar{j}}\Sigma^i\bar{\Sigma}^{\bar{j}} \quad (2.14)$$

can be only guessed, but at present cannot be rigorously proved. Indeed, for static, spherically symmetric, asymptotically flat dyonic nonextremal BHs a first order formalism is currently unavailable, so there is no way to compute the scalar charges (besides the direct integration of the equations of motion of the scalars, as far as we know at present feasible only for the (axion-) dilaton BH [77], and—partially—for the *stu* model [54]).

III. $\mathcal{N} = 2$ MINIMALLY COUPLED SUPERGRAVITY

We consider $\mathcal{N} = 2$, $d = 4$ ungauged supergravity minimally coupled (mc) [75] to n_V Abelian vector multiplets, in the case in which the scalar manifold is given by the sequence of homogeneous symmetric *rank*-1 special Kähler manifolds

$$\mathcal{M}_{\mathcal{N}=2,mc,n} = \frac{G_{\mathcal{N}=2,mc,n}}{H_{\mathcal{N}=2,mc,n}} = \frac{SU(1,n)}{SU(n)\times U(1)}, \qquad \dim_{\mathbb{R}} = 2n, \qquad n = n_V \in \mathbb{N}. \quad (3.1)$$

The $1 + n$ vector field strengths and their duals, as well as their asymptotical fluxes, sit in the fundamental $\mathbf{1} + \mathbf{n}$ representation of the U-duality group $G_{\mathcal{N}=2,\mathrm{mc},n} = SU(1, n)$, in turn embedded in the symplectic group[4] $Sp(2 + 2n, \mathbb{R})$.

The general analysis of the attractor equations, BH charge orbits, and attractor moduli spaces of such a theory has been performed in [25,41].

By fixing the Kähler gauge such that $X^0 = 1$ and in a suitable system of local symplectic *special* coordinates, the geometry of $\mathcal{M}_{\mathcal{N}=2,\mathrm{mc},n}$ is determined by the *holomorphic prepotential function*:

$$\mathcal{F}(z) \equiv -\frac{i}{2}[1 - (z^i)^2]. \quad (3.2)$$

The Kähler potential of $\mathcal{M}_{\mathcal{N}=2,\mathrm{mc},n}$ can be computed to be ($\Lambda = 0, 1, \ldots, n$ throughout all the present section, and $|z|^2 \equiv \sum_{i=1}^{n_V}|z^i|^2$)

$$K(z, \bar{z}) = -\log[i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)] = -\log[2(1 - |z|^2)], \quad (3.3)$$

yielding the *metric constraint* $1 - |z|^2 > 0$, and the covariant and contravariant metric tensors to be, respectively, ($G_{i\bar{j}}(z, \bar{z})G^{i\bar{k}}(z, \bar{z}) = \delta_{\bar{j}}^{\bar{k}}$):

$$G_{i\bar{j}}(z, \bar{z}) = \frac{(1 - |z|^2)\delta_{i\bar{j}} + \bar{z}^{\bar{i}}z^j}{(1 - |z|^2)^2} = 2e^K\delta_{i\bar{j}} + 4e^{2K}\bar{z}^{\bar{i}}z^j; \quad (3.4)$$

$$G^{i\bar{j}}(z, \bar{z}) = (1 - |z|^2)(\delta^{i\bar{j}} - z^i\bar{z}^{\bar{j}}) = \tfrac{1}{2}e^{-K}(\delta^{i\bar{j}} - z^i\bar{z}^{\bar{j}}). \quad (3.5)$$

From its very definition (see e.g. [83], and references therein), the covariantly holomorphic $\mathcal{N} = 2$, $d = 4$ central charge function can be computed to be

$$Z = e^{K/2}W = e^{K/2}[q_0 + ip^0 + (q_i - ip^i)z^i] = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1 - |z|^2}}[q_0 + ip^0 + (q_i - ip^i)z^i], \quad (3.6)$$

where W is the $\mathcal{N} = 2$, $d = 4$ superpotential [also named holomorphic central charge function, with Kähler weights (2, 0)].

On the other hand, the so-called matter charges read

$$Z_i \equiv D_iZ = \partial_iZ + \frac{1}{2}(\partial_iK)Z = e^{K/2}[\partial_iW + (\partial_iK)W] = \frac{1}{\sqrt{2}(1 - |z|^2)^{3/2}}[(q_i - ip^i)(1 - |z|^2) + (q_0 + ip^0)\bar{z}^{\bar{i}} + (q_j - ip^j)z^j\bar{z}^{\bar{i}}]. \quad (3.7)$$

Here, D denotes the $U(1)$-Kähler and $H_{\mathcal{N}=2,\mathrm{mc},n}$-covariant differential operator. Because of the global vanishing of the C_{ijk}-tensor of special Kähler geometry, there are only two ($U(1)$-Kähler) and $H_{\mathcal{N}=2,\mathrm{mc},n}$ invariants, namely,

$$\alpha_1 \equiv |Z|; \quad (3.8)$$

$$\alpha_2 \equiv \sqrt{Z_i\bar{Z}^i} = \sqrt{G^{i\bar{j}}Z_i\bar{Z}_{\bar{j}}} = \sqrt{G^{i\bar{j}}(D_iZ)\bar{D}_{\bar{j}}\bar{Z}}, \quad (3.9)$$

both (homogeneous) of degree 1 in BH charges (p, q) [in particular, square roots of quantities quadratic in (p, q)]. By a suitable rotation of $U(n)$, the vector Z_i of *matter charges* can be chosen real and pointing in a given direction, e.g.

$$Z_i = \sqrt{Z_j\bar{Z}^j}\delta_{i1} = \alpha_2\delta_{i1}. \quad (3.10)$$

As recalled at the start of the next subsection, only ($\frac{1}{2}$)-BPS and non-BPS ($Z = 0$) attractor flows are nondegenerate (i.e. corresponding to large BHs, see below) [25], and the corresponding (squared) first order fake superpotentials are ([40]; recall Eq. (4.3) and (4.5), respectively)

[4] In all our analysis we consider the (semi)classical limit of continuous (unquantized), large BH charges.

$$\mathcal{W}^2_{((1/2)-)\mathrm{BPS}} = |Z|^2 = \alpha_1^2 = \frac{[q_0 + ip^0 + (q_i - ip^i)z^i][q_0 - ip^0 + (q_j + ip^j)\bar{z}^{\bar{j}}]}{2(1-|z|^2)}; \tag{3.11}$$

$$\begin{aligned}\mathcal{W}^2_{\text{non-BPS}(,Z=0)} &= G^{i\bar{j}}(D_iZ)\bar{D}_{\bar{j}}\bar{Z} = \alpha_2^2 \\ &= \frac{1}{2(1-|z|^2)^2}(\delta^{i\bar{j}} - z^i\bar{z}^{\bar{j}})\cdot[(q_i - ip^i)(1-|z|^2) + (q_0+ip^0)\bar{z}^{\bar{i}} + (q_r - ip^r)z^r\bar{z}^{\bar{i}}] \\ &\quad \cdot[(q_j + ip^j)(1-|z|^2) + (q_0 - ip^0)z^j + (q_n + ip^n)\bar{z}^{\bar{n}}z^j],\end{aligned} \tag{3.12}$$

where use of Eqs. (3.6) and (3.7) was made.

A. Attractor equations and their solutions

The BH effective potential can be written as

$$V_{\mathrm{BH}} = |Z|^2 + G^{i\bar{j}}(D_iZ)\bar{D}_{\bar{j}}\bar{Z} = \alpha_1^2 + \alpha_2^2. \tag{3.13}$$

The $\mathcal{N}=2$, $d=4$ attractor equations in the case of minimal coupling to Abelian vector multiplets are nothing but the criticality conditions for such an $H_{\mathcal{N}=2,\mathrm{mc},n}$-invariant (and Kähler-gauge-invariant) quantity. Such criticality conditions are satisfied for two classes of critical points:

(i) ($\frac{1}{2}$)-BPS:

$$D_iZ = 0 \quad \forall\ i = 1,\ldots,n \Leftrightarrow \alpha_2 = 0, \qquad Z \neq 0; \tag{3.14}$$

(ii) nonsupersymmetric (non-BPS with $Z=0$):

$$\begin{gathered} D_iZ \neq 0 \text{ (at least for some } i), \\ Z = 0 \Leftrightarrow \alpha_1 = 0. \end{gathered} \tag{3.15}$$

It is worth counting here the degrees of freedom related to Eqs. (3.14) and (3.15). The $\frac{1}{2}$-BPS criticality conditions (3.14) are n complex independent ones, thus all scalars are stabilized by such conditions. On the other hand, there is only one complex non-BPS $Z=0$ criticality condition (3.15). This fact paves the way to the possibility to have a moduli space of non-BPS $Z=0$ attractors, spanned by the $n-1$ complex scalars unstabilized by Eq. (3.15); this actually holds true [41], as it will be explicitly found below for the first time (see Subsection III A 2).

1. $\frac{1}{2}$-BPS attractors

An *algebraic*, equivalent approach to the direct resolution of the n complex $\frac{1}{2}$-BPS criticality conditions (3.14) is based on the resolution of the special Kähler geometry identities evaluated along the geometrical locus in $\mathcal{M}_{\mathcal{N}=2,\mathrm{quadr},n}$ defined by the constraints (3.14). By following such an approach, the electric and magnetic BH charges are constrained as follows [2]:

$$\begin{cases} p^\Lambda = ie^{K/2}(\bar{Z}X^\Lambda - Z\bar{X}^\Lambda); \\ q_\Lambda = ie^{K/2}(\bar{Z}F_\Lambda - Z\bar{F}_\Lambda). \end{cases} \tag{3.16}$$

Summing such two sets of symplectic-covariant equations, one gets

$$X^\Lambda q_\Sigma - p^\Lambda F_\Sigma = ie^{K/2}Z(\bar{X}^\Lambda F_\Sigma - X^\Lambda \bar{F}_\Sigma), \tag{3.17}$$

in which the scalars z^i and $\bar{z}^{\bar{i}}$ and the central charge function Z are understood to be evaluated at the BH horizon. Then we can proceed to solve for the scalars, stabilized at the BH horizon in terms of the BH charges; by rewriting Eqs. (3.17) in components, one achieves the following result:

$$\begin{aligned} &(\Lambda,\Sigma) = (0,0){:}\, q_0 + ip^0 = 2e^{K/2}Z; \\ &(\Lambda,\Sigma) = (0,i){:}\, q_i - iz^ip^0 = -e^{K/2}Z(z^i + \bar{z}^{\bar{i}}); \\ &(\Lambda,\Sigma) = (i,0){:}\, z^iq_0 + ip^i = e^{K/2}Z(z^i + \bar{z}^{\bar{i}}); \\ &(\Lambda,\Sigma) = (i,i){:}\, z^i(q_i - ip^i) = -2e^{K/2}Z|z|^2. \end{aligned} \tag{3.18}$$

The decoupling of such $2n_V + 2$ real algebraic equations in terms of the n_V complex unknowns z^i [the two additional real degrees of freedom residing in the homogeneity of degree 1 of the system (3.18) in BH charges] allows for an effortless resolution, yielding the following explicit expression of the n complex scalars determining the $\frac{1}{2}$-BPS attractor scalar horizon configurations:

$$z^i_{\mathrm{BPS}} = -\frac{(q_i + ip^i)}{q_0 - ip^0}, \quad \forall\ i = 1,\ldots,n. \tag{3.19}$$

Notice that all n_V complex scalars z^i are stabilized in terms of the BH charges, and thus, as is well known, no classical moduli space for $\frac{1}{2}$-BPS attractors exists at all. By recalling Eqs. (1.2) and (3.13), and plugging Eqs. (3.19) into Eqs. (3.6) and (3.7), one obtains that

$$\begin{aligned}\frac{S_{\mathrm{BH,BPS}}}{\pi} &= \frac{A_{H,\mathrm{BPS}}}{4} = V_{\mathrm{BH}}|_{\mathrm{BPS}} = \alpha^2_{1,\mathrm{BPS}} = |Z|^2_{\mathrm{BPS}} \\ &= I_2 > 0,\end{aligned} \tag{3.20}$$

where I_2 is the (unique) invariant of the fundamental/antifundamental $(\mathbf{1}+\mathbf{n},\ \overline{\mathbf{1}+\mathbf{n}})$ representation of the U-duality group $G_{\mathcal{N}=2,\mathrm{mc},n}$ (not irreducible with respect to $G_{\mathcal{N}=2,\mathrm{mc},n}$ itself), quadratic in BH charges [see Eq. (5.1) below]:

$$\mathcal{I}_2 = \tfrac{1}{2}[q_0^2 - q_i^2 + (p^0)^2 - (p^i)^2] = \tfrac{1}{2}(q^2 + p^2), \quad (3.21)$$

where $q^2 \equiv \eta^{\Lambda\Sigma} q_\Lambda q_\Sigma$ and $p^2 \equiv \eta_{\Lambda\Sigma} p^\Lambda p^\Sigma$, $\eta^{\Lambda\Sigma} = \eta_{\Lambda\Sigma}$ being the $(1+n)$-dim. Lorentzian metric with signature $(+, -, \ldots, -)$ (see the discussion in Sec. V). In terms of the dressed charges Z and D_iZ, the (only apparently moduli-dependent) expression of $\mathcal{I}_2$ reads (see e.g. [25]):

$$\mathcal{I}_2 = |Z|^2 - G^{i\bar{j}}(D_iZ)\bar{D}_{\bar{j}}\bar{Z} = \alpha_1^2 - \alpha_2^2. \quad (3.22)$$

It can be explicitly checked that z^i_{BPS} given by Eqs. (3.19) satisfy the metric constraint $1 - |z^i|^2 > 0$ [yielded by Eq. (3.3)].

It is well known that $\frac{1}{2}$-BPS critical points of V_{BH} are stable, at least as far as the metric $G_{i\bar{j}}$ of the special Kähler scalar manifold is positive definite (at such points); indeed, the $2n_V \times 2n_V$ (covariant) Hessian matrix $\mathcal{H}^{V_{\mathrm{BH}}}_{\mathrm{BPS}}$ of V_{BH} at its $\frac{1}{2}$-BPS critical points has rank $2n_V$, and it reads [4]:

$$\begin{aligned}\mathcal{H}^{V_{\mathrm{BH}}}_{\mathrm{BPS}} &= 2|Z|^2_{\mathrm{BPS}}\begin{pmatrix} 0 & G_{i\bar{j}} \\ G_{\bar{j}i} & 0 \end{pmatrix}_{\mathrm{BPS}} \\ &= [q_0^2 - q_k^2 + (p^0)^2 - (p^k)^2]\cdot\begin{pmatrix} 0 & e^{K_{\mathrm{BPS}}}\delta_{i\bar{j}} + e^{2K_{\mathrm{BPS}}}\bar{z}_{i,\mathrm{BPS}} z_{\bar{j},\mathrm{BPS}} \\ e^{K_{\mathrm{BPS}}}\delta_{\bar{j}i} + e^{2K_{\mathrm{BPS}}}\bar{z}_{j,\mathrm{BPS}} z_{\bar{i},\mathrm{BPS}} & 0 \end{pmatrix},\end{aligned} \quad (3.23)$$

where use of the stabilization equations (3.19) was made.

2. *Non-BPS* ($Z = 0$) *attractors and their moduli space*

As yielded by Eqs. (3.15), non-BPS ($Z = 0$) attractor solutions are given by $D_iZ \neq 0$ for at least some $i \in \{1, \ldots, n_V\}$, and by the vanishing of the central charge Z, which in the considered theory reads as follows [(within the metric constraint $1 - |z|^2 > 0$; recall Eq. (3.6)]:

$$Z = 0 \Leftrightarrow q_0 + ip^0 = -(q_i - ip^i) z^i_{\text{non-BPS}}. \quad (3.24)$$

As noticed above, this is one complex equation in terms of n_V complex unknowns z^i, thus at most only one of them will be stabilized in terms of the BH charges. Indeed, one can choose, without any loss of generality, to solve Eq. (3.24) for z^1, getting ($\hat{i} = 2, \ldots, n_V$):

$$z^1_{\text{non-BPS}} = -\frac{(q_{\hat{i}} - ip^{\hat{i}}) z^{\hat{i}}_{\text{non-BPS}}}{q_1 - ip^1} - \frac{q_0 + ip^0}{q_1 - ip^1}. \quad (3.25)$$

The remaining scalars $z^{\hat{i}}$ are not stabilized at the considered (nondegenerate) non-BPS $Z = 0$ critical points of V_{BH}. As known from group-theoretical arguments (see Table 3 of [41]), such scalars span a moduli space given by the rank-1 symmetric special Kähler manifold

$$\begin{aligned}\mathcal{M}_{\mathcal{N}=2,\mathrm{mc},n,\text{non-BPS}} &= \frac{SU(1, n-1)}{SU(n-1)\times U(1)} \\ &= \mathcal{M}_{\mathcal{N}=2,\mathrm{mc},n-1}, \\ \dim_{\mathbb{R}} &= 2(n-1).\end{aligned} \quad (3.26)$$

The unique element of the sequence $\mathcal{M}_{\mathcal{N}=2,\mathrm{mc},n}$, $n \in \mathbb{N}$, in which the non-BPS $Z = 0$ attractors have no associated moduli space is the $n = 1$ case (the so-called t^2 model), in which all nondegenerate critical points of V_{BH} are stable, with no flat directions at all.

The existence of $n - 1$ flat directions at all orders in the (covariant) differentiation of V_{BH} at its *nondegenerate* non-BPS $Z = 0$ critical points in the considered theory can be realized also by the following argument.

First, it can be explicitly computed that the application of an odd number of covariant differential operators on V_{BH} always yields a vanishing result (here the tilded indices can be either holomorphic or antiholomorphic; $m \in \mathbb{N}$ throughout):

$$(D_{\tilde{i}_1} D_{\tilde{i}_2} \ldots D_{\tilde{i}_{2m-1}} V_{\mathrm{BH}})_{\text{non-BPS}} = 0. \quad (3.27)$$

Then, the $2n_V \times 2n_V$ (covariant) Hessian matrix $\mathcal{H}^{V_{\mathrm{BH}}}_{\text{non-BPS}}$ of V_{BH} at its non-BPS $Z = 0$ critical points can be computed to be

$$\mathcal{H}^{V_{\mathrm{BH}}}_{\text{non-BPS}} = 2\begin{pmatrix} 0 & (D_iZ)\bar{D}_{\bar{j}}\bar{Z} \\ (D_jZ)\bar{D}_{\bar{i}}\bar{Z} & 0 \end{pmatrix}_{\text{non-BPS}}, \quad (3.28)$$

and it has thus rank 2, with 2 strictly positive and $2n_V - 2$ vanishing real eigenvalues (massless "Hessian modes").

In order to investigate the persistence of such $2n_V - 2$ massless "Hessian modes" to higher order in the covariant differentiation of V_{BH}, one can define a *"putative" mass matrix* $\mathcal{H}^{V_{\mathrm{BH}}}_m$ for scalars, such that $\mathcal{H}^{V_{\mathrm{BH}}}_{m=0} = \mathcal{H}^{V_{\mathrm{BH}}}$ (covariant Hessian matrix of V_{BH}), in the following way:

$$\mathcal{H}^{V_{\mathrm{BH}}}_m \equiv (D_{\bar{i}} D_{\bar{j}} D_{\tilde{i}_1} \ldots D_{\tilde{i}_{2m}} V_{\mathrm{BH}}) Z^{\tilde{i}_1} \ldots Z^{\tilde{i}_{2m}}, \quad (3.29)$$

where $Z^{\tilde{i}}$ denotes the relevant contravariant matter charge. It can be thus calculated that

$$\mathcal{H}^{V_{\mathrm{BH}}}_{m,\text{non-BPS}} = 2^{2m}(V_{\mathrm{BH,non\text{-}BPS}})^m \mathcal{H}^{V_{\mathrm{BH}}}_{\text{non-BPS}}. \quad (3.30)$$

Therefore, regardless of m the putative mass matrix $\mathcal{H}^{V_{\mathrm{BH}}}_m$ has rank 2, with 2 strictly positive and $2n_V - 2$ vanishing real eigenvalues, and these latter thus span a moduli space.

By recalling Eq. (3.7) and plugging Eq. (3.24) into the matter charges D_iZ, one obtains

$$D_iZ|_{\text{non-BPS}} = \frac{q_i - ip^i}{\sqrt{2}\sqrt{1 - |z|^2_{\text{non-BPS}}}}. \quad (3.31)$$

By such a result, $\mathcal{H}^{V_{\rm BH}}_{\text{non-BPS}}$ can be rewritten as follows:

$$\mathcal{H}^{V_{\rm BH}}_{\text{non-BPS}} = \frac{1}{1-|z|^2_{\text{non-BPS}}}\begin{pmatrix} 0 & (q_i - ip^i)(q_j + ip^j) \\ (q_j - ip^j)(q_i + ip^i) & 0 \end{pmatrix}. \tag{3.32}$$

Because of Eqs. (3.24) and (3.25), in general the matter charges D_iZ and $\mathcal{H}^{V_{\rm BH}}$ are not stabilized in terms of the BH charges at the considered non-BPS $Z=0$ critical points, but nevertheless this does not affect the moduli independence of the BH entropy. Indeed, by recalling Eqs. (1.2) and (3.13), and plugging Eqs. (3.15) and (3.24) into Eqs. (3.6) and (3.7), one obtains that

$$\begin{aligned}\frac{S_{\text{BH,non-BPS}}}{\pi} &= \frac{A_{H,\text{non-BPS}}}{4} = V_{\rm BH}|_{\text{non-BPS}} = \alpha^2_{2,\text{non-BPS}} \\ &= [G^{i\bar{j}}(D_iZ)\bar{D}_{\bar{j}}\bar{Z}]_{\text{non-BPS}} \\ &= [G^{i\bar{j}}(\partial_iZ)\bar{\partial}_{\bar{j}}\bar{Z}]_{\text{non-BPS}} = -\mathcal{I}_2 > 0,\end{aligned} \tag{3.33}$$

where $\mathcal{I}_2$ is the (unique) quadratic $G_{\mathcal{N}=2,\text{mc},n}$-invariant given by Eqs. (3.21) and (3.22).

Thus, in $\mathcal{N}=2$, $d=4$ supergravity minimally coupled to Abelian vector multiplets, the BH charges supporting nondegenerate critical points of V_{BH} are split in two branches: the $(\frac{1}{2})$-BPS one, defined by $\mathcal{I}_2>0$, and the non-BPS ($Z=0$) one, corresponding to $\mathcal{I}_2<0$.

B. Black hole parameters for $\frac{1}{2}$-BPS flow

By using the explicit expressions of $\mathcal{W}^2_{\rm BPS}$ given by Eq. (6.32), using the differential relations of special Kähler geometry of $\mathcal{M}_{\mathcal{N}=2,mc,n}$ (see e.g. [83], and references therein), and exploiting the first order (fake supergravity) formalism discussed in Sec. II, one, respectively, obtains the following expressions of the (square) ADM mass covariant scalar charges, and (square) effective horizon radius for the $\frac{1}{2}$-BPS attractor flow[5] [63]:

$$\begin{aligned} r^2_{H,\rm BPS}(z_\infty,\bar{z}_\infty,p,q) &= M^2_{\rm ADM,BPS}(z_\infty,\bar{z}_\infty,p,q) = \mathcal{W}^2_{\rm BPS}(z_\infty,\bar{z}_\infty,p,q) = \lim_{\tau\to 0^-}|Z|^2(z(\tau),\bar{z}(\tau),p,q) \\ &= \frac{[q_0+ip^0+(q_i-ip^i)z^i_\infty][q_0-ip^0+(q_j+ip^j)\bar{z}^{\bar{j}}_\infty]}{2(1-|z_\infty|^2)};\end{aligned} \tag{3.34}$$

$$\begin{aligned}\Sigma_{i,\rm BPS}(z_\infty,\bar{z}_\infty,p,q) &= 2\lim_{\tau\to 0^-}(\partial_i\mathcal{W}_{\rm BPS})(z(\tau),\bar{z}(\tau),p,q) = \frac{1}{M_{\rm ADM,BPS}(z_\infty,\bar{z}_\infty,p,q)}\lim_{\tau\to 0^-}(\bar{Z}D_iZ)(z(\tau),\bar{z}(\tau),p,q) \\ &= \frac{1}{\sqrt{2}(1-|z_\infty|^2)^{3/2}}\sqrt{\frac{q_0-ip^0+(q_j+ip^j)\bar{z}^{\bar{j}}_\infty}{q_0+ip^0+(q_k-ip^k)z^k_\infty}}\cdot[(q_i-ip^i)(1-|z_\infty|^2)+(q_0+ip^0)\bar{z}^{\bar{i}}_\infty \\ &\quad +(q_r-ip^r)z^r_\infty\bar{z}^{\bar{i}}_\infty];\end{aligned} \tag{3.35}$$

$$\begin{aligned} R^2_{H,\rm BPS} &= \lim_{\tau\to 0^-}[\mathcal{W}^2_{\rm BPS}(z(\tau),\bar{z}(\tau),p,q) - 4G^{i\bar{j}}(z(\tau),\bar{z}(\tau))(\partial_i\mathcal{W}_{\rm BPS})(z(\tau),\bar{z}(\tau),p,q)\cdot(\bar{\partial}_{\bar{j}}\mathcal{W}_{\rm BPS})(z(\tau),\bar{z}(\tau),p,q)] \\ &= \mathcal{I}_2(p,q) = V_{\rm BH,BPS} = \frac{S_{\rm BH,BPS}(p,q)}{\pi}.\end{aligned} \tag{3.36}$$

Equation (3.36) proves Equation (2.13) for the $\frac{1}{2}$-BPS attractor flow of the $\mathcal{N}=2$, $d=4$ supergravity minimally coupled to $n\equiv n_V$ Abelian vector multiplets.

Notice that in the extremality regime ($c=0$) the effective horizon radius R_H, and thus A_H and the Bekenstein-Hawking entropy $S_{\rm BH}$ are independent on the particular vacuum or ground state of the considered theory, i.e. on $(z^i_\infty,\bar{z}^{\bar{i}}_\infty)$, but rather they depend only on the electric and magnetic charges q_Λ and p^Λ, which are conserved due to the overall $(U(1))^{n+1}$ gauge invariance. The independence on $(z^i_\infty,\bar{z}^{\bar{i}}_\infty)$ is of crucial importance for the consistency of the *microscopic state counting interpretation* of $S_{\rm BH}$, as well as for the overall consistency of the macroscopic thermodynamic picture of the BH. However, it is worth recalling that the ADM mass $M_{\rm ADM}$ generally does depend on $(z^i_\infty,\bar{z}^{\bar{i}}_\infty)$ also in the extremal case, as yielded by Eq. (3.34) for the considered $\frac{1}{2}$-BPS attractor flow.

Furthermore, Eq. (3.34) yields that the $\frac{1}{2}$-BPS attractor flow of the $\mathcal{N}=2$, $d=4$ supergravity minimally coupled

[5]Throughout the whole paper, for all the considered functions $f(z,\bar{z},p,q)$ we assume

$$(f(z,\bar{z},p,q))_\infty \equiv \lim_{\tau\to 0^-}f(z(\tau),\bar{z}(\tau),p,q) = f(z_\infty,\bar{z}_\infty,p,q).$$

Furthermore, we assume $f(z,\bar{z},p,q)$ to be smooth enough to split the asymptotical limit of a product into the product of the asymptotical limits of the factors.

to $n \equiv n_V$ Abelian vector multiplets does not saturate the marginal stability bound (see [54,84]).

C. Black hole parameters for non-BPS ($Z = 0$) flow

By using the explicit expressions of $\mathcal{W}^2_{\text{non-BPS}}$ given by Eq. (3.12), using the differential relations of special Kähler geometry of $\mathcal{M}_{\mathcal{N}=2,\text{mc},n}$ (see e.g. [83] and references therein), and exploiting the first order (fake supergravity) formalism discussed in Sec. II, one, respectively, obtains the following expressions of the (square) ADM mass, covariant scalar charges, and (square) effective horizon radius for the non-BPS $Z = 0$ attractor flow [63]:

$$\begin{aligned} r^2_{H,\,\text{non-BPS}}(z_\infty, \bar{z}_\infty, p, q) &= M^2_{\text{ADM, non-BPS}}(z_\infty, \bar{z}_\infty, p, q) = \mathcal{W}^2_{\text{non-BPS}}(z_\infty, \bar{z}_\infty, p, q) = \lim_{\tau\to 0^-}[G^{i\bar{j}}(D_i Z)\bar{D}_{\bar{j}}\bar{Z}](z(\tau), \bar{z}(\tau), p, q) \\ &= \frac{1}{2(1-|z_\infty|^2)^2}(\delta^{i\bar{j}} - z^i_\infty \bar{z}^{\bar{j}}_\infty)\cdot[(q_i - ip^i)(1-|z_\infty|^2) + (q_0 + ip^0)\bar{z}^{\bar{i}}_\infty + (q_r - ip^r)z^r_\infty \bar{z}^{\bar{i}}_\infty] \\ &\quad \cdot [(q_j + ip^j)(1-|z_\infty|^2) + (q_0 - ip^0)z^j_\infty + (q_n + ip^n)\bar{z}^{\bar{n}}_\infty z^j_\infty]; \end{aligned} \tag{3.37}$$

$$\begin{aligned} \Sigma_{i,\,\text{non-BPS}}(z_\infty, \bar{z}_\infty, p, q) &= 2\lim_{\tau\to 0^-}(\partial_i \mathcal{W}_{\text{non-BPS}})(z(\tau), \bar{z}(\tau), p, q) = \frac{1}{M_{\text{ADM, non-BPS}}(z_\infty, \bar{z}_\infty, p, q)}\lim_{\tau\to 0^-}(\bar{Z}D_i Z)(z(\tau), \bar{z}(\tau), p, q) \\ &= \frac{1}{\sqrt{2}}\frac{1}{(1-|z_\infty|^2)}\cdot[q_0 - ip^0 + (q_j + ip^j)\bar{z}^{\bar{j}}_\infty]\cdot[(q_i - ip^i)(1-|z_\infty|^2) + (q_0 + ip^0)\bar{z}^{\bar{i}}_\infty \\ &\quad + (q_m - ip^m)z^m_\infty \bar{z}^{\bar{i}}_\infty]\cdot[(\delta^{n\bar{p}} - z^n_\infty \bar{z}^{\bar{p}}_\infty)\cdot[(q_n - ip^n)(1-|z_\infty|^2) + (q_0 + ip^0)\bar{z}^{\bar{n}}_\infty \\ &\quad + (q_s - ip^s)z^s_\infty \bar{z}^{\bar{n}}_\infty]\cdot[(q_p + ip^p)(1-|z_\infty|^2) + (q_0 - ip^0)z^p_\infty + (q_w + ip^w)\bar{z}^{\bar{w}}_\infty z^p_\infty]^{-1/2}, \end{aligned} \tag{3.38}$$

$$\begin{aligned} R^2_{H,\text{non-BPS}} &= \lim_{\tau\to 0^-}[\mathcal{W}^2_{\text{non-BPS}}(z(\tau), \bar{z}(\tau), p, q) - 4G^{i\bar{j}}(z(\tau), \bar{z}(\tau))(\partial_i \mathcal{W}_{\text{non-BPS}})(z(\tau), \bar{z}(\tau), p, q)\cdot(\bar{\partial}_{\bar{j}}\mathcal{W}_{\text{non-BPS}}) \\ &\quad \times (z(\tau), \bar{z}(\tau), p, q)] \\ &= -\mathcal{I}_2(p, q) = V_{\text{BH, non-BPS}} = \frac{S_{\text{BH, non-BPS}}(p, q)}{\pi}. \end{aligned} \tag{3.39}$$

Equation (3.39) proves Eq. (2.13) for the non-BPS $Z = 0$ attractor flow of the $\mathcal{N} = 2$, $d = 4$ supergravity minimally coupled to $n \equiv n_V$ Abelian vector multiplets. The considerations made at the end of Subsection III B hold also for the considered attractor flow.

It is worth noticing out that Eqs. (3.36) and (3.39) are consistent, because, as pointed out above, the ($\frac{1}{2}$)-BPS- and non-BPS ($Z = 0$)-supporting BH charge configurations in the considered theory are, respectively, defined by the quadratic constraints $\mathcal{I}_2(p, q) > 0$ and $\mathcal{I}_2(p, q) < 0$.

As yielded by Eqs. (3.35) and (3.38) for both nondegenerate attractor flows of the considered theory it holds the following relation among scalar charges and ADM mass:

$$\Sigma_i = \frac{1}{M_{\text{ADM}}}\lim_{\tau\to 0^-} D_i(|Z|^2). \tag{3.40}$$

Furthermore, Eq. (3.37) yields that the non-BPS $Z = 0$ attractor flow of the $\mathcal{N} = 2$, $d = 4$ supergravity minimally coupled to $n \equiv n_V$ Abelian vector multiplets does not saturate the marginal stability bound (see [54,84]).

As it will be proved in the next sections, for all nondegenerate attractor flows of the considered $d = 4$ supergravities the marginal stability bound is not saturated. A more detailed discussion of such an issue falls beyond the scope of the present investigation, and it will be given elsewhere [85].

IV. $\mathcal{N} = 3$ SUPERGRAVITY

The (Kähler) scalar manifold is [76]

$$\mathcal{M}_{\mathcal{N}=3,n} = \frac{G_{\mathcal{N}=3,n}}{H_{\mathcal{N}=3,n}} = \frac{SU(3,n)}{SU(3)\times SU(n)\times U(1)}, \qquad \dim_{\mathbb{R}} = 6n. \tag{4.1}$$

The $3 + n$ vector field strengths and their duals, as well as their asymptotical fluxes, sit in the fundamental $\mathbf{3} + \mathbf{n}$ representation of the U-duality group $G_{\mathcal{N}=3,n} = SU(3, n)$, in turn embedded in the symplectic group $Sp(6 + 2n, \mathbb{R})$.

$Z_{AB} = Z_{[AB]}$ ($A, B = 1, 2, 3 = \mathcal{N}$) is the central charge matrix, and Z_I ($I = 1, \ldots, n$) are the matter charges, where $n \in \mathbb{N}$ is the number of matter (Abelian vector) multiplets coupled to the gravity multiplet. By a suitable transformation of the $\mathcal{R}$-symmetry $U(3)$, Z_{AB} can be skew-diagonalized by putting it in the *normal form* (see e.g. [40] and references therein):

$$Z_{AB} = \begin{pmatrix} \mathcal{Z}_1\epsilon & \\ & 0 \end{pmatrix}, \tag{4.2}$$

where ϵ is the 2×2 symplectic metric, and $\mathcal{Z}_1 \in \mathbb{R}^+_0$ is the unique $\mathcal{N} = 3$ (moduli-dependent) skew-eigenvalue, which can be expressed in terms of the unique $U(3)$ (and also $H_{\mathcal{N}=3,n} = U(3) \times U(n)$)-invariant as follows:

$$\mathcal{Z}_1 = \sqrt{\tfrac{1}{2} Z_{AB} \bar{Z}^{AB}}. \tag{4.3}$$

On the other hand, by a suitable rotation of $U(n)$, the vector Z_I of matter charges can be chosen real and pointing in a given direction, e.g.

$$Z_I = \rho \delta_{I1}, \tag{4.4}$$

where ρ can be expressed the unique $U(n)$ [and also $H_{\mathcal{N}=3,n} = U(3) \times U(n)$]-invariant as

$$\rho = \sqrt{Z_I \bar{Z}^I}. \tag{4.5}$$

The simplest holomorphic parametrization of $\mathcal{M}_{\mathcal{N}=3,n}$ can be written in terms of the $(3+n) \times (3+n)$ coset representative [86,87]

$$L = \begin{pmatrix} \sqrt{1 + XX^\dagger} & X \\ X^\dagger & \sqrt{1 + X^\dagger X} \end{pmatrix}, \tag{4.6}$$

where X is a complex $n \times 3$ matrix in the bifundamental of $SU(3) \times SU(n) = H_{\mathcal{N}=3,n} \backslash U(1)$, whose components are nothing but the $3n$ complex scalars z^i ($i = 1, \ldots, 3n$) spanning $\mathcal{M}_{\mathcal{N}=3,n}$. The embedding of $G_{\mathcal{N}=3,n}$ into the symplectic group $Sp(6+2n, \mathbb{R})$:

$$\begin{aligned} &SU(3,n) \longrightarrow Sp(6+2n, \mathbb{R}), \\ &g \equiv L(z) \longrightarrow S(g) \equiv S(L(z)), \end{aligned} \tag{4.7}$$

is determined by the $(6+2n) \times (6+2n)$ matrix

$$S(g) = \begin{pmatrix} \phi_0 & \bar{\phi}_1 \\ \phi_1 & \bar{\phi}_0 \end{pmatrix} \in SU(3,n) \subset Sp(6+2n, \mathbb{R}), \tag{4.8}$$

such that the $(3+n) \times (3+n)$ sub-blocks ϕ_0 and ϕ_1 satisfy the relations

$$\phi_0^\dagger \phi_0 - \phi_1^\dagger \phi_1 = 1, \qquad \phi_0^\dagger \bar{\phi}_1 - \phi_1^\dagger \bar{\phi}_0 = 0. \tag{4.9}$$

Let us here recall that in the *Gaillard-Zumino formalism* [88], the vector kinetic matrix can be written as ($\Lambda = 1, 2, 3, 4, \ldots, 3+n$ throughout all the present section)

$$\mathcal{N}_{\Lambda\Sigma} = (\phi_0^\dagger + \phi_1^\dagger)^{-1}(\phi_0^\dagger - \phi_1^\dagger). \tag{4.10}$$

The embedding $SU(3,n) \longrightarrow Sp(6+2n, \mathbb{R})$ is determined once S is written as a function of $X(z)$, namely,

$$S(X) = \begin{pmatrix} \sqrt{1 + XX^\dagger} & 0 & 0 & X \\ 0 & \sqrt{1 + X^T \bar{X}} & X^T & 0 \\ 0 & \bar{X} & \sqrt{1 + \bar{X} X^T} & 0 \\ X^T & 0 & 0 & \sqrt{1 + X^\dagger X} \end{pmatrix}, \tag{4.11}$$

that is

$$\phi_1 = \begin{pmatrix} 0 & \bar{X} \\ X^\dagger & 0 \end{pmatrix}; \tag{4.12}$$

$$\phi_0 = \sqrt{1 + \bar{\phi}_1 \phi_1} = \begin{pmatrix} \sqrt{1 + XX^\dagger} & 0 \\ 0 & \sqrt{1 + X^T \bar{X}} \end{pmatrix}. \tag{4.13}$$

The vector kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ can be written in terms of the $(3+n) \times (3+n)$ symplectic sections (and their inverse) as follows (see e.g. [83], and references therein):

$$\mathcal{N}_{\Lambda\Sigma} = h_{\Omega\Lambda} (f^{-1})^{\Omega}_{\Sigma}. \tag{4.14}$$

The explicit dependence of the symplectic sections on the sub-blocks of $S(X)$ is simply

$$h_{\Lambda\Sigma} = -\frac{i}{\sqrt{2}}(\phi_0 - \phi_1), \qquad f^{\Lambda}_{\Sigma} = \frac{1}{\sqrt{2}}(\phi_0 + \phi_1), \tag{4.15}$$

whereas in terms of the matrix $X(z)$ they read

$$\begin{aligned} h_{\Lambda\Sigma} &= -\frac{i}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + XX^\dagger} & -\bar{X} \\ -X^\dagger & \sqrt{1 + X^T \bar{X}} \end{pmatrix} \\ &\equiv (h_{\Lambda|AB}, \bar{h}_{\Lambda|I}) \equiv \mathbf{h}; \end{aligned} \tag{4.16}$$

$$f^{\Lambda}_{\Sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + XX^\dagger} & \bar{X} \\ X^\dagger & \sqrt{1 + X^T \bar{X}} \end{pmatrix} \equiv (f^{\Lambda}_{AB}, \bar{f}^{\Lambda}_{\bar{I}}) \equiv \mathbf{f}. \tag{4.17}$$

By rewriting Eqs. (4.9) in terms of the symplectic sections, one finds [89,90]

$$i(\mathbf{f}^\dagger \mathbf{h} - \mathbf{h}^\dagger \mathbf{f}) = 1; \tag{4.18}$$

$$\mathbf{h}^T \mathbf{f} - \mathbf{h} \mathbf{f}^T = 0. \tag{4.19}$$

The central charge matrix Z_{AB} and the matter charges Z_I are, respectively, defined as the integral over the 2-sphere at infinity S^2_∞ of the dressed graviphoton and matter field strengths [89–91]:

$$Z_{AB} \equiv -\int_{S^2_\infty} T_{AB} = -\int_{S^2_\infty} T^-_{AB} = f^{\Lambda}_{AB} q_\Lambda - h_{\Lambda|AB} p^\Lambda; \tag{4.20}$$

$$Z_I \equiv -\int_{S^2_\infty} T_I = -\int_{S^2_\infty} T_I^- = f_I^\Lambda q_\Lambda - h_{\Lambda|I} p^\Lambda. \quad (4.21)$$

Using the explicit expression for the symplectic sections given in Eqs. (4.16) and (4.17), one obtains

$$Z_{AB} = \frac{1}{\sqrt{2}}[(\sqrt{1+XX^\dagger})^C_{AB}(q_C + ip^C) + \bar{X}^i_{AB}(q_i - ip^i)]; \quad (4.22)$$

$$Z_I = \frac{1}{\sqrt{2}}[(X^\dagger)^A_I(q_A - ip^A) + (\sqrt{1+X^T\bar{X}})_I{}^i(q_i + ip^i)]. \quad (4.23)$$

As recalled at the start of the next subsection, only ($\frac{1}{3}$)-BPS and non-BPS ($Z_{AB}=0$) attractor flows are nondegenerate (i.e. corresponding to large BHs), and the corresponding (square) first order fake superpotentials are ([40]; recall Eq. (4.3) and (4.5), respectively)

$$\begin{aligned}\mathcal{W}^2_{((1/3)-)\mathrm{BPS}} &= \frac{1}{2}Z_{AB}\bar{z}^{AB} = \mathcal{Z}_1^2\\ &= \frac{1}{2}[[(q_C - ip^C)(\sqrt{1+XX^\dagger})^C + (q_i+ip^i)(X^T)^i]\cdot[(\sqrt{1+XX^\dagger})^D(q_D+ip^D) + \bar{X}^j(q_j - ip^j)]]\\ &= \frac{1}{2}[(1+XX^\dagger)^{AB}(q_A - ip^A)(q_B+ip^B) + (\sqrt{1+XX^\dagger}X)^{Ai}(q_i+ip^i)(q_A+ip^A)\\ &\quad + (X^\dagger\sqrt{1+XX^\dagger})^{jB}(q_A - ip^B)(q_j - ip^j) + (X^\dagger X)^{kl}(q_l+ip^l)(q_k - ip^k)];\end{aligned} \quad (4.24)$$

$$\begin{aligned}\mathcal{W}^2_{\mathrm{non\text{-}BPS}(,Z_{AB}=0)} &= Z_I\bar{z}^I = \rho^2\\ &= \frac{1}{2}[[(q_D+ip^D)X^D + (q_l - ip^l)(\sqrt{1+X^T\bar{X}})^l]\cdot[(X^\dagger)^C(q_C - ip^C) + (\sqrt{1+X^T\bar{X}})^i(q_i+ip^i)]]\\ &= \frac{1}{2}[(XX^\dagger)^{CD}(q_C+ip^C)(q_D - ip^D) + (\sqrt{1+X^\dagger XX^\dagger})^{lC}(q_l - ip^l)(q_C - ip^C)\\ &\quad + (X\sqrt{1+X^\dagger X})^{Di}(q_D+ip^D)(q_i+ip^i) + (1+X^\dagger X)^{li}(q_l - ip^l)(q_i+ip^i).],\end{aligned} \quad (4.25)$$

where Eqs. (4.22) and (4.23) were used. Notice that, since all the contractions of $SU(3)$ and $SU(n)$ indices of electric and magnetic BH charges are uniquely defined with respect to the row or columns of the matrix X, every transposition index has been suppressed in Eqs. (4.24) and (4.25).

By introducing the complexified graviphoton and matter BH charges, respectively, as follows:

$$Q_C \equiv q_C + ip^C; \quad (4.26)$$

$$Q_i \equiv q_i + ip^i, \quad (4.27)$$

Eqs. (4.24) and (4.25) can be rewritten as follows:

$$\begin{aligned}\mathcal{W}^2_{\mathrm{BPS}} &= \frac{1}{2}[(1+XX^\dagger)^{AB}\bar{Q}_AQ_B + (\sqrt{1+XX^\dagger}X)^{Ai}Q_iQ_A\\ &\quad + (X^\dagger\sqrt{1+XX^\dagger})^{jB}\bar{Q}_B\bar{Q}_j + (X^\dagger X)^{kl}\bar{Q}_kQ_l];\end{aligned} \quad (4.28)$$

$$\begin{aligned}\mathcal{W}^2_{\mathrm{non\text{-}BPS}} &= \frac{1}{2}[(XX^\dagger)^{AB}Q_A\bar{Q}_B + (\sqrt{1+X^\dagger XX^\dagger})^{iA}\bar{Q}_i\bar{Q}_A\\ &\quad + (X\sqrt{1+X^\dagger X})^{Bj}Q_BQ_j + (1+X^\dagger X)^{kl}\bar{Q}_kQ_l].\end{aligned} \quad (4.29)$$

A. Attractor equations and their solutions

The BH effective potential can be written as

$$V_{\mathrm{BH}} = \frac{1}{2}Z_{AB}\bar{Z}^{AB} + Z_I\bar{Z}^I = \mathcal{Z}_1^2 + \rho^2. \quad (4.30)$$

The $\mathcal{N}=3$, $d=4$ attractor equations are nothing but the criticality conditions for such an $H_{\mathcal{N}=3,n}$-invariant (and Kähler-gauge-invariant) quantity. Such criticality conditions are satisfied for two classes of critical points:

(i) ($\frac{1}{3}$)-BPS:

$$Z_I = 0 \quad \forall\ I = 1,\ldots,n \Leftrightarrow \rho = 0; \qquad Z_{AB} \neq 0; \quad (4.31)$$

(ii) nonsupersymmetric (non-BPS with $Z_{AB}=0$):

$$\begin{aligned}&Z_I \neq 0 \text{ (at least for some } I);\\ &Z_{AB} = 0 \Leftrightarrow \mathcal{Z}_1 = 0.\end{aligned} \quad (4.32)$$

It is worth counting here the degrees of freedom related to Eqs. (4.31) and (4.32).

The $\frac{1}{3}$-BPS criticality conditions (4.31) are n complex independent ones, thus a moduli space of $\frac{1}{3}$-BPS attractors, spanned by the $2n$ complex scalars unstabilized by Eq. (4.31) might—and actually does [25]—exist.

Furthermore, there are only three complex non-BPS $Z=0$ criticality conditions (4.32). This fact paves the way to the possibility to have a moduli space of non-BPS ($Z_{AB}=0$) attractors, spanned by the $3(n-1)$ complex scalars unstabilized by Eq. (4.32); this actually holds true [41], as it will be explicitly found below (see Subsection IV A 2).

1. $\frac{1}{3}$*-BPS attractors and their moduli space*

By inserting the ($\frac{1}{3}$)-BPS criticality conditions (4.31) into the expression (4.23) of the matter charges Z_I and recalling the definitions (4.26) and (4.27) of the complexified BH charges, one obtains

$$(X^{\dagger})_I^A \bar{Q}_A = -(\sqrt{1 + X^T \bar{X}})_I{}^i Q_i, \quad \forall\ I = 1, \ldots, n. \tag{4.33}$$

By plugging such an expression and its Hermitian conjugate into Eq. (4.28), and using the identity (holding for any matrix A)

$$A^{\dagger}\sqrt{1 + AA^{\dagger}} = \sqrt{1 + A^{\dagger}A}A^{\dagger} \tag{4.34}$$

and its Hermitian conjugate

$$\sqrt{1 + AA^{\dagger}}A = A\sqrt{1 + A^{\dagger}A}, \tag{4.35}$$

one obtains that

$$\begin{aligned}(Z_{AB}\bar{Z}^{AB})_{\mathrm{BPS}} &= (1 + XX^{\dagger})^{AB}\bar{Q}_A Q_B + (X\sqrt{1 + X^{\dagger}X})^{Ai} Q_i Q_A + (\sqrt{1 + X^{\dagger}X}X^{\dagger})^{jB}\bar{Q}_B\bar{Q}_j + (X^{\dagger}X)^{kl}\bar{Q}_k Q_l \\ &= Q_A\bar{Q}_A + (XX^{\dagger})^{AB}\bar{Q}_A Q_B - (XX^{\dagger})^{AB} Q_A\bar{Q}_B - (1 + X^{\dagger}X)^{ij}\bar{Q}_i Q_j + (X^{\dagger}X)^{ij}\bar{Q}_i Q_j = Q_A\bar{Q}_A - Q_i\bar{Q}_i,\end{aligned} \tag{4.36}$$

where in the last step we exploited the Hermiticity of $XX^{\dagger}$, yielding that ($\langle\cdot,\cdot\rangle_{XX^{\dagger}}$ denotes the $XX^{\dagger}$-dependent square norm of complexified BH charges)

$$\begin{aligned}(XX^{\dagger})^{AB} Q_A\bar{Q}_B &\equiv \langle Q, Q\rangle_{XX^{\dagger}} = \langle\bar{Q},\bar{Q}\rangle_{XX^{\dagger}} \\ &= (XX^{\dagger})^{AB}\bar{Q}_A Q_B.\end{aligned} \tag{4.37}$$

By recalling Eqs. (1.2) and (4.30), and using Eq. (4.36), one achieves the following result:

$$\begin{aligned}\frac{S_{\mathrm{BH,BPS}}}{\pi} &= \frac{A_{H,\mathrm{BPS}}}{4} = V_{\mathrm{BH}}|_{\mathrm{BPS}} = \mathcal{Z}^2_{1,\mathrm{BPS}} = \frac{1}{2}(Z_{AB}\bar{Z}^{AB})_{\mathrm{BPS}} \\ &= \frac{1}{2}(Q_A\bar{Q}_A - Q_i\bar{Q}_i) = \mathcal{I}_2 > 0.\end{aligned} \tag{4.38}$$

Here $\mathcal{I}_2$ denotes the (unique) invariant of the fundamental/antifundamental $(\mathbf{3} + \mathbf{n}, \overline{\mathbf{3} + \mathbf{n}})$ representation of the U-duality group $G_{\mathcal{N}=3,n}$ (not irreducible with respect to $G_{\mathcal{N}=3,n}$ itself), quadratic in BH charges [see Eq. (5.1) below]:

$$\mathcal{I}_2 = \tfrac{1}{2}[q_A^2 - q_i^2 + (p^A)^2 - (p^i)^2] = \tfrac{1}{2}(q^2 + p^2), \tag{4.39}$$

where $q^2 \equiv \eta^{\Lambda\Sigma} q_{\Lambda} q_{\Sigma}$ and $p^2 \equiv \eta_{\Lambda\Sigma} p^{\Lambda} p^{\Sigma}$, $\eta^{\Lambda\Sigma} = \eta_{\Lambda\Sigma}$ being the $(3 + n)$-dim. Lorentzian metric with signature $(3, n)$ (see the discussion in Sec. V). In terms of the dressed charges Z_{AB} and Z_I, the (only apparently moduli-dependent) expression of $\mathcal{I}_2$ reads (see e.g. [89–91]):

$$\mathcal{I}_2 = \tfrac{1}{2}Z_{AB}\bar{Z}^{AB} - Z_I\bar{Z}^I = \mathcal{Z}_1^2 - \rho^2. \tag{4.40}$$

As mentioned above, the $\frac{1}{3}$-BPS criticality conditions (4.31) or (4.33) are a set of n complex equations, which thus does not stabilize all the $3n$ complex scalar fields z^i in terms of the BH electric and magnetic charges. In [56] the residual $2n$ unstabilized scalars have been shown to span the $\frac{1}{3}$-BPS moduli space

$$\mathcal{M}_{\mathcal{N}=3,n,\mathrm{BPS}} = \frac{SU(2,n)}{SU(2)\times SU(n)\times U(1)}, \qquad \dim_{\mathbb{R}} = 4n. \tag{4.41}$$

2. Non-BPS ($Z_{AB} = 0$) *attractors and their moduli space*

By inserting the non-BPS ($Z_{AB} = 0$) criticality conditions (4.32) into the expression (4.22) of the central charge matrix Z_{AB} and recalling the definitions (4.26) and (4.27) of the complexified BH charges, one obtains

$$(\sqrt{1 + XX^{\dagger}})^C_{AB} Q_C = -(\bar{X})^i_{AB}\bar{Q}_i, \quad \forall\ A, B = 1, 2, 3. \tag{4.42}$$

By plugging such an expression and its Hermitian conjugate into Eq. (4.29), and using the identities (4.34) and (4.35), one obtains that

$$\begin{aligned}2(Z_I\bar{Z}^I)_{\text{non-BPS}} &= (XX^{\dagger})^{AB} Q_A\bar{Q}_B + (X^{\dagger}\sqrt{1 + XX^{\dagger}})^{iA}\bar{Q}_i\bar{Q}_A + (\sqrt{1 + XX^{\dagger}}X)^{Bj} Q_B Q_j + (1 + X^{\dagger}X)^{kl}\bar{Q}_k Q_l \\ &= (XX^{\dagger})^{AB} Q_A\bar{Q}_B - (1 + XX^{\dagger})^{AB} Q_A\bar{Q}_B - (X^{\dagger}X)^{ij}\bar{Q}_i Q_j + (1 + X^{\dagger}X)^{ij}\bar{Q}_i Q_j = Q_i\bar{Q}_i - Q_A\bar{Q}_A.\end{aligned} \tag{4.43}$$

where once again Eq. (4.37) was used.

By recalling Eqs. (1.2) and (4.30), and using Eq. (4.43), one achieves the following result:

$$\frac{S_{\mathrm{BH,non\text{-}BPS}}}{\pi} = \frac{A_{H,\mathrm{non\text{-}BPS}}}{4} = V_{\mathrm{BH}}|_{\mathrm{non\text{-}BPS}} = \rho^2_{\mathrm{non\text{-}BPS}} = (Z_I\bar{Z}^I)_{\mathrm{non\text{-}BPS}} = -\frac{1}{2}(Q_A\bar{Q}_A - Q_i\bar{Q}_i) = -\mathcal{I}_2 > 0, \tag{4.44}$$

where $\mathcal{I}_2$ is the quadratic $G_{\mathcal{N}=3,n}$-invariant, given by Eqs. (4.39) and (4.40).

As mentioned above, the non-BPS criticality conditions (4.32) or (4.42) are a set of 3 complex equations, which thus does not stabilize all the $3n$ complex scalar fields z^i in terms of the BH electric and magnetic charges. In [56] the residual $3(n-1)$ unstabilized scalars have been shown to span the non-BPS ($Z_{AB}=0$) moduli space

$$\mathcal{M}_{\mathcal{N}=3,n,\mathrm{non\text{-}BPS}} = \frac{SU(3,n-1)}{SU(3)\times SU(n-1)\times U(1)} = \mathcal{M}_{\mathcal{N}=3,n-1}, \quad \dim_{\mathbb{R}} = 6(n-1). \tag{4.45}$$

Thus, as it holds in symmetric $\mathcal{N}=2$, $d=4$ supergravity minimally coupled to $n_V=n$ Abelian vector multiplets, also in the considered $\mathcal{N}=3$, $d=4$ supergravity the BH charges supporting nondegenerate critical points of V_{BH} are split in two branches: the $(\frac{1}{3})$-BPS one, defined by $\mathcal{I}_2>0$, and the non-BPS ($Z_{AB}=0$) one, corresponding to $\mathcal{I}_2<0$.

B. Black hole parameters for $\frac{1}{3}$-BPS flow

By using the Maurer-Cartan equations of $\mathcal{N}=3$, $d=4$ supergravity (see e.g. [89–91]), one gets [40]

$$\partial_i Z_1 = \partial_i \mathcal{W}_{\mathrm{BPS}} = \frac{1}{2\sqrt{2}}\frac{P_{IAB,i}\bar{Z}^I\bar{Z}^{AB}}{\sqrt{Z_{CD}\bar{Z}^{CD}}} = \frac{1}{4Z_1}P_{IAB,i}\bar{Z}^I\bar{Z}^{AB}, \tag{4.46}$$

where $P_{IAB}\equiv P_{IAB,i}dz^i$ is the holomorphic vielbein of $\mathcal{M}_{\mathcal{N}=3,n}$. Here, ∇ denotes the $U(1)$-Kähler and $H_{\mathcal{N}=3,n}$-covariant differential operator.

Thus, by using the explicit expressions of $\mathcal{W}^2_{\mathrm{BPS}}$ given by Eq. (4.28), using the Maurer-Cartan equations of $\mathcal{N}=3$, $d=4$ supergravity (see e.g. [89–91]), and exploiting the first order (fake supergravity) formalism discussed in Sec. II, one, respectively, obtains the following expressions of the (square) ADM mass, covariant scalar charges, and (square) effective horizon radius for the $\frac{1}{3}$-BPS attractor flow:

$$r^2_{H,\,\mathrm{BPS}}(z_\infty,\bar{z}_\infty,p,q) = M^2_{\mathrm{ADM,\,BPS}}(z_\infty,\bar{z}_\infty,p,q) = \mathcal{W}^2_{\mathrm{BPS}}(z_\infty,\bar{z}_\infty,p,q) = \tfrac{1}{2}\lim_{\tau\to 0^-}(Z_{AB}\bar{Z}^{AB})(z(\tau),\bar{z}(\tau),p,q)$$
$$= \tfrac{1}{2}[(1+X_\infty X_\infty^\dagger)^{AB}\bar{Q}_A Q_B + (\sqrt{1+X_\infty X_\infty^\dagger}X_\infty)^{Ai}Q_iQ_A + (X_\infty^\dagger\sqrt{1+X_\infty X_\infty^\dagger})^{jB}\bar{Q}_B\bar{Q}_j + (X_\infty^\dagger X_\infty)^{kl}\bar{Q}_kQ_l]; \tag{4.47}$$

$$\Sigma_{i,\,\mathrm{BPS}}(z_\infty,\bar{z}_\infty,p,q) = 2\lim_{\tau\to 0^-}(\partial_i\mathcal{W}_{\mathrm{BPS}})(z(\tau),\bar{z}(\tau),p,q) = \frac{1}{\sqrt{2}}\left[\frac{P_{IAB,i}\bar{Z}^I\bar{Z}^{AB}}{\sqrt{Z_{CD}\bar{Z}^{CD}}}\right]_\infty = \frac{1}{2}\left[\frac{1}{Z_1}P_{IAB,i}\bar{Z}^I\bar{Z}^{AB}\right]_\infty$$
$$= \frac{1}{2M_{\mathrm{ADM,\,BPS}}(z_\infty,\bar{z}_\infty,p,q)}(P_{IAB,i}\bar{Z}^I\bar{Z}^{AB})_\infty; \tag{4.48}$$

$$R^2_{H,\,\mathrm{BPS}} = \lim_{\tau\to 0^-}[\mathcal{W}^2_{\mathrm{BPS}}(z(\tau),\bar{z}(\tau),p,q) + -4G^{i\bar{j}}(z(\tau),\bar{z}(\tau))(\partial_i\mathcal{W}_{\mathrm{BPS}})(z(\tau),\bar{z}(\tau),p,q)\cdot(\bar{\partial}_{\bar{j}}\mathcal{W}_{\mathrm{BPS}})(z(\tau),\bar{z}(\tau),p,q)]$$
$$= \mathcal{I}_2(p,q) = V_{\mathrm{BH,\,BPS}} = \frac{S_{\mathrm{BH,\,BPS}}(p,q)}{\pi}, \tag{4.49}$$

where

$$X_\infty \equiv \lim_{\tau\to 0^-}X(\tau). \tag{4.50}$$

Throughout all the treatment, the subscript "∞" indicates the evaluation at the scalars at *radial infinity* z^i_∞.

Equation (4.49) proves Eq. (2.13) for the $\frac{1}{3}$-BPS attractor flow of the considered $\mathcal{N}=3$, $d=4$ supergravity. Such a result was obtained by using Eq. (4.46) and computing that

$$4G^{i\bar{j}}(\partial_i\mathcal{W}_{\mathrm{BPS}})\bar{\partial}_{\bar{j}}\mathcal{W}_{\mathrm{BPS}} = 4G^{i\bar{j}}(\partial_iZ_1)\bar{\partial}_{\bar{j}}Z_1 = \frac{G^{i\bar{j}}P_{IAB,i}\bar{P}_{JEF,\bar{j}}\bar{Z}^IZ^J\bar{Z}^{AB}Z^{EF}}{2Z_{CD}\bar{Z}^{CD}} = Z_I\bar{Z}^I = \rho^2, \tag{4.51}$$

where the relation

$$G^{i\bar{j}}P_{IAB,i}\bar{P}_{JEF,\bar{j}} = \delta_{IJ}(\delta_{AE}\delta_{BF} - \delta_{AF}\delta_{BE}) \tag{4.52}$$

was exploited.

The considerations made at the end of Subsection III B hold also for the considered attractor flow.

As pointed out above, the same also holds for ($\frac{1}{2}$-BPS attractor flow of) $\mathcal{N}=2$, $d=4$ supergravity *minimally coupled* to Abelian vector multiplets (see Eq. (150) of [63]), in which the (unique) invariant of the U-duality group $SU(1,n)$ is quadratic in BH electric and magnetic charges. Such a similarity is ultimately due to the fact that $SU(m,n)$ is endowed with a pseudo-Hermitian quadratic form built out of the fundamental $\mathbf{m}+\mathbf{n}$ and antifundamental $\overline{\mathbf{m}+\mathbf{n}}$ representations.

Furthermore, Eq. (4.47) yields that the $\frac{1}{3}$-BPS attractor flow of the $\mathcal{N}=3$, $d=4$ supergravity does not saturate the marginal stability bound (see [54,84]; see also the discussion at the end of Subsection III C).

C. Black hole parameters for non-BPS ($Z_{AB}=0$) flow

By using the Maurer-Cartan equations of $\mathcal{N}=3$, $d=4$ supergravity (see e.g. [89–91]), one gets [40]

$$\partial_i\rho=\partial_i\mathcal{W}_{\text{non-BPS}}=\frac{1}{4}\frac{P_{IAB,i}\bar{Z}^I\bar{Z}^{AB}}{\sqrt{Z_J\bar{Z}^J}}=\frac{P_{IAB,i}\bar{Z}^I\bar{Z}^{AB}}{4\rho}. \tag{4.53}$$

Thus, by using the explicit expressions of $\mathcal{W}^2_{\text{non-BPS}}$ given by Eq. (4.29), using the Maurer-Cartan equations of $\mathcal{N}=3$, $d=4$ supergravity (see e.g. [89–91]), and exploiting the first order (fake supergravity) formalism discussed in Sec. II, one, respectively, obtains the following expressions of the (square) ADM mass, covariant scalar charges, and (square) effective horizon radius for the non-BPS attractor flow:

$$\begin{aligned}r^2_{H,\text{non-BPS}}(z_\infty,\bar{z}_\infty,p,q)&=M^2_{\text{ADM, non-BPS}}(z_\infty,\bar{z}_\infty,p,q)=\mathcal{W}^2_{\text{non-BPS}}(z_\infty,\bar{z}_\infty,p,q)=\lim_{\tau\to0^-}(Z_I\bar{Z}^I)(z(\tau),\bar{z}(\tau),p,q)\\&=\tfrac{1}{2}[(X_\infty X_\infty^\dagger)^{AB}Q_A\bar{Q}_B+(\sqrt{1+X_\infty^\dagger X_\infty}X_\infty^\dagger)^{iA}\bar{Q}_i\bar{Q}_A+(X_\infty\sqrt{1+X_\infty^\dagger X_\infty})^{Bj}Q_BQ_j\\&\quad+(1+X_\infty^\dagger X_\infty)^{kl}\bar{Q}_kQ_l];\end{aligned} \tag{4.54}$$

$$\begin{aligned}\Sigma_{i,\text{ non-BPS}}(z_\infty,\bar{z}_\infty,p,q)&=2\lim_{\tau\to0^-}(\partial_i\mathcal{W}_{\text{non-BPS}})(z(\tau),\bar{z}(\tau),p,q)=\frac{1}{2}\left[\frac{P_{IAB,i}\bar{Z}^I\bar{Z}^{AB}}{\sqrt{Z_J\bar{Z}^J}}\right]_\infty=\frac{1}{2}\left[\frac{P_{IAB,i}\bar{Z}^I\bar{Z}^{AB}}{\rho}\right]_\infty\\&=\frac{1}{2M_{\text{ADM, non-BPS}}(z_\infty,\bar{z}_\infty,p,q)}(P_{IAB,i}\bar{Z}^I\bar{Z}^{AB})_\infty;\end{aligned} \tag{4.55}$$

$$\begin{aligned}R^2_{H,\text{non-BPS}}&=\lim_{\tau\to0^-}[\mathcal{W}^2_{\text{non-BPS}}(z(\tau),\bar{z}(\tau),p,q)+-4G^{i\bar{j}}(z(\tau),\bar{z}(\tau))(\partial_i\mathcal{W}_{\text{non-BPS}})(z(\tau),\bar{z}(\tau),p,q)\cdot(\bar{\partial}_{\bar{j}}\mathcal{W}_{\text{non-BPS}})\\&\quad\times(z(\tau),\bar{z}(\tau),p,q)]\\&=-I_2(p,q)=V_{\text{BH, non-BPS}}=\frac{S_{\text{BH, non-BPS}}(p,q)}{\pi}.\end{aligned} \tag{4.56}$$

Equation (4.49) proves Eq. (2.13) for the non-BPS ($Z_{AB}=0$) attractor flow of the considered $\mathcal{N}=3$, $d=4$ supergravity. Such a result was obtained by using Eq. (4.53) and computing that

$$4G^{i\bar{j}}(\partial_i\mathcal{W}_{\text{non-BPS}})\bar{\partial}_{\bar{j}}\mathcal{W}_{\text{non-BPS}}=4G^{i\bar{j}}(\partial_i\rho)\bar{\partial}_{\bar{j}}\rho=\frac{1}{4}\delta_{IK}(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC})\frac{\bar{Z}^{AB}Z^{CD}\bar{Z}^IZ^K}{Z_J\bar{Z}^J}=\frac{1}{2}Z_{AB}\bar{Z}^{AB}=Z_1^2. \tag{4.57}$$

The considerations made at the end of Subsection III B hold also for the considered attractor flow.

It is worth noticing that Eqs. (4.49) and (4.56) are consistent, because, as pointed out above, the ($\frac{1}{3}$-BPS)- and non-BPS ($Z_{AB}=0$)- supporting BH charge configurations in the considered theory are, respectively, defined by the quadratic constraints $I_2(p,q)>0$ and $I_2(p,q)<0$.

As yielded by Eqs. (4.48) and (4.55) for both nondegenerate attractor flows of the considered theory it holds the following relation among scalar charges and ADM mass:

$$\Sigma_i=\frac{1}{2M_{\text{ADM}}}\lim_{\tau\to0^-}P_{IAB,i}\bar{Z}^I\bar{Z}^{AB}. \tag{4.58}$$

Furthermore, Eq. (4.54) yields that the non-BPS $Z_{AB}=0$ attractor flow of the $\mathcal{N}=3$, $d=4$ supergravity does not saturate the marginal stability bound (see [54,84]; see also the discussion at the end of Subsection III C).

V. BLACK HOLE ENTROPY IN MINIMALLY COUPLED $\mathcal{N}=2$ AND $\mathcal{N}=3$ SUPERGRAVITY

It is here worth remarking that the classical Bekenstein-Hawking [72] $d=4$ BH entropy S_{BH} for minimally coupled $\mathcal{N}=2$ and $\mathcal{N}=3$ supergravity is given by the following $SU(m,n)$-invariant expression:

$$\frac{S_{\text{BH}}}{\pi}=\frac{1}{2}|q^2+p^2|, \tag{5.1}$$

where $q^2 \equiv \eta^{\Lambda\Sigma}q_\Lambda q_\Sigma$ and $p^2 \equiv \eta_{\Lambda\Sigma}p^\Lambda p^\Sigma$, $\eta^{\Lambda\Sigma} = \eta_{\Lambda\Sigma}$ being the Lorentzian metric with signature (m, n). As said above, $\mathcal{N} = 2$ is obtained by putting $m = 1$, whereas $\mathcal{N} = 3$ is given by $m = 3$ [see Eqs. (3.21) and (4.39) above, respectively]. Thus, in Eq. (5.1) the positive signature pertains to the graviphoton charges, while the negative signature corresponds to the charges given by the fluxes of the vector field strengths from the matter multiplets.

The supersymmetry-preserving features of the attractor solution depend on the sign of $q^2 + p^2$. The limit case $q^2 + p^2 = 0$ corresponds to the so-called small BHs (which, however, in the case $\mathcal{N} = 3$, do not enjoy an enhancement of supersymmetry, contrary to what usually happens in $\mathcal{N} \geq 4$, $d = 4$ supergravities; see e.g. the treatment in [71]).

By setting $n = 0$ in $\mathcal{N} = 3$, $d = 4$ supergravity (with resulting U-duality $U(3)$ which, due to the absence of scalars, coincides with the $\mathcal{N} = 3$ $\mathcal{R}$-symmetry $U(3)$ [92]), one gets

$$\frac{S_{\rm BH}}{\pi} = \frac{1}{2}[q_1^2 + q_2^2 + q_3^2 + (p^1)^2 + (p^2)^2 + (p^3)^2], \quad (5.2)$$

which is nothing but the sum of the entropies of three extremal (and thus BPS; see e.g. the discussion in [63]) Reissner-Nördstrom BHs, without any interference terms. Such a result can be simply understood by recalling that the generalization of the Maxwell electric-magnetic duality $U(1)$ to the case of n Abelian gauge fields is $U(n)$ [88], and that the expression in the right-hand side of Eq. (5.2) is the unique possible $U(3)$-invariant combination of charges.

Moreover, it is here worth noticing that $\mathcal{N} = 3$, $d = 4$ supergravity is the only $\mathcal{N} > 2$ supergravity in which the gravity multiplet does not contain any scalar field at all, analogously to what happens in the case $\mathcal{N} = 2$. Thus, in minimally coupled $\mathcal{N} = 2$[6] and $\mathcal{N} = 3$, $d = 4$ supergravity the pure supergravity theory, obtained by setting $n = 0$, is *scalarless*, with the U-duality coinciding with the $\mathcal{R}$-symmetry [92].

This does not happen for all other $\mathcal{N} > 2$ theories. For instance, the $\mathcal{N} = 4$, $d = 4$ gravity multiplet does contain one complex scalar field (usually named *axion-dilaton*) and six Abelian vectors; thus, the pure $\mathcal{N} = 4$ theory, obtained by setting $n = 0$ [see Eqs. (A1) and (7.1)], is not scalarless. By further truncating four vectors out [i.e. by performing a $(U(1))^6 \to (U(1))^2$ gauge truncation] and analyzing the bosonic field content, one gets the bosonic sector of $\mathcal{N} = 2$, $d = 4$ supergravity minimally coupled to one vector multiplet, the so-called axion-dilaton supergravity (for a discussion of the invariance properties of the classical BH entropy in such cases, see e.g. [63] and references therein).

[6]Let us notice also that $\mathcal{N} = 2$ minimally coupled theory is the only (symmetric) $\mathcal{N} = 2$, $d = 4$ supergravity which yields the pure $\mathcal{N} = 2$ supergravity simply by setting $n = 0$.

VI. $\mathcal{N} = 5$ SUPERGRAVITY

The (special Kähler) scalar manifold is [74]

$$\mathcal{M}_{\mathcal{N}=5} = \frac{G_{\mathcal{N}=5}}{H_{\mathcal{N}=5}} = \frac{SU(1,5)}{SU(5)\times U(1)}, \qquad \dim_{\mathbb{R}} = 10. \quad (6.1)$$

As previously mentioned, no matter coupling is allowed (pure supergravity).

The 10 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the *three-fold antisymmetric* irrepr. **20** of the U-duality group $G_{\mathcal{N}=5} = SU(1,5)$ (or equivalently of the compact form $SU(6)_{\mathbb{C}}$), and not in its fundamental repr. **6**.

$Z_{AB} = Z_{[AB]}$, $A, B = 1, 2, 3, 4, 5 = \mathcal{N}$ is the central charge matrix. By means of a suitable transformation of the $\mathcal{R}$-symmetry $H_{\mathcal{N}=5} = U(5)$, Z_{AB} can be skew-diagonalized by putting it in the normal form (see e.g. [40] and references therein):

$$Z_{AB} = \begin{pmatrix} \mathcal{Z}_1\epsilon & & \\ & \mathcal{Z}_2\epsilon & \\ & & 0 \end{pmatrix}, \quad (6.2)$$

where $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{R}_0^+$ are the $\mathcal{N} = 5$ (moduli-dependent) skew-eigenvalues, which can be ordered as $\mathcal{Z}_1 \geq \mathcal{Z}_2$ without any loss of generality (up to renamings; see e.g. [40]), and can be expressed as follows (see also the treatment of [93]):

$$\begin{cases} \mathcal{Z}_1 = \frac{1}{\sqrt{2}}\sqrt{I_1 + \sqrt{2I_2 - I_1^2}}; \\ \mathcal{Z}_2 = \frac{1}{\sqrt{2}}\sqrt{I_1 - \sqrt{2I_2 - I_1^2}}; \end{cases} \Leftrightarrow \begin{cases} I_1 = \mathcal{Z}_1^2 + \mathcal{Z}_2^2; \\ I_2 = \mathcal{Z}_1^4 + \mathcal{Z}_2^4, \end{cases} \quad (6.3)$$

where

$$I_1 \equiv \tfrac{1}{2}Z_{AB}\bar{Z}^{AB}; \quad (6.4)$$

$$I_2 \equiv \tfrac{1}{2}Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} \quad (6.5)$$

are the two unique (moduli-dependent) $H_{\mathcal{N}=5}$-invariants.

From the Lagrangian density of $\mathcal{N} = 5$, $d = 4$ supergravity [74], the vector kinetic terms read as follows:

$$\mathcal{L}_{\rm kin}^{\rm vec} = -\tfrac{1}{4}\sqrt{-g}(\mathcal{S}^{ij|kl} - \tfrac{1}{2}\delta^{ik}\delta^{jl})F^+_{\mu\nu|ij}F^{+\mu\nu}{}_{kl} + \text{H.c.}; \quad (6.6)$$

here and below $i = 1, \ldots, 5$ and antisymmetrization of a couple of i-indices is understood. In the Gaillard-Zumino formalism, it also holds (see Eq. (36) of [71])

$$\mathcal{L}_{\rm kin}^{\rm vec} = \sqrt{-g}\frac{i}{4}\mathcal{N}_{ij|kl}F^+_{\mu\nu|ij}F^{+\mu\nu}{}_{kl} + \text{H.c.} \quad (6.7)$$

By comparing Eqs. (6.6) and (6.7), can write the 10×10 vector kinetic matrix $\mathcal{N}^{ij,kl}$ as follows:

$$\mathcal{N}^{ij|kl} = i(\mathcal{S}^{ij|kl} - \tfrac{1}{2}\delta^{ik}\delta^{jl}). \quad (6.8)$$

The matrix $\mathcal{S}$ satisfies the relation

$$(\delta^{ij}_{kl} - \bar{\mathcal{S}}^{ij|kl})\mathcal{S}^{kl|mn} = \delta^{ij}_{mn}, \quad (6.9)$$

and, in a suitable choice of the scalar fields, it holds that

$$\bar{\mathcal{S}}^{ij|kl} = -\tfrac{1}{2}\epsilon^{ijklm}z_m. \quad (6.10)$$

Thus, one finds (square brackets denote antisymmetrization of enclosed indices throughout)

$$\mathcal{S}^{ij|kl} = \frac{1}{1-(z_n)^2}\left[\delta^{ij}_{kl} - \frac{1}{2}\epsilon^{ijklm}z_m - 2\delta_{[i[k}z_{l]}z_{j]}\right], \quad (6.11)$$

where the last term is normalized as

$$\delta^{[i}_{[k}z^{j]}z_{l]} = \tfrac{1}{4}(\delta^i_k z^j z_l \pm \text{permutations}\ldots). \quad (6.12)$$

Consequently, one achieves the following expression of the vector kinetic matrix:

$$\mathcal{N}_{ij|kl} = \frac{\alpha}{1-(z_m)^2}\left(\frac{1}{2}[1+(z_n)^2]\delta^{ij}_{kl} - \frac{1}{2}\epsilon^{ijklp}z_p - 2\delta_{[i[k}z_{l]}z_{j]}\right), \quad (6.13)$$

where α is a factor to be determined by the relations satisfied by the symplectic sections **h** and **f** defined in the last steps of Eqs. (4.16) and (4.17) above.

The supersymmetry transformation of the vector field can be written as [94]

$$\delta A^{ij}_\mu = 2f^{ij|AB}\bar{\psi}_{A\mu} + 2f^{ij}_{AB}\bar{\psi}^A_\mu\epsilon^B. \quad (6.14)$$

Such a formula is equivalent to the following expression [74] (see also [95]):

$$\delta A^{ij}_\mu = (\mathcal{S}^{ij|kl} - \delta^{ij|kl})(\mathcal{C}^{-1})_{kl}{}^{AB}(\bar{\epsilon}^C\gamma_\mu\chi_{ABC} + 2\sqrt{2}\bar{\epsilon}_A\psi_B), \quad (6.15)$$

where

$$\mathcal{C}_{ij}{}^{kl} \equiv \frac{1}{e_1}\delta^{ij}_{kl} - 2\frac{e_2}{e_1}\delta^{[i}_{[k}z^{j]}z_{l]}; \quad (6.16)$$

$$(\mathcal{C}^{-1})_{ij}{}^{kl} \equiv (e_1\delta^{kl}_{ij} + 2e_2\delta^{[i}_{[k}z^{j]}z_{l]}), \quad (6.17)$$

and

$$e_1^2 \equiv \frac{1}{1-|z|^2}; \quad (6.18)$$

$$e_2 \equiv \frac{1}{|z|^2}(1-e_1). \quad (6.19)$$

By comparing Eqs. (6.14) and (6.15), one obtains that

$$f^{ij}{}_{AB} = \left(e_1\delta^{ij}_{AB} + \frac{e_1}{2}\epsilon^{ijABm}z_m + 2e_2\delta^{[A}_{[i}z^{B]}z_{j]}\right). \quad (6.20)$$

The symplectic section **h** is

$$h_{ij|AB} = \mathcal{N}_{ij|mn}f^{mn}_{AB}, \quad (6.21)$$

and thus it explicitly reads

$$\begin{aligned} h_{ij|AB} &= \frac{\alpha}{1-(z_p)^2}\left\{\left[\frac{1}{2}(1-(z_q)^2)\delta^{ij}_{kl} - \frac{1}{2}\epsilon^{ijklm}z_m - 2\delta_{[i[k}z_{l]}z_{j]}\right]\cdot\left[e_1\delta^{kl}_{AB} + \frac{e_1}{2}\epsilon^{klABn}z_n + 2e_2\delta^{[A}_{[k}z^{B]}z_{l]}\right]\right\} \\ &= \alpha\left[\frac{e_1}{2}\delta^{ij}_{AB} - \frac{e_1}{4}\epsilon^{ijABk}z_k + e_2\delta^{[A}_{[i}z^{B]}z_{j]}\right]. \end{aligned} \quad (6.22)$$

Next, we check the above results and determine the overall numerical factor α of the matrix $\mathcal{N}$, by explicitly writing down the identities (4.18) and (4.19) in the considered framework.

By recalling that

$$\tfrac{1}{4}\epsilon^{ijABk}\epsilon^{ijCDl}z_k z_l = (z_m)^2\delta^{AB}_{CD} - 2\delta_{[A[C}z_{D]}z_{B]}; \quad (6.23)$$

$$\delta^{[A}_{[i}z^{B]}z_{j]}\delta^{[C}_{[i}z^{D]}z_{j]} = \tfrac{1}{2}(z_m)^2\delta^{[A[C}z^{D]}z^{B]}, \quad (6.24)$$

the identity (4.19) is easily verified, because

$$\begin{aligned} (\mathbf{f}^T\mathbf{h})^{AB}_{CD} = (\mathbf{h}^T\mathbf{f})^{AB}_{CD} &= \alpha\left\{\left[\frac{e_1}{2}\delta^{ij}_{AB} - \frac{e_1}{4}\epsilon^{ijABk}z_k + e_2\delta^{[A}_{[i}z^{B]}z_{j]}\right]\cdot\left[e_1\delta^{ij}_{CD} + \frac{e_1}{2}\epsilon^{ijCDl}z_l + 2e_2\delta^{[C}_{[i}z^{D]}z_{j]}\right]\right\} \\ &= \alpha\left\{\frac{e_1^2}{2}\delta^{AB}_{CD}(1-(z_i)^2) + e_1^2\delta_{[A[C}z_{D]}z_{B]} + e_2^2(z_i)^2\delta^{[A[C}z^{D]}z^{B]} + 2e_1e_2\mathrm{Re}(\delta^{[C}_{[A}z^{D]}z_{B]})\right\}. \end{aligned} \quad (6.25)$$

In order to check the identity (4.18) and determine α, we compute

$$\mathbf{f}^{\dagger}\mathbf{h} = f_{ij}{}^{AB}h_{ij|CD} = \alpha\left\{\left[e_1\delta^{AB}_{ij} + \frac{e_1}{2}\epsilon_{ijABk}z^k + 2e_2\delta^{[i}_{[A}z^{j]}z_{B]}\right]\cdot\left[\frac{e_1}{2}\delta^{ij}_{CD} - \frac{e_1}{4}\epsilon^{ijCDl}z_l + e_2\delta^{[C}_{[i}z^{D]}z_{j]}\right]\right\}$$
$$= \alpha\left[\frac{1}{2}\delta^{AB}_{CD} + \frac{e_1^2}{4}(\epsilon_{ijABk}z^k - \epsilon^{ijABk}z_k) - (e_1^2 + e_1e_2 + e_2)\delta^{[C}_{[A}z^{D]}z_{B]}\right] = \alpha\left[\frac{1}{2}\delta^{AB}_{CD} + i\frac{e_1^2}{2}\,\mathrm{Im}(\epsilon_{ijABk}z^k)\right], \qquad (6.26)$$

where we used the facts that $|z|^2e_2 = 1 - e_1$, and $e_1^2 + e_1e_2 + e_2 = 0$, directly following from the definitions (6.18) and (6.19).

By an analogous calculation, one achieves the following result:

$$\mathbf{h}^{\dagger}\mathbf{f} = (h_{ij|CD})^{\dagger}f_{ij}{}^{AB} = \bar{\alpha}\left[\frac{1}{2}\delta^{AB}_{CD} + i\frac{e_1^2}{2}\,\mathrm{Im}(\epsilon^{ijABk}z_k)\right]. \qquad (6.27)$$

Consequently, the identity (4.18) is satisfied iff $\alpha = -i$. By substituting such a value into Eq. (6.13), one obtains the following expression of the vector kinetic matrix of $\mathcal{N} = 5$, $d = 4$ ungauged supergravity:

$$\mathcal{N}_{ij|kl} = -\frac{i}{1-(z_m)^2}\left(\frac{1}{2}[1+(z_n)^2]\delta^{ij}_{kl} - \frac{1}{2}\epsilon^{ijklp}z_p - 2\delta_{[i[k}z_{l]}z_{j]}\right). \qquad (6.28)$$

From the general definition[7] (4.20), the central charge matrix Z_{AB} is defined as the integral over the 2-sphere at infinity S^2_∞ of the dressed graviphoton field strength [89–91]:

$$Z_{AB} \equiv -\int_{S^2_\infty} T_{AB} = -\int_{S^2_\infty} T^-_{AB} = f^{\Lambda}_{AB}q_\Lambda - h_{\Lambda|AB}p^\Lambda = f^{ij}_{AB}q_{ij} - h_{ij|AB}p^{ij}. \qquad (6.29)$$

By recalling Eqs. (6.20) and (6.22) with $\alpha = -i$, and, analogously to definitions (4.26) and (4.27), by introducing the complexified BH charges as

$$Q^{ij} \equiv \tfrac{1}{2}(q_{ij} + ip^{ij}), \qquad (6.30)$$

one gets

$$Z_{AB}(z,\bar{z},q,p) = e_1Q^{AB} + \frac{e_1}{2}\epsilon^{ABijk}\bar{Q}^{ij}z_k - 2e_2z^{[A}Q^{B]C}z_C. \qquad (6.31)$$

As recalled at the start of the next subsection, only $(\frac{1}{5})$-BPS attractor flow is nondegenerate (i.e. corresponding to large BHs; see e.g. the discussion in [71]), and the corresponding (squared) first order fake superpotential is ([40]; recall Eqs. (6.3), (6.4), and (6.5))

$$\mathcal{W}^2_{((1/5)-)\mathrm{BPS}} = \tfrac{1}{2}\left[\tfrac{1}{2}Z_{AB}\bar{Z}^{AB} + \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}\right] = \tfrac{1}{2}\left[I_1 + \sqrt{2I_2 - I_1^2}\right] = \mathcal{Z}_1^2. \qquad (6.32)$$

A. Attractor equations and their solutions

Because of the absence of matter charges, the BH effective potential can be written as

$$V_{\mathrm{BH}} = \tfrac{1}{2}Z_{AB}\bar{Z}^{AB} = \mathcal{Z}_1^2 + \mathcal{Z}_2^2 = I_1. \qquad (6.33)$$

The $\mathcal{N} = 5$, $d = 4$ attractor equations are nothing but the criticality conditions for the $H_{\mathcal{N}=5}$-invariant I_1. By using the Maurer-Cartan equations of $\mathcal{N} = 5$, $d = 4$ supergravity (see e.g. [89–91]), such attractor equations can be written as follows:

$$\epsilon^{ABCDE}Z_{AB}Z_{CD} = 0; \qquad (6.34)$$

$$\epsilon_{ABCDE}\bar{Z}^{AB}\bar{Z}^{CD} = 0, \qquad (6.35)$$

where the two lines are reciprocally complex conjugated. They can also be written more explicitly, by using the expression of central charge matrix Z_{AB} given by Eq. (6.31). For instance, Eqs. (6.34) can be rewritten in the following way:

$$\begin{aligned} 0 &= \epsilon^{ABCDE}Z_{AB}Z_{CD} \\ &= e_1^2\epsilon^{ABCDE}Q^{AB}Q^{CD} + e_1^2(\epsilon^{ABCDi}\bar{Q}^{AB}\bar{Q}^{CD}z_i)z_E \\ &\quad + 8e_1^2(Q^{AB}Q^{AB})z_E + 16e_1^2z_iQ^{iA}\bar{Q}^{AE} \\ &\quad - 4e_1e_2\epsilon^{ABCDE}Q^{AB}z^CQ^{Di}z_i - 8e_1e_2(z_iQ^{Di}\bar{Q}^{CD}z^C)z_E \\ &\quad - 16e_1e_2|z|^2z_jQ^{ij}\bar{Q}^{iE}. \end{aligned} \qquad (6.36)$$

The criticality conditions (6.35) and (6.36) are satisfied for a unique class of critical points (for further elucidation, see e.g. the treatment in [71]):

(i) $(\frac{1}{5})$-BPS:

$$\mathcal{Z}_2 = 0, \qquad \mathcal{Z}_1 > 0. \qquad (6.37)$$

It is worth counting here the degrees of freedom related to Eqs. (6.34) and (6.35), or equivalently to the unique $\frac{1}{5}$-BPS solution given by Eq. (6.37). Equations (6.34) and (6.35) are 10 real equations, but actually only 6 real among them are independent. Thus a moduli space of $\frac{1}{5}$-BPS attractors, spanned by the 2 complex scalars unstabilized

[7] We rescale the symplectic section $\mathbf{f}$ by a factor $\frac{1}{2}$, and the kinetic matrix $\mathcal{N}$ correspondingly by a factor 2. The definition of the symplectic section $\mathbf{h}$ through Eq. (6.21) and the identities (4.18) and (4.19) are left unchanged. Such a redefinition is performed in order to avoid an unsuitable rescaling of the magnetic charges, and thus to define the complexified BH charges as in Eq. (6.30).

by Eq. (6.37), might—and actually does [56]—exist. Such a counting of flat directions of $V_{\rm BH}$ at its $\frac{1}{5}$-BPS critical points was given in terms of the leftover would-be $\mathcal{N} = 2$ hyperscalars' degrees of freedom in the $\mathcal{N} = 5 \rightarrow \mathcal{N} = 2$ supersymmetry reduction in [91].

Entropy of $\frac{1}{5}$-BPS attractors and their moduli space

By recalling Eqs. (1.1) and (6.33), and using Eq. (6.37), one achieves the following result:

$$\frac{S_{\rm BH,\,BPS}}{\pi} = \frac{A_{H,\,\rm BPS}}{4} = V_{\rm BH}|_{\rm BPS} = Z^2_{1,\,\rm BPS} = \sqrt{I_4}. \tag{6.38}$$

I_4 here denotes the (unique) invariant of the threefold antisymmetric **20** representation of the U-duality group $G_{\mathcal{N}=5}$. Such a representation is symplectic, containing the singlet $\mathbf{1}_a$ in the tensor product $\mathbf{20} \times \mathbf{20}$ [96], and it is thus irreducible with respect to both $G_{\mathcal{N}=5}$ and $Sp(20, \mathbb{R})$). I_4 is quartic in BH charges.

In terms of the dressed charges, i.e. of the central charge matrix Z_{AB}, I_4 can be written as follows:

$$\begin{aligned} I_4 &= Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2 = 2I_2 - I_1^2 \\ &= {\rm Tr}(A^2) - \tfrac{1}{4}({\rm Tr}(A))^2, \end{aligned} \tag{6.39}$$

where the (moduli-dependent) matrix $A_A{}^B \equiv Z_{AC}\bar{Z}^{BC}$ and the related quantities [71]

$${\rm Tr}\,(A) = Z_{AB}\bar{Z}^{AB} = 2I_1 = 2V_{\rm BH}; \tag{6.40}$$

$${\rm Tr}\,(A^2) = Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} = 2I_2, \tag{6.41}$$

were introduced, I_1 and I_2 being the two unique (moduli-dependent) $H_{\mathcal{N}=5}$-invariants, respectively, defined by Eqs. (6.4) and (6.5).

The U-invariant I_4, introduced in Eq. (6.38) and expressed in terms of dressed charges by Eq. (6.39), is the unique (moduli-independent) independent $G_{\mathcal{N}=5} =$ -invariant combination of (moduli-dependent) $H_{\mathcal{N}=5}$-invariant quantities (see e.g. [91], and [71], as well as references therein). Its quarticity in BH charges is ultimately due to the symplectic nature of the **20** of $SU(1, 5)$ (irreducible with respect to both $SU(1, 5)$ and $Sp(20, \mathbb{R})$), i.e. to the fact that the tensor product $\mathbf{20} \times \mathbf{20}$ contains an antisymmetric singlet $\mathbf{1}_a$ [96], thus yielding a vanishing quadratic invariant of $SU(1, 5)$.

Thus, the expression (6.39) is moduli dependent only apparently. Being moduli independent, I_4 can actually be written only in terms of the BH charges. In order to determine such an expression, one can use the fact that, in the considered coordinate parametrization, the origin O of $\mathcal{M}_{\mathcal{N}=5}$ (determined by $z^i = 0\ \forall\ i = 1, \ldots, 5$) is the invariant point under the action of $H_{\mathcal{N}=5} = U(5)$ (see e.g. Eq. (2.17) of [74]). Thus, the explicit dependence of I_4 on BH charges is obtained simply by computing $Z_{AB}|_O$ and using Eq. (6.39). By recalling Eqs. (6.31), (6.40), and (6.41), one obtains

$$\begin{aligned} I_4 &= Q_{AB}Q^{BC}Q_{CD}Q^{DA} - \tfrac{1}{4}(Q_{AB}Q^{AB})^2 \\ &= \tfrac{1}{2^4}(q_{AB} - ip^{AB})(q_{BC} + ip^{BC})(q_{CD} - ip^{CD}) \\ &\quad \times (q_{DA} + ip^{DA}) - \tfrac{1}{2^6}[(q_{AB} - ip^{AB})(q_{AB} + ip^{AB})]^2, \end{aligned} \tag{6.42}$$

where the complexified BH charges Q^{ij} defined by Eq. (6.30) have been used, with $Q_{AB} \equiv \bar{Q}^{AB}$. Equation (6.42) is manifestly $H_{N=5}$-invariant, the $10 + 10$ real BH charges being arranged in the (reciprocally conjugated) complex rank-2 tensors Q^{ij} and Q_{ij} in the twofold antisymmetric **10** and **10**$'$ of $SU(5)$, or equivalently in the real rank-2 tensors q_{ij} and p^{ij} in a pair of twofold antisymmetric **10** of $SO(5)$.

Clearly, the BH electric and magnetic BH charges, being the asymptotical fluxes of the vector field strengths and of their duals, can actually be arranged to sit in the threefold antisymmetric, symplectic representation **20** of the U-duality group $G_{\mathcal{N}=5} = SU(1, 5)$. The embedding of the twofold antisymmetric $\mathbf{10}^{(\prime)}$ of $SU(5)$ into the threefold antisymmetric **20** of $SU(1, 5)$ is given by the formula (a, b, c, d, $e = 1, \ldots, 5$)

$$t^{abc} \equiv \frac{1}{3!}\epsilon^{abcde6}t_{de}, \tag{6.43}$$

or, more precisely (recall $A, B, C = 1, \ldots, 5$, and moreover $\hat{A}, \hat{B}, \hat{C} = 1, \ldots, 6$, here and below):

$$Q_{\hat{A}\hat{B}\hat{C}} \equiv q_{\hat{A}\hat{B}\hat{C}} + ip^{\hat{A}\hat{B}\hat{C}} = (Q_{ABC}, Q_{AB6}); \tag{6.44}$$

$$Q_{ABC} \equiv \frac{1}{3!}\epsilon_{ABCDE6}Q^{DE} \equiv \frac{1}{2}(q_{ABC} + ip^{ABC}); \tag{6.45}$$

$$Q_{6AB} \equiv Q_{AB} \equiv \frac{1}{2}(q_{6AB} + ip^{6AB}). \tag{6.46}$$

Equation (6.44) describes the splitting of the **20** of $SU(1, 5)$ into the **10** and **10**$'$ of $SU(5)$, whose embedding is determined by Eqs. (6.45) and (6.46), respectively. As given by Eq. (6.46), by identifying $t_{de} \equiv t_{de6}$, Eq. (6.43) can be seen as part of a self-reality condition [admitting solutions in $SU(1, 5)$, but not in $SU(6)$].

Thus, it can be easily shown that Eq. (6.42) can be recast in a manifestly $U = G_{\mathcal{N}=5}$-invariant way, with BH charges in the threefold antisymmetric **20** of $SU(1, 5)$, as follows:

$$\begin{aligned} I_4 &= \frac{1}{3\cdot 2^5}\epsilon^{\hat{A}\hat{B}\hat{C}\hat{A}'\hat{B}'\hat{C}'''}\epsilon^{\hat{A}''\hat{B}''\hat{C}''\hat{A}'''\hat{B}'''\hat{C}'}Q_{\hat{A}\hat{B}\hat{C}}Q_{\hat{A}'\hat{B}'\hat{C}'}Q_{\hat{A}''\hat{B}''\hat{C}''} \\ &\quad \times Q_{\hat{A}'''\hat{B}'''\hat{C}'''} \end{aligned} \tag{6.47}$$

$$\begin{aligned} &= \frac{1}{3\cdot 2^5}\epsilon^{\hat{A}\hat{B}\hat{C}\hat{A}'\hat{B}'\hat{C}'''}\epsilon^{\hat{A}''\hat{B}''\hat{C}''\hat{A}'''\hat{B}'''\hat{C}'}(q_{\hat{A}\hat{B}\hat{C}} + ip^{\hat{A}\hat{B}\hat{C}}) \\ &\quad \times (q_{\hat{A}'\hat{B}'\hat{C}'} + ip^{\hat{A}'\hat{B}'\hat{C}'}) \cdot (q_{\hat{A}''\hat{B}''\hat{C}''} + ip^{\hat{A}''\hat{B}''\hat{C}''}) \\ &\quad \times (q_{\hat{A}'''\hat{B}'''\hat{C}'''} + ip^{\hat{A}'''\hat{B}'''\hat{C}'''}). \end{aligned} \tag{6.48}$$

In order to prove such a formula, let us explicit the entries "6" in Eq. (6.47), obtaining

$$I_4 = \frac{1}{3\cdot 2^5}[9\epsilon^{6BCA'B'C'''}\epsilon^{6B''C''A'''B'''C'}Q_{6BC}Q_{A'B'C'}Q_{6B''C''}Q_{A'''B'''C'''} + 12\epsilon^{ABC6B'C'''}\epsilon^{6B''C''A'''B'''C'}Q_{ABC}Q_{6B'C'}Q_{6B''C''}Q_{A'''B'''C'''} + 6\epsilon^{ABCA'B'6}\epsilon^{6B''C''A'''B'''C'}Q_{ABC}Q_{6B'C'}Q_{A''B''C''}Q_{A'''B'''6} + 4\epsilon^{ABC6B'C'''}\epsilon^{A''B''C''6B'''C'}Q_{ABC}Q_{6B'C'}Q_{A''B''C''}Q_{6B'''C'''} + \epsilon^{ABCA'B'6}\epsilon^{A''B''C''A'''B'''6}Q_{ABC}Q_{A'B'6}Q_{A''B''C''}Q_{A'''B'''6}]. \tag{6.49}$$

By using the embedding equations (6.45) and (6.46), it is immediate to check that Eq. (6.49) yields Eq. (6.42).

As mentioned above, only 6 out of the 10 real $\frac{1}{5}$-BPS criticality conditions (6.34) and (6.35) are actually independent. Thus, they do not stabilize all the 5 complex scalar fields z^i in terms of the BH electric and magnetic charges, but only 3 of them. In [56] the residual 2 unstabilized complex scalar degrees of freedom have been shown to span the $\frac{1}{5}$-BPS moduli space

$$\mathcal{M}_{\mathcal{N}=5,\,\mathrm{BPS}} = \frac{SU(2,1)}{SU(2)\times SU(1)} = \mathcal{M}_{\mathcal{N}=3,n=1,\,\mathrm{BPS}}, \quad \dim_{\mathbb{R}} = 4. \tag{6.50}$$

B. Black hole parameters for $\frac{1}{5}$-BPS flow

By using the Maurer-Cartan equations of $\mathcal{N}=5$, $d=4$ supergravity (see e.g. [89–91]), one gets [40]

$$\partial_i Z_1 = \partial_i \mathcal{W}_{\mathrm{BPS}} = \frac{P_{,i}}{\sqrt{2}}\sqrt{\frac{1}{2}Z_{AB}\bar{Z}^{AB} - \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \frac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}} = P_{,i}Z_2, \tag{6.51}$$

where $P \equiv P_{1234}$, $P_{ABCD} \equiv P_{ABCD,i}dz^i = \epsilon_{ABCDE}P^E$ being the holomorphic vielbein of $\mathcal{M}_{\mathcal{N}=5}$. Here, ∇ denotes the $U(1)$-Kähler and $H_{\mathcal{N}=5}$-covariant differential operator.

Thus, by using the explicit expressions of $\mathcal{W}^2_{\mathrm{BPS}}$ given by Eq. (6.32), using the Maurer-Cartan equations of $\mathcal{N}=5$, $d=4$ supergravity (see e.g. [89–91]), and exploiting the first order (fake supergravity) formalism discussed in Sec. II, one, respectively, obtains the following expressions of the (square) ADM mass, covariant scalar charges, and (square) effective horizon radius for the $\frac{1}{5}$-BPS attractor flow:

$$r^2_{H,\,\mathrm{BPS}}(z_\infty,\bar{z}_\infty,p,q) = M^2_{\mathrm{ADM},\,\mathrm{BPS}}(z_\infty,\bar{z}_\infty,p,q) = \mathcal{W}^2_{\mathrm{BPS}}(z_\infty,\bar{z}_\infty,p,q) = \frac{1}{2}\lim_{\tau\to0^-}[\tfrac{1}{2}Z_{AB}\bar{Z}^{AB} + \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}] = Z_1^2|_\infty; \tag{6.52}$$

$$\Sigma_{i,\,\mathrm{BPS}}(z_\infty,\bar{z}_\infty,p,q) \equiv 2\lim_{\tau\to0^-}(\partial_i\mathcal{W}_{\mathrm{BPS}})(z(\tau),\bar{z}(\tau),p,q) = \sqrt{2}\lim_{\tau\to0^-}[P_{,i}\sqrt{\tfrac{1}{2}Z_{AB}\bar{Z}^{AB} - \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}}] = 2(P_{,i}Z_2)_\infty; \tag{6.53}$$

$$R^2_{H,\,\mathrm{BPS}} = \mathcal{W}^2_{\mathrm{BPS}}(z_\infty,\bar{z}_\infty,p,q) - 4G^{i\bar{j}}(z_\infty,\bar{z}_\infty)(\partial_i\mathcal{W}_{\mathrm{BPS}})(z_\infty,\bar{z}_\infty,p,q)(\bar{\partial}_{\bar{j}}\mathcal{W}_{\mathrm{BPS}})(z_\infty,\bar{z}_\infty,p,q) = \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2} = \sqrt{2Z_1^4 + 2Z_2^4 - (Z_1^2+Z_2^2)^2} = Z_1^2 - Z_2^2 = \sqrt{I_4(p,q)} > 0. \tag{6.54}$$

Equation (6.54) proves Eq. (2.13) for the $\frac{1}{5}$-BPS attractor flow of the considered $\mathcal{N}=5$, $d=4$ supergravity. Such a result was obtained by using Eq. (6.51) and computing that

$$4G^{i\bar{j}}(\partial_i\mathcal{W}_{\mathrm{BPS}})\bar{\partial}_{\bar{j}}\mathcal{W}_{\mathrm{BPS}} = 4G^{i\bar{j}}(\partial_i Z_1)\bar{\partial}_{\bar{j}}Z_1 = 2G^{i\bar{j}}P_{,i}\bar{P}_{,\bar{j}}[\tfrac{1}{2}Z_{AB}\bar{Z}^{AB} - \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}] = \tfrac{1}{2}[\tfrac{1}{2}Z_{AB}\bar{Z}^{AB} - \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}] = Z_2^2, \tag{6.55}$$

where the relation

$$4G^{i\bar{j}}P_{,i}\bar{P}_{,\bar{j}} = 1 \tag{6.56}$$

was used.

The considerations made at the end of Subsection III B hold also for the considered attractor flow.

It is worth noticing that Eq. (6.54) is consistent, because, as pointed out above, the $\frac{1}{5}$-BPS-supporting BH charge configurations in the considered theory is defined by the quartic constraints $I_4(p, q) > 0$.

Furthermore, Eq. (6.52) yields that the $\frac{1}{5}$-BPS attractor flow of $\mathcal{N} = 5$, $d = 4$ supergravity does not saturate the marginal stability bound (see [54,84]; see also the discussion at the end of Subsection III C).

VII. $\mathcal{N} = 4$ PURE SUPERGRAVITY REVISITED

The treatment of $\mathcal{N} = 4$, $d = 4$ pure supergravity is pretty similar to the one given for $\mathcal{N} = 5$, $d = 4$ supergravity in Sec. VI.

The (special Kähler) scalar manifold is [73]

$$\mathcal{M}_{\mathcal{N}=4,\,\mathrm{pure}} = \frac{G_{\mathcal{N}=4,\,\mathrm{pure}}}{H_{\mathcal{N}=4,\,\mathrm{pure}}} = \frac{SU(1,1) \times SU(4)}{U(1) \times SU(4)} = \frac{SU(1,1)}{U(1)},$$
$$\dim_{\mathbb{R}} = 2, \tag{7.1}$$

spanned by the complex scalar

$$s \equiv a + ie^{-2\varphi}, \qquad a, \varphi \in \mathbb{R}, \tag{7.2}$$

where a and φ are usually named axion and dilaton, respectively.

The six vector field strengths and their duals, as well as their asymptotical fluxes, sit in the bifundamental irrepr. $(\mathbf{2}, \mathbf{6})$ of the U-duality group $G_{\mathcal{N}=4,\,\mathrm{pure}} = SU(1,1) \times SO(6) \sim SU(1,1) \times SU(4)$.

$Z_{AB} = Z_{[AB]}$, A, $B = 1, 2, 3, 4 = \mathcal{N}$ is the central charge matrix. By means of a suitable transformation of the $\mathcal{R}$-symmetry $H_{\mathcal{N}=4,\,\mathrm{pure}} = U(1) \times SO(6) \sim U(1) \times SU(4)$, Z_{AB} can be skew-diagonalized by putting it in the normal form (see e.g. [40] and references therein):

$$Z_{AB} = \begin{pmatrix} \mathcal{Z}_1 \epsilon & \\ & \mathcal{Z}_2 \epsilon \end{pmatrix}, \tag{7.3}$$

where $\mathcal{Z}_1$, $\mathcal{Z}_1 \in \mathbb{R}_0^+$ are the $\mathcal{N} = 4$ (moduli-dependent) skew-eigenvalues, which can be ordered as $\mathcal{Z}_1 \geq \mathcal{Z}_2$ without any loss of generality (up to renamings; see e.g. [40]), and can be formally expressed by the very same Eqs. (6.3), (6.4), and (6.5), where now I_1 and I_2 are the two unique (moduli-dependent) $H_{\mathcal{N}=4,\,\mathrm{pure}}$-invariants.

The symplectic sections read as follows ($\Lambda \equiv [AB] = 1, \ldots, 6$ throughout) [71,89,90]

$$f^{\Lambda}_{AB} = e^{\varphi} \delta^{\Lambda}_{AB},$$
$$h_{\Lambda|AB} = se^{\varphi} \delta_{\Lambda|AB} = (ae^{\varphi} + ie^{-\varphi}) \delta_{\Lambda|AB}, \tag{7.4}$$

such that the kinetic vector matrix is given by [recall Eq. (4.14)]

$$\mathcal{N}_{\Lambda\Sigma} = (\mathbf{h}\mathbf{f}^{-1})_{\Lambda\Sigma} = s\delta_{\Lambda\Sigma}. \tag{7.5}$$

By the general definition (4.20), the central charge matrix is given by

$$Z_{AB} = f^{\Lambda}_{AB} q_{\Lambda} - h_{\Lambda|AB} p^{\Lambda} = e^{\varphi} \delta^{\Lambda}_{AB} q_{\Lambda} - se^{\varphi} \delta_{\Lambda|AB} p^{\Lambda}$$
$$= -e^{\varphi}(sp_{AB} - q_{AB}). \tag{7.6}$$

Such an explicit expression allows one to elaborate Eqs. (6.3), (6.4), and (6.5) further, obtaining

$$I_1 \equiv \tfrac{1}{2} Z_{AB} \bar{Z}^{AB} = \mathcal{Z}_1^2 + \mathcal{Z}_2^2$$
$$= \tfrac{1}{2} e^{2\varphi}(sp_{AB} - q_{AB})(\bar{s}p^{AB} - q^{AB})$$
$$= e^{2\varphi}(sp_{\Lambda} - q_{\Lambda})(\bar{s}p^{\Lambda} - q^{\Lambda})$$
$$= (e^{2\varphi}a^2 + e^{-2\varphi})p^2 + e^{2\varphi}q^2 - 2ae^{2\varphi} p \cdot q; \tag{7.7}$$

$$I_2 \equiv \tfrac{1}{2} Z_{AB} \bar{Z}^{BC} Z_{CD} \bar{Z}^{DA} = \mathcal{Z}_1^4 + \mathcal{Z}_2^4$$
$$= \tfrac{1}{2} e^{4\varphi}(sp_{AB} - q_{AB})(\bar{s}p^{BC} - q^{BC})(sp_{CD} - q_{CD})$$
$$\times (\bar{s}p^{DA} - q^{DA}), \tag{7.8}$$

where $p^2 \equiv (p^1)^2 + \ldots + (p^6)^2$, $q^2 \equiv q_1^2 + \ldots + q_6^2$, and $p \cdot q \equiv p^{\Lambda} q_{\Lambda}$ (see Eq. (7.1) of [63], fixing a typo in Eq. (225) of [71]).

Only $(\frac{1}{4})$-BPS attractor flow is nondegenerate (i.e. corresponding to large BHs; see e.g. the $n = 0$ limit of the discussion in [71]), and the corresponding (squared) first order fake superpotential is identical to the one of the $(\frac{1}{5})$-BPS attractor flow in $\mathcal{N} = 5$, $d = 4$ supergravity [40], given by Eq. (6.32) above, which in the considered framework can be further elaborated as follows:

$$\mathcal{W}^2_{((1/4)-)\mathrm{BPS}} = \frac{1}{2}\left[I_1 + \sqrt{2I_2 - I_1^2}\right] = \mathcal{Z}_1^2$$
$$= \frac{e^{2\varphi}}{4}\Big[(sp_{AB} - q_{AB})(\bar{s}p^{AB} - q^{AB})$$
$$+ \sqrt{4(sp_{AB} - q_{AB})(\bar{s}p^{BC} - q^{BC})(sp_{CD} - q_{CD})(\bar{s}p^{DA} - q^{DA}) - [(sp_{AB} - q_{AB})(\bar{s}p^{AB} - q^{AB})]^2}\Big]. \tag{7.9}$$

A. Attractor equations and their solutions: $\frac{1}{4}$-BPS attractors and their entropy

Because of the absence of matter charges, the BH effective potential V_{BH} reads

$$V_{BH} = \tfrac{1}{2}Z_{AB}\bar{Z}^{AB} = Z_1^2 + Z_2^2 = I_1$$
$$= (e^{2\varphi}a^2 + e^{-2\varphi})p^2 + e^{2\varphi}q^2 - 2ae^{2\varphi}p \cdot q, \quad (7.10)$$

where in the second line we recalled Eq. (7.7) (see also the treatments of [63,71]). The complex attractor equation of $\mathcal{N} = 4$, $d = 4$ pure supergravity is nothing but the criticality condition for the $H_{\mathcal{N}=4,\,\text{pure}}$-invariant I_1. By using the relevant Maurer-Cartan equations (see e.g. [89–91]), such a complex attractor equation can be written as follows ($|\varphi| < \infty$):

$$\epsilon_{ABCD}\bar{Z}^{AB}\bar{Z}^{CD} \Leftrightarrow \epsilon_{ABCD}(\bar{s}p^{AB} - q^{AB})^{\varphi}(\bar{s}p^{CD} - q^{CD}) = 0. \quad (7.11)$$

An equivalent set of two real equations is given by the system

$$\frac{\partial V_{BH}}{\partial a} = 0, \qquad \frac{\partial V_{BH}}{\partial \varphi} = 0, \quad (7.12)$$

with V_{BH} given by Eqs. (7.7) or (7.10), yielding Eqs. (7.2)–(7.3) of [63].

Thus, the criticality conditions (7.11), or equivalently (7.12), are satisfied for a unique class of critical points:

($\frac{1}{4}$)-BPS:

$$Z_2 = 0, \qquad Z_1 > 0, \quad (7.13)$$

yielding Eqs. (7.2)–(7.3) of [63]. Equations (7.12) are 2 real equations in 2 real unknowns, namely, the axion a and the dilaton φ, which both are stabilized solely in terms of the magnetic and electric BH charges. Thus no moduli space of $\frac{1}{4}$-BPS attractors in $\mathcal{N} = 4$, $d = 4$ pure supergravity exists at all [56,63,71].

By recalling Eqs. (1.1) and (7.10), and using Eq. (7.13), one achieves the following result [97]:

$$\frac{S_{BH,\,BPS}}{\pi} = \frac{A_{H,\,BPS}}{4} = V_{BH}|_{BPS} = Z_{1,\,BPS}^2 = \sqrt{I_4}. \quad (7.14)$$

I_4 denotes the (unique) invariant of the bifundamental representation $(\mathbf{2}, \mathbf{6})$ of the U-duality group $G_{\mathcal{N}=4,\,\text{pure}}$. Such a representation is symplectic,[8] containing the singlet $\mathbf{1}_a$ in the tensor product $(\mathbf{2}, \mathbf{6}) \times (\mathbf{2}, \mathbf{6})$ [96] (and thus yielding a vanishing quadratic U-invariant); consequently, it is irreducible with respect to both $G_{\mathcal{N}=4,\,\text{pure}}$ and $Sp(12, \mathbb{R})$. I_4 is quartic in BH charges (see Eq. (7.4), and the related discussion of [63]) [97]:

$$I_4 = 4[p^2q^2 - (p \cdot q)^2]. \quad (7.15)$$

In terms of the dressed charges, i.e. of the central charge matrix Z_{AB}, I_4 is formally given by the very same Eqs. (6.39), (6.40), and (6.41). I_4 is the unique (moduli-independent) independent $G_{\mathcal{N}=4,\,\text{pure}}$-invariant combination of (moduli-dependent) $H_{\mathcal{N}=4,\,\text{pure}}$-invariant quantities (see e.g. the $n = 0$ limit of the discussion in [71,91], and references therein).

It is worth pointing out that only when $\Lambda = 1, 2$ (corresponding to the truncation $(U(1))^6 \to (U(1))^2$ of the gauge group) I_4 is a perfect square, thus reproducing the quadratic invariant I_2 of the (1-modulus, $n = 1$ element of the) minimally coupled $\mathcal{N} = 2$, $d = 4$ sequence, given by Eq. (3.21) (see e.g. [25,63], and references therein).

B. Black hole parameters for $\frac{1}{4}$-BPS flow

By using the Maurer-Cartan equations of $\mathcal{N} = 4$, $d = 4$ pure supergravity (see e.g. [89–91]), one gets [40]

$$\partial_s Z_1 = \partial_s \mathcal{W}_{BPS} = \frac{P_{,s}}{\sqrt{2}}\sqrt{\frac{1}{2}Z_{AB}\bar{Z}^{AB} - \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \frac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}} = P_{,s}Z_2, \quad (7.16)$$

where $P \equiv P_{,s}ds$ is the holomorphic vielbein of $\mathcal{M}_{\mathcal{N}=5}$. Here, ∇ denotes the $U(1)$-Kähler and $H_{\mathcal{N}=4,\,\text{pure}}$-covariant differential operator.

Thus, by using the explicit expressions of $\mathcal{W}_{BPS}^2$ given by Eq. (7.9), using the Maurer-Cartan equations of $\mathcal{N} = 5$, $d = 4$ supergravity (see e.g. [89–91]), and exploiting the first order (fake supergravity) formalism discussed in Sec. II, one, respectively, obtains the following expressions of the (square) ADM mass, axion-dilaton charge, and (square) effective horizon radius for the $\frac{1}{4}$-BPS attractor flow:

[8]This fact has been observed above also for $\mathcal{N} = 5$, $d = 4$ supergravity.

Even though in general the (unique) quartic invariant of a (semisimple) Lie group can be built from a nonsymplectic representation, this never happens for the U-duality groups of $\mathcal{N} = 2$ symmetric and $\mathcal{N} > 2$, $d = 4$ theories. Thus, for all such supergravities having a (unique) U-invariant quartic in BH charges, the relevant representation of the U-duality group is symplectic (irreducible to both U-duality and relevant symplectic group).

$$
\begin{aligned}
&r^2_{H,\,\mathrm{BPS}}(s_\infty,\bar{s}_\infty,p,q)\\
&= M^2_{\mathrm{ADM,\,BPS}}(s_\infty,\bar{s}_\infty,p,q) = \mathcal{W}^2_{\mathrm{BPS}}(s_\infty,\bar{s}_\infty,p,q) = \frac{1}{2}\lim_{\tau\to0^-}\left[\frac{1}{2}Z_{AB}\bar{Z}^{AB} + \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \frac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}\right]\\
&= \mathcal{Z}_1^2|_\infty = \frac{i}{2(s_\infty-\bar{s}_\infty)}\cdot[(s_\infty p_{AB}-q_{AB})(\bar{s}_\infty p^{AB}-q^{AB})\\
&\quad + \sqrt{4(s_\infty p_{AB}-q_{AB})(\bar{s}_\infty p^{BC}-q^{BC})\cdot(s_\infty p_{CD}-q_{CD})(\bar{s}_\infty p^{DA}-q^{DA}) - [(s_\infty p_{AB}-q_{AB})(\bar{s}_\infty p^{AB}-q^{AB})]^2}];
\end{aligned}
\tag{7.17}
$$

$$
\begin{aligned}
\Sigma_{s,\,\mathrm{BPS}}(s_\infty,\bar{s}_\infty,p,q) &\equiv 2\lim_{\tau\to0^-}(\partial_s\mathcal{W}_{\mathrm{BPS}})(s(\tau),\bar{s}(\tau),p,q) = \sqrt{2}[P_{,s}\sqrt{\tfrac{1}{2}Z_{AB}\bar{Z}^{AB} - \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}}]_\infty\\
&= 2(P_{,s}\mathcal{Z}_2)_\infty;
\end{aligned}
\tag{7.18}
$$

$$
\begin{aligned}
R^2_{H,\,\mathrm{BPS}} &= \mathcal{W}^2_{\mathrm{BPS}}(z_\infty,\bar{z}_\infty,p,q) - 4G^{i\bar{j}}(z_\infty,\bar{z}_\infty)(\partial_i\mathcal{W}_{\mathrm{BPS}})(z_\infty,\bar{z}_\infty,p,q)(\bar{\partial}_{\bar{j}}\mathcal{W}_{\mathrm{BPS}})(z_\infty,\bar{z}_\infty,p,q)\\
&= \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2} = \sqrt{2\mathcal{Z}_1^4 + 2\mathcal{Z}_2^4 - (\mathcal{Z}_1^2+\mathcal{Z}_2^2)^2} = \mathcal{Z}_1^2 - \mathcal{Z}_2^2 = \sqrt{\mathrm{Tr}(A^2) - \tfrac{1}{4}(\mathrm{Tr}(A))^2}\\
&= \sqrt{\mathcal{I}_4(p,q)} > 0,
\end{aligned}
\tag{7.19}
$$

where, with suitable changes, the matrix $A_A{}^B$ and related quantities are defined by Eqs. (6.40) and (6.41).

Equation (7.19) proves Eq. (2.13) for the $\frac{1}{4}$-BPS attractor flow of $\mathcal{N}=4$, $d=4$ pure supergravity. Such a result was obtained by using Eq. (7.16) and computing that

$$
\begin{aligned}
4G^{s\bar{s}}(\partial_s\mathcal{W}_{\mathrm{BPS}})\bar{\partial}_{\bar{s}}\mathcal{W}_{\mathrm{BPS}} &= 4G^{s\bar{s}}(\partial_s\mathcal{Z}_1)\bar{\partial}_{\bar{s}}\mathcal{Z}_1 = 2G^{s\bar{s}}P_{,s}\bar{P}_{,\bar{s}}[\tfrac{1}{2}Z_{AB}\bar{Z}^{AB} - \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}]\\
&= \tfrac{1}{2}[\tfrac{1}{2}Z_{AB}\bar{Z}^{AB} - \sqrt{Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2}] = \mathcal{Z}_2^2,
\end{aligned}
\tag{7.20}
$$

where the relation

$$
4G^{s\bar{s}}P_{,s}\bar{P}_{,\bar{s}} = 1 \tag{7.21}
$$

was used.

From its very definition, by using Eqs. (7.9) or (7.17), the axion-dilaton charge $\Sigma_{s,\,\mathrm{BPS}}$ can be explicitly computed as follows:

$$
\begin{aligned}
\Sigma_{s,\,\mathrm{BPS}} &= \frac{1}{M_{\mathrm{ADM,\,BPS}}}(\partial_s\mathcal{W}^2_{\mathrm{BPS}})_\infty\\
&= \frac{1}{M_{\mathrm{ADM,\,BPS}}}\cdot\Big\{-\frac{1}{s_\infty-\bar{s}_\infty}M^2_{\mathrm{ADM,\,BPS}} + \frac{i}{2(s_\infty-\bar{s}_\infty)}\cdot\Big[p_{AB}(\bar{s}_\infty p^{AB}-q^{AB})\\
&\quad + \frac{1}{2}[4(s_\infty p_{AB}-q_{AB})(\bar{s}_\infty p^{BC}-q^{BC})(s_\infty p_{CD}-q_{CD})(\bar{s}_\infty p^{DA}-q^{DA})\\
&\quad - [(s_\infty p_{AB}-q_{AB})(\bar{s}_\infty p^{AB}-q^{AB})]^2]^{-1/2}[4p_{AB}(\bar{s}_\infty p^{BC}-q^{BC})(s_\infty p_{CD}-q_{CD})(\bar{s}_\infty p^{DA}-q^{DA})\\
&\quad + 4(s_\infty p_{AB}-q_{AB})(\bar{s}_\infty p^{BC}-q^{BC})p_{CD}(\bar{s}_\infty p^{DA}-q^{DA}) + -2(s_\infty p_{AB}-q_{AB})(\bar{s}_\infty p^{AB}-q^{AB})\\
&\quad \times p_{CD}(\bar{s}_\infty p^{CD}-q^{CD})]\Big]\Big\}.
\end{aligned}
\tag{7.22}
$$

Furthermore, from the definition (7.2) and Eq. (7.18), it follows that

$$
\Sigma_{s,\,\mathrm{BPS}} \equiv 2\lim_{\tau\to0^-}(\partial_s\mathcal{W}_{\mathrm{BPS}}) = \Sigma_{a,\,\mathrm{BPS}} + \frac{i}{2}e^{2\varphi_\infty}\Sigma_{\varphi,\,\mathrm{BPS}}, \tag{7.23}
$$

where

$$
\Sigma_{a,\mathrm{BPS}} \equiv 2\lim_{\tau\to0^-}(\partial_a\mathcal{W}_{\mathrm{BPS}}) = \mathrm{Re}(\Sigma_{s,\,\mathrm{BPS}}); \tag{7.24}
$$

$$\Sigma_{\varphi,\,\mathrm{BPS}} \equiv 2\lim_{\tau\to 0^-}(\partial_\varphi \mathcal{W}_{\mathrm{BPS}}) = 2e^{-2\varphi_\infty}\,\mathrm{Im}(\Sigma_{s,\,\mathrm{BPS}}) = -i(s_\infty - \bar{s}_\infty)\,\mathrm{Im}(\Sigma_{s,\,\mathrm{BPS}}), \tag{7.25}$$

respectively, are the axionic and dilatonic charges pertaining to the $\frac{1}{4}$-BPS attractor flow.

The considerations made at the end of Subsection III B hold also for the considered attractor flow.

It is worth noticing that Eq. (7.19) is consistent, because, as pointed out above, the $\frac{1}{4}$-BPS-supporting BH charge configurations in the considered theory is defined by the quartic constraints $\mathcal{I}_4(p,q) > 0$.

Furthermore, Eq. (7.17) yields that the $\frac{1}{4}$-BPS attractor flow of $\mathcal{N}=4$, $d=4$ pure supergravity does not saturate the marginal stability bound (see [54,84]; see also the discussion at the end of Subsection III C).

VIII. PECULIARITY OF PURE $\mathcal{N}=4$ AND $\mathcal{N}=5$ SUPERGRAVITY

By exploiting the first order (fake supergravity) formalism discussed in Sec. II, the expression of the squared effective horizon radius (in the extremal case $c=0$) R_H^2 given by Eq. (2.12) has been shown to hold for the following $d=4$ supergravity theories:

(i) minimally coupled $\mathcal{N}=2$ theory, whose scalar manifold is given by the sequence $\frac{SU(1,n)}{SU(n)\times U(1)}$ ([75], also named multidilaton system in [63]; see Subsections III B and III C);

(ii) $\mathcal{N}=3$ ([76], see Subsections IV B and IV C);

(iii) $\mathcal{N}=5$ ([74], see Subsection VI B);

(iv) $\mathcal{N}=4$ pure ([73], see Subsection VII B).

Such theories differ by a crucial fact: whereas the U-invariant of minimally coupled $\mathcal{N}=2$ and $\mathcal{N}=3$ supergravity is quadratic, the U-invariant of $\mathcal{N}=5$ and pure $\mathcal{N}=4$ theories is quartic in BH charges.

Thus, among all $d=4$ supergravities with U-invariant quartic in BH charges, the cases $\mathcal{N}=5$ and pure $\mathcal{N}=4$ turn out to be peculiar ones.

Such a peculiarity can be traced back to the form of their attractor equations, which are structurally identical to the ones of the minimally coupled $\mathcal{N}=2$ and $\mathcal{N}=3$ cases (see e.g. the treatments in [40,71]), and actually also to the very structure of $\mathcal{W}^2_{\mathrm{BPS}}$, as given by Eqs. (6.32) and (7.9), respectively.

This is ultimately due to a remarkable property, expressed by the last two lines of Eqs. (6.54) and (7.19): the (unique) invariant $\mathcal{I}_4(p,q)$ of $G_{\mathcal{N}=5}$ and $G_{\mathcal{N}=4,\,\mathrm{pure}}$, which is quartic in the electric and magnetic BH charges (p,q), is a perfect square of a quadratic expression when written in terms of the moduli-dependent skew-eigenvalues $\mathcal{Z}_1$ and $\mathcal{Z}_2$:

$$\mathcal{I}_4(p,q) \equiv Z_{AB}\bar{Z}^{BC}Z_{CD}\bar{Z}^{DA} - \tfrac{1}{4}(Z_{AB}\bar{Z}^{AB})^2 = \mathrm{Tr}(A^2) - \tfrac{1}{4}(\mathrm{Tr}(A))^2 = (\mathcal{Z}_1^2 - \mathcal{Z}_2^2)^2. \tag{8.1}$$

Such a result, which is true in the whole scalar manifolds $\mathcal{M}_{\mathcal{N}=5}$ and $\mathcal{M}_{\mathcal{N}=4,\,\mathrm{pure}}$, does not generally hold for all other $\mathcal{N}>2$, $d=4$ supergravities with (unique) quartic U-invariant, i.e. for $\mathcal{N}=4$ matter coupled and $\mathcal{N}=6,8$ theories, as well as for $\mathcal{N}=2$ supergravity whose scalar manifold does not belong to the aforementioned sequence of complex Grassmannians $\frac{SU(1,n)}{SU(n)\times U(1)}$.

This allows one to state that the relation (in the extremal case $c=0$) between the square effective horizon radius R_H^2 and the square BH event horizon radius r_H^2 for the nondegenerate attractor flows of such supergravities, if any, is structurally different from the one given by Eq. (2.12). Of course, in such theories one can still construct the quantity $r_H^2(z_\infty,\bar{z}_\infty,p,q) - G_{i\bar{j}}\Sigma^i\bar{\Sigma}^{\bar{j}}$ (eventually within a real parametrization of the scalar fields), but, also in the extremal case, it will be moduli dependent, thus not determining $R_H^2(p,q)$.

IX. $\mathcal{N}\geq 2$ SUPERGRAVITIES WITH THE SAME BOSONIC SECTOR AND DUALITIES

In the present section we consider $\mathcal{N}\geq 2$, $d=4$ supergravities[9] sharing the same bosonic sector, and thus with the same number of fermion fields, but with different supersymmetric completions.

(I)

(i) $\mathcal{N}=2$ (matter coupled) magic supergravity based on the degree 3 complex Jordan algebra $J_3^{\mathbb{H}}$;

(ii) $\mathcal{N}=6$ supergravity.

The scalar manifold of both such theories (upliftable to $d=5$) is $\frac{SO^*(12)}{SU(6)\times U(1)}$ (rank-3 homogeneous symmetric special Kähler space). In both theories the 16 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the left-handed spinor repr. $\mathbf{32}$ of the U-duality group $SO^*(12)$, which is symplectic, containing the symmetric singlet $\mathbf{1}_a$ in the tensor product $\mathbf{32}\times\mathbf{32}$, and thus irreducible with respect to both $SO^*(12)$ and $Sp(32,\mathbb{R})$. For a discussion of the spin/field content, see e.g. [89,90].

The correspondences among the various classes of nondegenerate extremal BH attractors of such two theories have been studied in [25] (see e.g. Table 9 therein).

(II)

(i) $\mathcal{N}=2$ supergravity minimally coupled to $n=n_V=3$ Abelian vector multiplets;

(ii) $\mathcal{N}=3$ supergravity coupled to $m=1$ matter (Abelian vector) multiplet.

All such theories (matter coupled, with quadratic U-invariant, and not upliftable to $d=5$) share the same scalar manifold, namely, the rank-1 symmetric special Kähler space $\frac{SU(1,3)}{SU(3)\times U(1)}$. Furthermore, in both such theories

[9] The relation between $\mathcal{N}=1$ and $\mathcal{N}=2$, $d=4$ supergravities and their attractor solutions is discussed in [36].

the 4 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the fundamental **4** repr. of the U-duality group $SU(3,1)$ (not irreducible with respect to $SU(3,1)$ itself, but only with respect to $Sp(8,\mathbb{R})$).

By (local) supersymmetry, the number of fermion fields is the same in the three theories, namely, there are 8 bosons and 8 fermions, but with different relevant spin/field contents:

$$\begin{aligned} &\mathcal{N}=2 \text{ minimally coupled}, n_V=3\text{: } [1(2), 2(\tfrac{3}{2}), 1(1)], 3[1(1), 2(\tfrac{1}{2}), 1_C(0)];\\ &\mathcal{N}=3, m=1\text{: } [1(2), 3(\tfrac{3}{2}), 3(1), 1(\tfrac{1}{2})], 1[1(1), 4(\tfrac{1}{2}), 3_C(0)]. \end{aligned} \tag{9.1}$$

From this it follows that one can switch between two such theories by transforming 1 gravitino in 1 gaugino, and vice versa.

The relation among the various classes of nondegenerate extremal BH attractors of three such theories [56,71] is given in Table I.

When switching between $\mathcal{N}=2$ and $\mathcal{N}=3$, the flip in sign of the quadratic U-invariant $\mathcal{I}_2 = q^2 + p^2$ can be understood by recalling that $q^2 \equiv \eta^{\Lambda\Sigma} q_\Lambda q_\Sigma$ and $p^2 \equiv \eta_{\Lambda\Sigma} p^\Lambda p^\Sigma$, with $\eta^{\Lambda\Sigma} = \eta_{\Lambda\Sigma} = \mathrm{diag}(1,-1,-1,-1)$ in the case $\mathcal{N}=2$, and $\eta^{\Lambda\Sigma} = \eta_{\Lambda\Sigma} = \mathrm{diag}(1,1,1,-1)$ in the case $\mathcal{N}=3$ [recall Eq. (5.1)]. It is here worth pointing out once again that the positive signature pertains to the graviphoton charges, while the negative signature corresponds to the charges given by the asymptotical fluxes of the vector field strengths from the matter multiplets (see also the discussion in Sec. V). As yielded by Table I, the supersymmetry-preserving features of the attractor solutions depend on the sign of $\mathcal{I}_2$.

(III)

(i) $\mathcal{N}=2$ supergravity coupled to $n_V = n+1 = 7$ Abelian vector multiplets, with scalar manifold $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,6)}{SO(2)\times SO(6)}$ [shortly named "cubic," $n_V = 7$ in Eq. (9.2)];

(ii) $\mathcal{N}=4$ supergravity coupled to $n_m = 2$ matter (Abelian vector) multiplets.

The scalar manifold of both such theories (upliftable to $d=5$) is $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,6)}{SO(2)\times SO(6)}$ (homogeneous symmetric reducible special Kähler, with rank 3). In both theories the 8 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the (spinor/doublet)-vector (bi-fundamental) repr. $(\mathbf{2},\mathbf{8})$ of the U-duality group $SU(1,1)\times SO(2,6)$, which is symplectic, containing the antisymmetric singlet $\mathbf{1}_a$ in the tensor product $(\mathbf{2},\mathbf{8})\times(\mathbf{2},\mathbf{8})$, and thus irreducible with respect to both $SU(1,1)\times SO(2,6)$ and $Sp(16,\mathbb{R})$.

Notice that, due to the isomorphism $\mathfrak{so}(6,2)\sim\mathfrak{so}^*(8)$ (see e.g. [86]), the dual supersymmetric interpretation of the scalar manifold $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,6)}{SO(2)\times SO(6)}$ can be considered, disregarding the axion-dilaton sector $\frac{SU(1,1)}{U(1)}$, as a "subduality" of the duality discussed in point (I).

By (local) supersymmetry, the number of fermion fields is the same in the three theories, namely, there are 16 bosons and 16 fermions, but with different relevant spin/field contents:

$$\begin{aligned} &\mathcal{N}=2\text{“cubic”},\\ &n_V=7\text{: } [1(2), 2(\tfrac{3}{2}), 1(1)], 7[1(1), 2(\tfrac{1}{2}), 1_C(0)];\\ &\mathcal{N}=4, n_m=2\text{: } [1(2), 4(\tfrac{3}{2}), 6(1), 4(\tfrac{1}{2}), 1_C(0)], 2[1(1), 4(\tfrac{1}{2}), 3_C(0)]. \end{aligned} \tag{9.2}$$

From this it follows that one can switch between two such theories by transforming 2 gravitinos in 2 gauginos, and vice versa.

The correspondence among the various classes of nondegenerate extremal BH attractors of two such theories have been studied in [25,41,56,71], and it is given in Table II.

As yielded by the comparison of Table 9 of [25] and Table II, such a duality is pretty similar to the duality between $\mathcal{N}=2$ $J_3^{\mathbb{H}}$ and $\mathcal{N}=6$ considered at point (I), also because the sign of the quartic U-invariant is unchanged by the duality relation (this is also consistent with the subduality relation mentioned above). In this sense, it differs from the duality between $\mathcal{N}=2$ mini-

TABLE I. $\mathcal{N}$-dependent BPS-interpretations of the classes of nondegenerate orbits of the symmetric special Kähler manifold $\frac{SU(1,3)}{SU(3)\times U(1)}$.

Orbit	$\mathcal{N}=2$ minimally coupled, $n_V=3$	$\mathcal{N}=3$, $m=1$
$\frac{SU(1,3)}{SU(3)}$	$\mathcal{O}_{1/2\text{-BPS}}$, no mod. space, $\mathcal{I}_{2,\mathcal{N}=2}>0$	$\mathcal{O}_{\text{non-BPS},Z_{AB}=0}$, no mod. space, $\mathcal{I}_{2,\mathcal{N}=3}<0$
$\frac{SU(1,3)}{SU(1,2)}$	$\mathcal{O}_{\text{non-BPS},Z=0}$, mod.space $=\frac{SU(1,2)}{SU(2)\times U(1)}$, $\mathcal{I}_{2,\mathcal{N}=2}<0$	$\mathcal{O}_{1/3\text{-BPS}}$, mod.space $=\frac{SU(1,2)}{SU(2)\times U(1)}$, $\mathcal{I}_{2,\mathcal{N}=3}>0$

TABLE II. $\mathcal{N}$-dependent BPS-interpretations of the classes of nondegenerate orbits of the reducible symmetric special Kähler manifold $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,6)}{SO(2)\times SO(6)}$. The structure of the duality is analogous to the one pertaining to the manifold $\frac{SO^*(12)}{SU(6)\times U(1)}$ (see point (I) above, as well as Table 9 of [25]).

Orbit	$\mathcal{N}=2$, $n_V=7$	$\mathcal{N}=4$, $n=2$
$\frac{SU(1,1)\times SO(2,6)}{SO(2)\times SO(6)}$	$\mathcal{O}_{1/2\text{-BPS}}$, no mod. space, $I_{4,\ \mathcal{N}=2}>0$	$\mathcal{O}_{\text{non-BPS},\ Z_{AB}=0}$, no mod. space, $I_{4,\ \mathcal{N}=4}>0$
$\frac{SU(1,1)\times SO(2,6)}{SO(2)\times SO(2,4)}$	$\mathcal{O}_{\text{non-BPS},\ Z=0}$, mod.space $=\frac{SO(2,4)}{SO(2)\times SO(4)}$ $I_{4,\ \mathcal{N}=2}>0$	$\mathcal{O}_{1/4\text{-BPS}}$, mod.space $=\frac{SO(2,4)}{SO(2)\times SO(4)}$ $I_{4,\ \mathcal{N}=4}>0$
$\frac{SU(1,1)\times SO(2,6)}{SO(1,1)\times SO(1,5)}$	$\mathcal{O}_{\text{non-BPS},\ Z\neq 0}$, mod.space $=SO(1,1)\times\frac{SO(1,5)}{SO(5)}$ $I_{4,\ \mathcal{N}=2}<0$	$\mathcal{O}_{\text{non-BPS},\ Z_{AB}\neq 0}$, mod.space $=SO(1,1)\times\frac{SO(1,5)}{SO(5)}$ $I_{4,\ \mathcal{N}=4}<0$

mally coupled, $n_V=3$ and $\mathcal{N}=3$, $m=1$ considered at point (II), because in both such theories the U-invariant is quadratic, and its sign is flipped by the duality relation (see Table I).

Points (I–III) present evidence against the conventional wisdom that interacting bosonic field theories have a unique supersymmetric extension. The sharing of the same bosonic backgrounds with different supersymmetric completions implies the dual interpretation with respect to the supersymmetry-preserving properties of nondegenerate extremal BH attractor solutions (see, respectively, Table 9 of [25], Table I and II).

X. CONCLUSION

In the present investigation we have considered the class of $\mathcal{N}\geq 2$, $d=4$ ungauged supergravity theories which do not have a counterpart in $d=5$ space-time dimensions. For such theories, the extremal BH parameters (namely, the ADM mass M_{ADM}, the scalar charges Σ^a, and the effective horizon radius R_H) pertaining to nondegenerate attractor flows have a simple formulation in the first order (fake supergravity) formalism.

All such theories share the property that an effective radial variable R can be defined such that the effective BH (horizon) area A is simply given by the surface of a sphere of radius $R_H(p,q)$, where $R_H(p,q)$ is the moduli-independent effective horizon radius of the extremal BH.

For $\mathcal{N}=2$ this holds for supergravity minimally coupled to Abelian vector multiplets, but it does not hold for more general matter couplings, such as symmetric spaces coming from degree three Jordan algebras [81] (with cubic holomorphic prepotential). In this respect, minimally coupled $\mathcal{N}=2$ supergravity can be considered as a multidilaton system in that it generalizes the Maxwell-Einstein-axion-dilaton system, studied in Refs. [77,78] (see also the recent treatment in [63]). Matter coupled supergravity with $\mathcal{N}=3$ shares similar properties. Both such theories exhibit two class of nondegenerate attractors (BPS and non-BPS), and a Bekenstein-Hawking classical BH entropy quadratic in the electric and magnetic BH charges. Furthermore, non-BPS ($Z=0$) $\mathcal{N}=2$ attractors and $\mathcal{N}=3$ [both $(\frac{1}{3})$-BPS and non-BPS ($Z_{AB}=0$)] attractors yield a related moduli space of solutions.

Pure $\mathcal{N}=4$ and $\mathcal{N}=5$ supergravities have also the same formula yielding to define $R_H(p,q)$, in spite of the fact that the classical BH entropy is not quadratic, but rather the square root of a quartic expression, in terms of the BH charges. Such theories have only BPS attractors, and for $\mathcal{N}=5$ a residual moduli space of solutions exists, as well.

It would be interesting to extend the notion of effective radius and of fake supergravity formalism to other $d=4$ theories, such as $\mathcal{N}=2$ not minimally coupled to Abelian vector multiplets, matter coupled $\mathcal{N}=4$, $\mathcal{N}=6$, and $\mathcal{N}=8$. In these cases, different formulae should occur to determine the BH parameters, such as ADM mass M_{ADM} and scalar charges Σ^a, in terms of the geometry of the underlying (asymptotical) scalar manifold.

Finally, one may wonder about a stringy realization of the theories discussed in the present paper, and their extremal BH states. At the string tree level, the massless spectrum of $\mathcal{N}=3$ and $\mathcal{N}=5$, $d=4$ supergravity can be obtained via asymmetric orbifolds of Type II superstrings [98–101], or by orientifolds [102]. Furthermore, it should be remarked that in string theory the attractor mechanism is essentially a nonperturbative phenomenon, either because it fixes the dilaton, or because it involves nonperturbative string states made out of D-brane bound states [103].

It is worth pointing out once again that the present analysis only covers nondegenerate extremal BH attractors, determining "large" BH horizon geometries with nonvanishing classical effective BH (horizon) area in the field theory limit. For degenerate extremal BHs, having "small" BH horizon geometries with vanishing classical effective BH (horizon) area, a departure from the Einsteinian approximation, including higher curvature terms in the gravity sector, is at least required [104]. Such corrections in supersymmetric theories of gravity have been considered in [11,105–107].

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APPENDIX: A COUNTEREXAMPLE: $\mathcal{N} = 4$ MATTER COUPLED SUPERGRAVITY

As an example of $d = 4$ supergravity in which $\mathcal{I}_4$ is not a perfect square of an expression quadratic in all the relevant geometrical quantities (such as the skew-eigenvalues of the central charge matrix and the matter charges), let us consider the $\mathcal{N} = 4$, $d = 4$ matter coupled theory [108,109].

The real scalar manifold is

$$\mathcal{M}_{\mathcal{N}=4} = \frac{G_{\mathcal{N}=4}}{H_{\mathcal{N}=4}} = \frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SO(6)\times SO(n)}, \qquad \dim_{\mathbb{R}} = 6n+2. \tag{A1}$$

The $6 + n$ vector field strengths and their duals, as well as their asymptotical fluxes, sit in the bifundamental irrepr. $[\mathbf{2}, (\mathbf{6}+\mathbf{n})]$ of the U-duality group $G_{\mathcal{N}=4} = SU(1,1) \times SO(6,n)$.

$Z_{AB} = Z_{[AB]}$, A, $B = 1, 2, 3, 4 = \mathcal{N}$ is the central charge matrix. As in the pure supergravity treated in Sec. VII [obtained by setting $n = 0$ in Eq. (A1)], by means of a suitable transformation of the $\mathcal{R}$-symmetry $U(1) \times SO(6) \sim U(1) \times SU(4)$, Z_{AB} can be skew-diagonalized by putting it in the normal form (see e.g. [40] and references therein):

$$Z_{AB} = \begin{pmatrix} \mathcal{Z}_1\epsilon & \\ & \mathcal{Z}_2\epsilon \end{pmatrix}, \tag{A2}$$

where $\mathcal{Z}_1$, $\mathcal{Z}_2 \in \mathbb{R}_0^+$ are the $\mathcal{N} = 4$ (moduli-dependent) skew-eigenvalues, which can be ordered as $\mathcal{Z}_1 \geq \mathcal{Z}_2$ without any loss of generality (up to renamings; see e.g. [40]), and can be formally expressed by the very same Eqs. (6.3), (6.4), and (6.5), where I_1 and I_2 are still the two (moduli-dependent) $H_{\mathcal{N}=4,\,\mathrm{pure}}$ (and also $H_{\mathcal{N}=4}$)-invariants.

In this case, similar to $\mathcal{N} = 3$ supergravity, also the matter charges Z_I ($I = 1, \ldots, n$) enter the game, $n \in \mathbb{N}$ denoting the number of matter multiplets coupled to the gravity multiplet. By a suitable rotation of $SO(n)$, the vector Z_I can be reduced in such a way that only one (strictly positive) real and one complex matter charge are nonvanishing [40]:

$$Z_I = (Z_1 = \rho_1, Z_2 = \rho_2 e^{i\theta}, Z_{\hat{I}} = 0), \qquad \rho_1, \rho_2 \in \mathbb{R}_0^+, \qquad \theta \in [0, 2\pi), \qquad \hat{I} = 3, \ldots, n. \tag{A3}$$

Thus, one can introduce the (moduli-dependent, unique) $SO(n)$ (and also $H_{\mathcal{N}=4}$)-invariants

$$I_3 \equiv Z_I \bar{Z}^I = \rho_1^2 + \rho_2^2; \tag{A4}$$

$$I_4 \equiv \mathrm{Re}(Z_I Z^I) = \rho_1^2 + \rho_2^2 \cos(2\theta). \tag{A5}$$

Now, there are only three (moduli-dependent) $SO(6,n)$-invariants, reading as follows [91]:

$$\mathbb{I}_1 \equiv I_1 - I_3 = \mathcal{Z}_1^2 + \mathcal{Z}_2^2 - \rho_1^2 - \rho_2^2; \tag{A6}$$

$$\mathbb{I}_2 \equiv \tfrac{1}{4}\epsilon^{ABCD} Z_{AB} Z_{CD} - \bar{Z}_I \bar{Z}^I = 2\mathcal{Z}_1\mathcal{Z}_2 - \rho_1^2 - \rho_2^2 e^{-2i\theta}; \tag{A7}$$

$$\mathbb{I}_3 \equiv \bar{\mathbb{I}}_2 = 2\mathcal{Z}_1\mathcal{Z}_2 - \rho_1^2 - \rho_2^2 e^{2i\theta}. \tag{A8}$$

The quartic $G_{\mathcal{N}=4}$-invariant $\mathcal{I}_4$ of $\mathcal{N} = 4$, $d = 4$ supergravity is the following unique (moduli-independent) $G_{\mathcal{N}=4}$-invariant combination of $\mathbb{I}_1$, $\mathbb{I}_2$, and $\mathbb{I}_3$ [91]:

$$\begin{aligned} \mathcal{I}_4 &\equiv \mathbb{I}_1^2 - \mathbb{I}_2\mathbb{I}_3 = \mathbb{I}_1^2 - |\mathbb{I}_2|^2 \\ &= (\mathcal{Z}_1^2 - \mathcal{Z}_2^2)^2 + (Z_I\bar{Z}^I)^2 - 2(\mathcal{Z}_1^2 + \mathcal{Z}_2^2) Z_I \bar{Z}^I \\ &\quad + 2\mathcal{Z}_1\mathcal{Z}_2(Z_I Z^I + \bar{Z}_I \bar{Z}^I) - |Z_I Z^I|^2 \\ &= (\mathcal{Z}_1^2 - \mathcal{Z}_2^2)^2 + (\rho_1^2 + \rho_2^2)^2 - 2(\mathcal{Z}_1^2 + \mathcal{Z}_2^2)(\rho_1^2 + \rho_2^2) \\ &\quad + 4\mathcal{Z}_1\mathcal{Z}_2[\rho_1^2 + \rho_2^2\cos(2\theta)] \\ &\quad - [\rho_1^4 + \rho_1^4 + 2\rho_1^2\rho_2^2\cos(2\theta)]. \end{aligned} \tag{A9}$$

On the other hand, in terms of the BH charges (q, p), $\mathcal{I}_4$ reads as follows [recall Eq. (7.15)]:

$$\mathcal{I}_4 = 4[p^2 q^2 - (p \cdot q)^2], \tag{A10}$$

where $p^2 \equiv p^\Lambda p^\Sigma \eta_{\Lambda\Sigma}$, $q^2 \equiv q_\Lambda q_\Sigma \eta^{\Lambda\Sigma}$, with Λ ranging $1, \ldots, n+6$, and the scalar product $\cdot$ is defined by $\eta_{\Lambda\Sigma} = \eta^{\Lambda\Sigma}$, the Lorentzian metric with signature $(n, 6)$ (see [63] and references therein).

Looking at Eq. (A9), it is easy to realize that $\mathcal{I}_4$ is a nontrivial perfect square of a function of degree 2 of $\mathcal{Z}_1$, $\mathcal{Z}_2$, ρ_1, ρ_2, and θ only in the pure supergravity theory (obtained by setting $n = 0$), i.e. only in the case $\rho_1 = \rho_2 = 0$. In such a limit, Eq. (A9) consistently reduces to Eq. (8.1).

As an example, we can work out the case $n = 1$ (which uplifts to pure $\mathcal{N} = 4$, $d = 5$ supergravity). In this case, only a (strictly positive) real matter charge $Z_1 = \rho_1 \in \mathbb{R}_0^+$ is present, and the quartic invariant $\mathcal{I}_4$ acquires the following form:

$$\begin{aligned} \mathcal{I}_4 &= (\mathcal{Z}_1^2 - \mathcal{Z}_2^2)^2 - 2(\mathcal{Z}_1^2 + \mathcal{Z}_2^2)Z_1^2 + 4\mathcal{Z}_1\mathcal{Z}_2 Z_1^2 \\ &= (\mathcal{Z}_1 - \mathcal{Z}_2)^2[(\mathcal{Z}_1 + \mathcal{Z}_2)^2 - 2Z_1^2](\mathcal{Z}_1 - \mathcal{Z}_2)^2 \\ &\quad \times (\mathcal{Z}_1 + \mathcal{Z}_2 + \sqrt{2}\rho_1)(\mathcal{Z}_1 + \mathcal{Z}_2 - \sqrt{2}\rho_1), \end{aligned} \tag{A11}$$

which is not a nontrivial perfect square of $\mathcal{Z}_1$, $\mathcal{Z}_2$, and ρ_1.

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