Chains of Interacting Brownian Particles under Strain

by

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Contents

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[1] LaTeX 2$_\varepsilon$ is an extension of LaTeX. LaTeX is a collection of macros for TeX. TeX is a trademark of the American Mathematical Society. The style package *warwickthesis* was used.

Declarations

A paper based on the material contained in Sections 3.1 and 3.2 has been published in Stochastic Processes and their Applications [AB09]. A paper based on Chapter 4 has been submitted to the same journal and is under consideration.

Prof. Martin Hairer provided the main ideas of Sections 4.2 and 4.4. He also provided the arguments for Propositions 4.3.1, 4.4.1 and 4.4.2.

I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. The material in this thesis has not been submitted for a degree at any other univeristy.

Abstract

We consider the behaviour of a one-dimensional chain of interacting Brownian particles being slowly pulled apart. More precisely, the leftmost particle is fixed, while the rightmost is pulled away at slow speed $\varepsilon > 0$. The interaction between particles is through a pairwise potential U of finite range. If we wait for a long enough time, the distance between a pair of neighbouring particles will exceed the range of U so that these two particles no longer interact. When this happens, we consider the chain broken at this point. Our aim is to investigate how the speed of pulling affects where the chain breaks, in the limit as $\sigma \to 0$, where $\sigma > 0$ is the noise intensity.

In Chapter 3, we begin by treating the case that U is cut-off strictly convex. In particular, it does not go smoothly to zero. We find, roughly, that if $\varepsilon > \sigma$ then the chain breaks at the end where it is pulled, while if $\varepsilon < \sigma$ it has an equal probability to break at either end. Then in Chapter 4, we consider the case that U goes smoothly to zero. After approximating the shape of the total energy function, we find, roughly, that the threshold between pulling regimes is given by $\varepsilon = \sigma^{4/3}$. Our approach is based on a careful analysis of sample path behaviour.

Although we mostly consider overdamped dynamics, we also show in Chapter 4 that if the particles have mass ε^{β} with $\beta > 2$, then the behaviour of the chain is well-approximated by that in the overdamped case.

Chapter 1

Introduction

This thesis concerns the behaviour of a one-dimensional chain of interacting Brownian particles being slowly pulled apart. Taking the interaction to be through a pairwise potential U of finite range, it follows that by pulling for a long enough time, the distances between neighbouring particles may exceed the range of U. In this case, the chain will consist of independent pieces, no longer interacting with each other, and we consider the chain to be broken. Our aim is to investigate how the speed of pulling affects where the chain breaks, in the limit of small noise. As we shall see, the answer depends on certain properties of U. In Chapter 3, we consider the case that U is cut-off strictly convex. In Chapter 4, we allow U to go smoothly to zero.

Such systems of Brownian particles are widely used to model (non-rigorously) the mechanical failure of molecular bonds arising in dynamic force spectroscopy (DFS) experiments (see Chapter 2 for more details). The main idea of these experiments, loosely speaking, is to clamp a given molecule at one end and attach a spring at the other, which is pulled at a constant speed until a break in the molecule occurs. The spring is usually assumed to be harmonic so that the force increases linearly with time. Upon repeating this several times and plotting the distribution of break forces, it is hoped to understand better various properties of the molecule. One of the main quantities of interest is an expression for the average break force and how it scales with the pulling speed of the spring. As we discuss in Chapter 2, there are two main pulling regimes: fast and slow. The scaling relation one obtains depends on the pulling regime. Both cases are treated within a large deviation framework with an adiabatic approximation that the system instantaneously adjusts itself to the increasing force.

The models we consider in Chapters 3 and 4 are similar to the DFS models.

However, instead of attaching a harmonic spring at one end and pulling this, we pull the end particle of the chain itself. This means that in our case, the external force does not increase linearly with time, but rather depends on the pairwise potential. Furthermore, we consider pulling speeds that are too fast for the large deviation framework to be relevant. We again find there are two pulling regimes, the threshold between which can be interpreted as that at which the adiabatic approximation becomes valid.

This thesis is organised as follows:

- In Chapter 2, we consider the DFS models mentioned above in more detail, as well as some other models of interacting Brownian particles arising in the physical and biological sciences.
- In Chapter 3, we introduce our model in the case that U is cut-off strictly convex. We treat chains with two and three breakable bonds (by which we mean three and four particles, respectively), assuming that the dynamics are overdamped, and discuss how the results obtained may be extended to arbitrarily long chains.
- In Chapter 4, we consider the case that U goes smoothly to zero. Furthermore, we no longer assume the dynamics to be overdamped. After treating two simplifications, including the overdamped equation, we show that for suitably small particle mass, the dynamics are well-approximated by those in the overdamped case.

Appendix A gives an overview of some of the main results concerning the exit of a diffusion from a domain, with an emphasis on identifying the exit location, which is most relevant in terms of the problems this thesis addresses. Furthermore, these results are the basis of much of Chapter 2. Appendix B gives some basic concepts and results from singular perturbation theory that are used in Chapter 3. Appendix C gives a result allowing the comparison of solutions of stochastic differential equations (SDEs). Appendix D gives a partial list of notation and acronyms used throughout this thesis. Other notation is explained within the thesis itself.

Chapter 2

Background

In this chapter we look at some models of chains of interacting Brownian particles arising in the physical and biological sciences. These models are typically considered from a mathematically non-rigorous point of view, and we follow the terminology commonly employed in these communities. In Appendix A, we review some of the mathematical theory underpinning the concepts arising in the present chapter.

2.1 Dynamic Force Spectroscopy

Dynamic force spectroscopy (DFS) is an experimental technique to analyse properties of molecular bonds [Mer01, Eva01]. The idea is to apply a slowly increasing, external force to a single bond and see how much force is required for it to rupture. This is commonly carried out using atomic force microscopes and optical tweezers. Due to small fluctuations on the molecular level, no two experiments give the same result and so the procedure is repeated many times to give a distribution of break forces. In order to interpret the experimental data and infer properties of the bond, a theoretical model of the experiment is required.

DFS experiments are typically modelled by an equation of the form

$$\dot{q}_t = -U'(q_t) + F(t) + \sqrt{2k_B T}\,\xi_t\ , \tag{2.1}$$

where q is a reaction coordinate for the experiment, e.g. length of bond, and U represents the intrinsic potential energy of the bond in the absence of an external force [LCST07, DFKU03]. Initially, q is assumed to be at the minimum of U so that the molecule is in its most stable state. The term $F(t)$ represents the external, time-dependent force. It is usually assumed that this force is applied through a harmonic spring linked to one end of the bond and so is taken of the form $F(t) = rt$, where $r > 0$ represents the loading rate, or pulling speed. The constants k_B and T are the Boltzmann constant and temperature, respectively. Finally, ξ_t is a standard Gaussian white noise process accounting for all other forces acting. Rigorously, this equation can be written in the form

$$\mathrm{d}q_t = [-U'(q_t) + F(t)]\,\mathrm{d}t + \sqrt{2k_BT}\,\mathrm{d}W_t\ ,$$

where W_t is a standard, one-dimensional Brownian motion. We shall use the formal equation (2.1) throughout this section.

Letting $H(q,t) = U(q) - F(t)q$ denote the effective, time-dependent potential in which q moves, rupture of the bond corresponds to the first time q overcomes the energy barrier keeping it within the potential well of H. The effect of the external force F is to lower this energy barrier and make escape more likely.

2.1.1 Kramers' Reaction Rate Theory

Let us first consider (2.1) with $F \equiv 0$. Then the equation is simply

$$\dot{q}_t = -U'(q_t) + \sqrt{2k_BT}\,\xi_t\ .$$

We shall think of q as a particle sitting in the potential U. Suppose that U has a minimum at q_- and a local maximum at q_+. In the absence of any stochastic force, the particle q, started from q_-, would just stay there for all times. However, the stochastic force changes this behaviour: q moves rapidly, staying close to q_- most of the time, but never still. The drift term $-U'(q_t)$ means it is very unlikely for q to move far from q_-. However, after a long enough time, the extremely unlikely event of q moving from a small neighbourhood of q_- to q_+ occurs. This and related phenomena, often referred to as "the first-exit problem", are the subject of Appendix A.

An immediate question one may ask is how long we have to wait for this unlikey event to happen. A first step was taken in 1889 by Arrhenius [Arr89], who observed empirically that the rate constant k_0 for a chemical reaction behaves like $k_0 = A_0\,\mathrm{e}^{-E_0/(k_BT)}$, where E_0 is the activation energy and A_0 is some constant exponential prefactor. In terms of our equation for q, a chemical reaction occurring corresponds to q overcoming the potential barrier between q_- and q_+. In that case, we have $E_0 = U(q_+) - U(q_-)$. The typical time needed would then be $1/k_0$. This suggests that we need to wait for a time that is exponentially long in the energy barrier height.

Eyring and Kramers determined theoretically the exponential prefactor in

Arrhenius' equation, which depends on the geometry of the potential at the minimum and local maximum [Eyr35, Kra40]. More precisely, they found

$$A_0 = \frac{\sqrt{|U''(q_+)|U''(q_-)}}{2\pi} \ ,$$

so that

$$k_0 = \frac{\sqrt{|U''(q_+)|U''(q_-)}}{2\pi} \, \mathrm{e}^{-E_0/(k_B T)} \ . \tag{2.2}$$

We shall use the term "Kramers' reaction rate theory" to indicate that k_0 is given by (2.2).

2.1.2 Instantaneous Rates of Rupture

We return to (2.1) with $F(t) = rt$:

$$\dot{q}_t = -U'(q_t) + rt + \sqrt{2k_B T}\, \xi_t \ .$$

As mentioned above, the drift term here has the form $-\partial_q H(q,t)$, with $H(q,t) = U(q) - rtq$. As H is time-dependent, Kramers' reaction rate theory does not immediately apply. However, it is still used to model DFS experiments, as we shall now see. Important steps in this direction were taken by Bell [Bel78] and later by Evans and Ritchie [ER97]. The main assumption is that the relaxation time of atoms is much faster than the rate of loading of the external force, in which case an adiabatic approximation is assumed: the particle (in the absence of noise) is always at the bottom of the potential well of H and instantaneously adjusts to the changing potential. Then, at any given time t, we apply (2.2) to the potential $H(q,t)$ to derive an instantaneous rate of escape $k(t)$. As described above, the effect of the additional force rt is to lower the energy barrier. Therefore, we expect $k(t)$ to increase with time. The probability that the bond has survived until time t, denoted $P(t)$, decays according to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(t) = -k(t)P(t) \ ,$$

which can be written in terms of force $F = rt$ as

$$\frac{\mathrm{d}}{\mathrm{d}F} P(F) = -\frac{1}{r} k(F) P(F) \ . \tag{2.3}$$

Note that instead of $H(q,t)$, we may equivalently consider $H(q,F) = U(q) - Fq$. The next step in the analysis is to estimate $k(F)$.

2.1.3 Estimating the Rate of Rupture

There are two main ways to estimate $k(F)$, depending on the loading rate r of the external force.

The first is Bell's phenomenological theory [Bel78], which says that the barrier height changes linearly with the external force. For small F, this can be derived using Taylor approximations for H, as is done in [LCST07, DHS06]. Recalling that $q_\pm$ denotes, respectively, the local maximum and minimum of U, we find that for q near $q_\pm$, $H(q,F) \approx H^*_\pm(q,F)$, where

$$H^*_\pm(q,F) = U(q_\pm) + \frac{1}{2}U''(q_\pm)(q-q_\pm)^2 - Fq \,.$$

Letting $q^*_\pm = q^*_\pm(F)$ be the solutions to $\partial_q H^*_\pm(q,F) = 0$ gives

$$q^*_\pm = q_\pm + \frac{F}{U''(q_\pm)} \,.$$

These are the approximate local maximum and minimum of $H(q,F)$. Let $\Delta = \Delta(F) = q^*_+ - q^*_-$. The energy barrier $E(F)$ between the maximum and minimum of H is approximately $H^*_+(q^*_+,F) - H^*_-(q^*_-,F)$, which gives

$$E(F) \approx E_0 - F\Delta + \frac{F^2}{2}\left(\frac{1}{U''(q_+)} - \frac{1}{U''(q_-)}\right) \approx E_0 - F\Delta \,,$$

where we have used that F is small. This shows the linear dependence of $E(F)$ on F. To approximate the exponential prefactor $A(F)$, we use $\partial_q^2 H^*_\pm(q^*_\pm,F) = U''(q_\pm)$, so that $A(F) \approx A_0$. Putting this all together, we arrive at $k(F) = k_0\, \mathrm{e}^{F\Delta/(k_B T)}$, commonly known as Bell's formula.

The derivation of Bell's formula suggests that it should only be valid if the bond ruptures when the external force F is small. This happens when the loading rate r is low and the effective potential H changes so slowly that a transition has enough time to occur before F becomes large. For higher loading rates, H changes significantly before rupture occurs and the energy barrier becomes shallow.

Garg considered the situation where at the time of rupture, the external force F is close to the critical force F_c at which the energy barrier of H vanishes [Gar95]. This means that $F_c := U'(q_c)$, where $q_c \in (q_-, q_+)$ is an inflection point $U''(q_c) = 0$. This situation has subsequently been studied by many other authors [LCST07, DFKU03, Fri08]. Following the presentation given in [LCST07], we find by a Taylor expansion that for q near q_c, $H(q,F) \approx H^*(q,F)$, where

$$H^*(q,F) = U(q_c) + F_c(q-q_c) + \frac{1}{6}U'''(q_c)(q-q_c)^3 - Fq \ .$$

Then if $q^*_\pm = q^*_\pm(F)$ solve $\partial_q H^*(q,F) = 0$, we find

$$q^*_\pm = q_c \pm \sqrt{\frac{2(F_c - F)}{|U'''(q_c)|}} = q_c \pm \sqrt{\frac{2F_c}{|U'''(q_c)|}(1 - F/F_c)} \ .$$

The energy barrier $E(F)$ of H is then given approximately by $H^*(q^*_+,F) - H^*(q^*_-,F)$, so that

$$\begin{aligned} E(F) &\approx (F_c - F)(q^*_+ - q^*_-) + \frac{1}{6}U'''(q_c)(q^*_+ - q_c)^3 - \frac{1}{6}U'''(q_c)(q^*_- - q_c)^3 \\ &= c_1(1 - F/F_c)^{3/2} \ , \end{aligned}$$

where $c_1 > 0$ is a constant depending on F_c and $U'''(q_c)$, which is given explicitly in [LCST07]. Note that $F < F_c$. We also have

$$\partial^2_q H^*(q^*_\pm, F) = \mp\sqrt{2F_c|U'''(q_c)|}(1 - F/F_c)^{1/2} \ ,$$

so that $A(F)$ is given approximately by

$$A(F) \approx \frac{\sqrt{|\partial^2_q H^*_+(q^*_+,F)|\partial^2_q H^*_-(q^*_-,F)}}{2\pi} \approx c_2(1 - F/F_c)^{1/2} \ ,$$

where $c_2 > 0$ is a constant also depending on F_c and $U'''(q_c)$. Putting this all together gives

$$k(F) = c_2(1 - F/F_c)^{1/2}\, \mathrm{e}^{-c_1(1-F/F_c)^{3/2}/(k_B T)} \ . \tag{2.4}$$

Similar results were obtained by Dudko et. al [DHS06] using a slightly different approach in which U is chosen explicitly and then certain integrals of the form $\int \mathrm{e}^{-H(q,F)}\,\mathrm{d}q$ around the well and barrier regions of H are analytically evaluated.

2.1.4 Average Rupture Force

One of the main quantities of interest is the mean rupture force $\langle F \rangle$, which can be calculated using (2.3). Solving this equation gives

$$P(F) = \exp\left(-\frac{1}{r}\int_0^F k(F')\mathrm{d}F'\right) \ . \tag{2.5}$$

The probability density function of rupture forces $p(F)$ is then given by

$$p(F) = -P'(F) = \frac{k(F)}{r} \exp\left(-\frac{1}{r}\int_0^F k(F')\mathrm{d}F'\right) ,$$

and the mean rupture force is given by

$$\langle F \rangle = \int_0^\infty F p(F)\,\mathrm{d}F = \int_0^\infty P(F)\,\mathrm{d}F , \tag{2.6}$$

where the second equality is obtained using integration by parts, noting that the boundary terms vanish since $P(F) \to 0$ exponentially fast as $F \to \infty$.

Substituting Bell's formula $k(F') = k_0\,\mathrm{e}^{\beta F'}$ into (2.5), where $\beta = \Delta/(k_B T)$, gives

$$P(F) = \exp\left(-\frac{k_0}{r}\frac{\mathrm{e}^{\beta F} - 1}{\beta}\right) ,$$

so that

$$\langle F \rangle = \int_0^\infty P(F)\,\mathrm{d}F = \mathrm{e}^{k_0/(\beta r)} \int_0^\infty \exp\left(-\frac{k_0\,\mathrm{e}^{\beta F}}{\beta r}\right)\mathrm{d}F \approx \frac{1}{\beta}\ln\left(\frac{\beta r}{k_0}\right) ,$$

which shows that $\langle F \rangle$ grows like $\ln r$.

In the case that F is near F_c at the time of rupture, so that $k(F)$ is given by (2.4), we first note that

$$k(F) = c_3 \frac{\mathrm{d}}{\mathrm{d}F}\,\mathrm{e}^{-c_1(1-F/F_c)^{3/2}/(k_B T)}$$

for some constant $c_3 > 0$ depending on U and T. Then

$$P(F) \approx \exp\left(-\frac{c_3}{r}\,\mathrm{e}^{-c_1(1-F/F_c)^{3/2}/(k_B T)}\right) .$$

Performing the integral in (2.6) with F near F_c then gives

$$\langle F \rangle \approx F_c\left(1 - \left[\frac{k_B T}{c_1}\ln\left(\frac{r}{c_3}\right)\right]^{2/3}\right) ,$$

so that now $\langle F \rangle$ scales like $(\ln r)^{2/3}$.

2.1.5 Random Rates of Rupture

In [REB+06], the authors propose an extension of the above theory in which the instantaneous rate of rupture $k(F)$ is random. Then any quantities calculated, such as the average rupture force, should be further integrated against the distribution of $k(F)$. The idea behind this is that upon repeating a DFS experiment on a given bond, there are inevitably small changes in experimental conditions that mean the rate of rupture changes too. However, they also conclude that when estimating certain quantities, such as k_0, from experimental data, there is little difference between using the above theory with randon or fixed $k(F)$, as both give approximately the same answer.

2.2 Multiple Bonds in Series

We now look at some models of multiple bonds in series in the presence of an external force, which may be time-independent or -dependent.

2.2.1 Time-Independent Force

Lee considers a one-dimensional chain of N particles connected by springs with identical spring constant κ [Lee09]. Each end of the chain is attached to a fixed wall. Initially they are equally spaced at unit distance in their minimal energy configuration. Letting $q(t) = (q_1(t), \ldots, q_N(t))$ denote the deviation of the chain from its initial configuration, the equations of motion are

$$\dot{q}(t) = -\frac{\kappa}{\gamma} A q(t) + \sqrt{\frac{2k_B T}{\gamma}} \xi_t \ ,$$

where $\gamma > 0$ is the damping coefficient and A is the matrix given by

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} .$$

Lee then introduces an orthogonal matrix P such that $P^{-1}AP = D$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ is a diagonal matrix with entries

$$\lambda_i = 2\left[1 - \cos\left(\frac{\pi i}{N+1}\right)\right] ,$$

and the orthogonal matrix P has entries

$$v_{ij} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{\pi i j}{N+1}\right) \ .$$

In the coordinates $p = P^{-1}q$, the equations become

$$\dot{p}(t) = -\frac{\kappa}{\gamma} D p(t) + \sqrt{\frac{2k_B T}{\gamma}} \xi_t \ ,$$

where the noise term remains unchanged under the orthogonal transformation P^{-1}. Letting e_i, $1 \leqslant i \leqslant N+1$, be the extension or contraction of the ith spring, Lee considers

$$\tau = \inf\{t \geqslant 0 : \max_i\{|e_i(t)|\} > b\} \ ,$$

where $b < 1$. He investigates the mean first breakage time $\langle\tau\rangle$ and the distribution of exit locations $e_i(\tau)$. The exit is more likely to occur where the energy, denoted by U, is minimal, which can be seen from (2.2). Lee finds that there are $2(N+1)$ such points, which all take the form of one spring undergoing breakage and all others being of equal length. He uses results obtained by Matkowsky and Schuss [MS77] using formal methods, although in the context of quadratic potentials they have been made rigorous by Kamin [Kam79] (see Appendix A). Using these results, he obtains

$$\langle\tau\rangle = \frac{\gamma\sqrt{2\pi k_B T} \exp(\frac{(N+1)\kappa b^2}{2Nk_B T})}{\phi_0^{1/2} \sum_{k=1}^{2N+2} \phi_k^{-1/2} |\kappa D z_k|} \ ,$$

where

$$\phi_0 = \det \left.\frac{\partial^2 U}{\partial p_i \partial p_j}\right|_{p=0} = \kappa^N \prod_{i=1}^{N} \lambda_i \ ,$$

and

$$\phi_k = \det \left.\frac{\partial^2 U}{\partial p_i' \partial p_j'}\right|_{p=z_k} \ , \quad 1 \leqslant i,j \leqslant N-1 \ ,$$

where the p_i', $1 \leqslant i \leqslant N-1$, are coordinates on the boundary of the domain. After some calculations, this is reduced to

$$\langle\tau\rangle = \sqrt{\frac{\pi k_B T N}{8(N+1)}} \frac{\gamma}{\kappa^{3/2} b(N+1)} \exp\left(\frac{(N+1)\kappa b^2}{2Nk_B T}\right) \ .$$

Lee shows that this formula is in good agreement with numerical simulations for large κ, which corresponds to small $k_B T$. Using further results from [MS77] (see

(A.5)), he finds that the probability to break at the ith spring, $\Pr(i)$, is given by

$$\Pr(i) = \frac{2\phi_i^{-1/2}|\kappa D z_i|}{\sum_{k=1}^{2N+2}\phi_k^{-1/2}|\kappa D z_k|} = \begin{cases} 1/(2N)\ , & i = 1, N+1\ , \\ 1/N\ , & \text{otherwise}\ . \end{cases} \tag{2.7}$$

This shows that the two end springs are half as likely to break as those in the middle. This behaviour comes from the fact that the end springs, joined to fixed walls at one end, are subject to half the thermal fluctuations.

Lee points out that it is possible to perform similar calculations when τ is defined using $e_i(t)$ instead of $|e_i(t)|$, which means that neighbouring particles cannot be too far apart, but are allowed to be close. In that case, the end springs will again have half the breaking probability as the middle ones. This is the type of stopping time we consider in Chapter 3.

2.2.2 Time-Dependent Force

Fugmann and Sokolov [FS09a, FS09b] consider a chain of N particles interacting via a Morse potential

$$U(q) = \frac{C}{2\alpha}(1 - \mathrm{e}^{-\alpha q})^2\ ,$$

where C, $\alpha > 0$ are constants. The chain is fixed to a wall at one end and at the other a linearly increasing force $F(t) = rt$ acts. The equations of motion are

$$\gamma \dot{q}_i = -U'(q_i - q_{i-1}) + U'(q_{i+1} - q_i) + \sqrt{2k_B T\gamma}\xi_i + rt\delta_{i,N}\ ,$$

where $q_0 \equiv 0$, representing the wall. They consider the chain to break as soon as the distance between any neighbouring particles overcomes an energy barrier in the total potential energy of the chain. Using numerical methods, the authors investigate the most probable rupture force $F_{\max}$ as a function of N and r, where ${p_N}'(F_{\max}) = 0$ for p_N the probability density function of break forces for the chain. For small N, the numerical results follow the relation

$$F_{\max} = F_c\left[1 - \left(\frac{\ln(Nv/r)}{w}\right)^{2/3}\right]\ , \tag{2.8}$$

where $w = C/(3\alpha k_B T)$, $v = C\alpha^2 k_B T/(8\pi\gamma^2)$ and F_c is again the critical force at which the energy barrier vanishes. This relation is obtained from the single bond rupture model of the previous section by taking each bond to be independent, as we shall now see. Let $P(F_i)$ be the probability that the ith bond is intact, where F_i is the force acting on the ith bond (at time t), with $F_N = F_N(t) = F(t)$, and let

$P_N(F)$ be the probability that the whole chain is intact under force F.

If the typical rupture time $t_{\max} = F_{\max}\gamma/r$ is larger than the relaxation time of the chain, given by $N^2\gamma/(C\alpha\pi^2)$, as is the case for short chains or slow pulling, it is assumed that each bond experiences the same force. Then by the independence assumption, $P_N(F) = \prod_{i=1}^{N} P(F_i) = P(F)^N$ so that the probability density function of rupture forces is given by

$$p_N(F) = -P_N{}'(F) = NP(F)^{N-1}p(F)\ ,$$

where $p(F)$ is the density function of rupture forces for a single bond. Following the single bond approach to estimate $P(F)$ and $p(F)$ leads to (2.8).

For faster pulling or longer chains, it is first noted that

$$P_N(F) = \prod_{i=1}^{N} P(F_i) = \exp\left(\sum_{i=1}^{N} \ln[P(F_i)]\right)\ .$$

Upon passing to the continuum limit $F_i(t) \to F(x,t)$, the relation

$$P_N(F(t)) = \exp\left(\int_0^{N(\Delta x)} \ln[P(F(x,t))]\frac{\mathrm{d}x}{\Delta x}\right) \tag{2.9}$$

is obtained, where Δx is the bond spacing and x the coordinate along the chain. The authors use a harmonic approximation for the chain dynamics in order to obtain an expression for $F(x,t)$. Taking into account that one end of the chain is fixed and linearising close to the pulled end, they insert the resulting approximation for F into (2.9), obtaining rather complicated expressions for $P_N(F)$ and $p_N(F)$, which agree well with numerical simulations.

Chapter 3

Convex Potentials

In this chapter, we consider a model of interacting Brownian particles in which the pairwise potential acting between particles is strictly convex on a certain subset of its support. In Sections 3.1 and 3.3, we treat a chain with two and three breakable bonds, respectively. Then in Section 3.5, we discuss how the results obtained in these cases may be extended to arbitrarily long chains. Throughout this chapter, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

3.1 Two Breakable Bonds

3.1.1 The Model and Main Result

Three particles $q_L < q < q_R$ in $\mathbb{R}$ interact with each other via a potential $\tilde{U}$ of finite range given by

$$\tilde{U}(y) = \begin{cases} U(y) & |y| \leqslant b\ , \\ 0 & \text{otherwise}\ , \end{cases} \tag{3.1}$$

where $U : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ has the following properties:

(U1) $U \in \mathcal{C}^3(0, \infty)$;

(U2) $U(-y) = U(y)$ for all $y \in \mathbb{R} \setminus \{0\}$;

(U3) There exists a unique $a > 0$ such that $U(a) = \inf_{y \geqslant 0}\{U(y)\} < 0$;

(U4) There exist constants $u_+ > u_- > 0$ such that $u_+ \geqslant U''(y) \geqslant u_-$ for all $y \geqslant a$;

(U5) $b < 2a$, where $b > a$ is uniquely defined by $U(b) = 0$.

An example of such a U is given by the quadratic function $U(y) = (|y|-a)^2-(b-a)^2$ for any $b < 2a$. Note that $\tilde{U}$ is not differentiable at b. We have chosen to define $\tilde{U}$ in terms of U for later convenience, as will become clear. As long as $|y| \leqslant b$, $\tilde{U}$ and U coincide so that one may work with either.

The particle q_L is fixed at the origin and the position of q_R at time $s \geqslant 0$ is given by $q_R(s) = 2a + \varepsilon s$, where $\varepsilon > 0$ is a small parameter denoting the speed of pulling. We study the behaviour of the middle particle, with position at time s given by q_s. Initially, it has position $q_0 = a$ so that the distance between neighbouring particles is a.

The potential energy $\tilde{H}$ of the configuration $(0, q, 2a + \varepsilon s)$ is given by

$$\tilde{H}(q, \varepsilon s) = \tilde{U}(q) + \tilde{U}(2a + \varepsilon s - q) \ .$$

As long as the arguments of $\tilde{U}$ above are smaller than b, we have $\tilde{H} = H$, where H is given by

$$H(q, \varepsilon s) = U(q) + U(2a + \varepsilon s - q) \ .$$

Henceforth, we shall work only with U and H.

The middle particle q moves according to the SDE

$$\mathrm{d}q_s = -\frac{\partial H}{\partial q}(q_s, \varepsilon s)\,\mathrm{d}s + \sigma\,\mathrm{d}W_s \ , \qquad q_0 = a \ , \tag{3.2}$$

where W_s is a standard Brownian motion and $\sigma > 0$ is the noise intensity. Rescaling time as $t = \varepsilon s$, this is the same in law as solving

$$\begin{aligned}\mathrm{d}q_t &= -\frac{1}{\varepsilon}\frac{\partial H}{\partial q}(q_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \\ &= \frac{1}{\varepsilon}[-U'(q_t) + U'(2a + t - q_t)]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \ ,\end{aligned} \tag{3.3}$$

with the same initial condition. We consider the chain to break as soon as there is no longer any interaction between q and one of its neighbours, i.e. when the distance between them is greater than b. Let

$$\tau = \inf\{t \geqslant 0 : q_t \notin (2a + t - b, b)\} \ . \tag{3.4}$$

This represents the breaking time of the chain. We say the left bond breaks if $q_\tau = b$ and the right bond breaks if $q_\tau = 2a + \tau - b$. It is clear from the definition of τ that $\tau \leqslant 2(b - a)$. This time corresponds to $q_R = 2b$, by which point at least one of the bonds must have reached length b. It is also clear that for $t \leqslant \tau$, $H = \tilde{H}$ and one

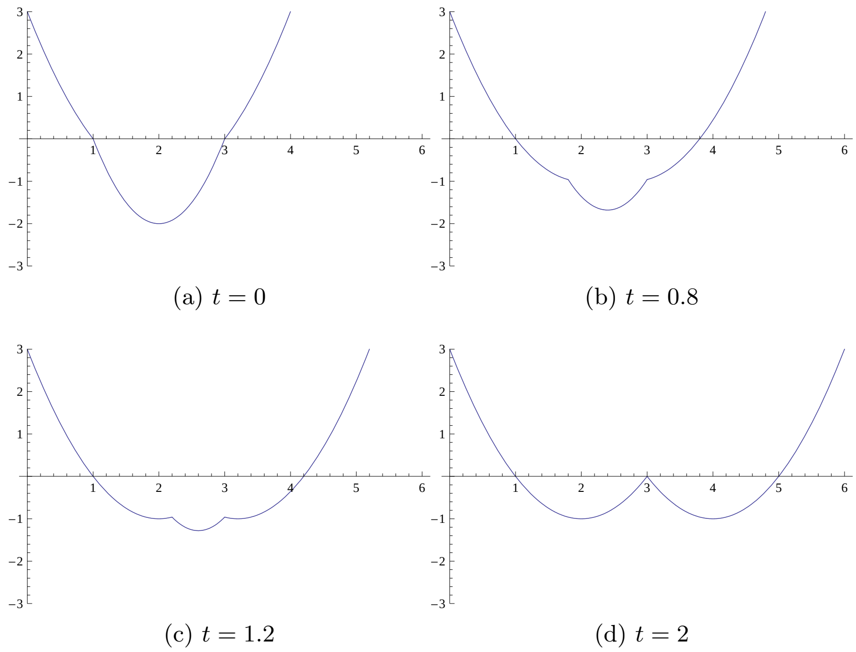


Figure 3.1: $H(q, t)$ plotted against q at various times for $U(q) = q^2 - 4|q| + 3$. Hitting one of the two cusps corresponds to the chain breaking.

may use either.

Letting $\varepsilon = \varepsilon(\sigma)$, our aim is to investigate, asymptotically as $\sigma \downarrow 0$, how the speed of pulling affects the location of the breakpoint. The following theorem gives our result.

Theorem 3.1.1. *Let q_t solve (3.3) and define τ as in (3.4).*

1. *(Fast pulling) There is a constant $k > 0$ such that if $k\sigma|\ln \sigma|^{1/2} \leqslant \varepsilon(\sigma) \ll 1$ then*

$$\lim_{\sigma \downarrow 0} \mathbb{P}(\textit{Right bond breaks}) = 1 \;.$$

2. *(Slow pulling) If*

$$\exp\left(-\frac{r(\sigma)}{\sigma^{2/3}}\right) \ll \varepsilon(\sigma) \ll \sigma|\ln \sigma|^{-1/2}$$

for some $\sigma^{2/3} \ll r(\sigma) \ll 1$, then

$$\lim_{\sigma \downarrow 0} \mathbb{P}(\textit{Left bond breaks}) = \lim_{\sigma \downarrow 0} \mathbb{P}(\textit{Right bond breaks}) = 1/2 \;.$$

- The proof of part (1) of this theorem will actually yield that

$$\mathbb{P}(\text{Right bond breaks}) > 1 - c_1\varepsilon^{-1}\,\mathrm{e}^{-c_2\varepsilon^2/\sigma^2}$$

  for constants $c_1, c_2 > 0$ and σ sufficiently small.

- The lower bound on ε in (2) arises because our proof applies on timescales shorter than that given by the Eyring-Kramers formula (see Chapter 2 and Appendix A). An exponentially small ε corresponds to an exponentially long time on our original timescale.

- We expect the result to hold without this lower bound on ε. Indeed, it can be proved in the case that U is quadratic. See Remark 3.2.4.

- The proof can easily be extended to the case that the chain is stretched according to some nonlinear function $p(t)$, that is, $q_R(t) = 2a + p(t)$, where $0 < p_0 < p'(t) < p_1$.

The theorem shows that when the pulling is fast, the chain will almost surely break on the right-hand side as $\sigma \downarrow 0$. This is the same behaviour as in the deterministic case when $\sigma = 0$ (see Section 3.2.1 below). However, when the pulling is sufficiently slow, there is an equal probability to break on either side, as when there is no pulling at all.

3.2 Proof of Theorem 3.1.1

In Section 3.2.1, we give an alternative formulation of Theorem 3.1.1 in terms of another stochastic process, leading to the equivalent Theorem 3.2.1, which will subsequently be proved. In Section 3.2.2, we isolate the linear part of our new process, bounding the nonlinear part using Lemma 3.2.2. In Sections 3.2.3 and 3.2.4, we prove Theorem 3.2.1 (1) and (2), respectively. Finally, in Section 3.2.5 we prove some auxillary results.

3.2.1 An Alternative Formulation

Let q_t^{det} be the solution of the ordinary differential equation (ODE) obtained from (3.3) by setting $\sigma = 0$:

$$\dot{q}_t^{\mathrm{det}} = -\frac{1}{\varepsilon}\frac{\partial H}{\partial q}(q_t^{\mathrm{det}}, t)\;, \qquad q_0^{\mathrm{det}} = a\;. \tag{3.5}$$

This slow-fast system has a slow manifold (see Appendix B) given by $q^*(t) = a+t/2$, corresponding to the configuration of equally spaced particles. In other words, for all $t \geqslant 0$, $\partial_q H(q^*(t), t) = 0$. We know by Tihonov's Theorem (see Theorem B.0.2) that $q_t^{\text{det}} = q^*(t) + \mathcal{O}(\varepsilon)$ for all $t \leqslant 2(b-a)$, so that q_t^{det} stays close to the chain midpoint. Furthermore, $(q^*)'(t) = 1/2 > 0$ implies $q_t^{\text{det}} \leqslant q^*(t)$ for all t.

A more precise expression for q_t^{det} is also possible. Firstly, there exists an adiabatic manifold $\bar{q}(t)$ given by

$$\bar{q}(t) = a + t/2 - \frac{\varepsilon}{4U''(a+t/2)} + \mathcal{O}(\varepsilon^2)$$

for all $0 \leqslant t \leqslant 2(b-a)$, where we have used (B.3). Secondly, as $q^*(t)$ is uniformly asymptotically stable, $\bar{q}(t)$ is locally attractive and we can use (B.2) to find

$$\begin{aligned} |q_t^{\text{det}} - \bar{q}(t)| &\leqslant |q_0^{\text{det}} - \bar{x}(0)|\, \mathrm{e}^{-\kappa t/\varepsilon} \\ &\leqslant C\varepsilon\, \mathrm{e}^{-\kappa t/\varepsilon} \end{aligned}$$

for some C, $\kappa > 0$. It follows that for $\varepsilon|\ln\varepsilon| \ll t \leqslant 2(b-a)$,

$$q_t^{\text{det}} = a + t/2 - \frac{\varepsilon}{4U''(a+t/2)} + \mathcal{O}(\varepsilon^2)\ . \tag{3.6}$$

Using this expansion and (U4), we see that after a very short time, the middle particle follows the chain midpoint at a distance of order ε. Therefore, in the deterministic case the right bond is always bigger than the left and the right bond breaks. Note also that if (3.5) were written in terms of $\tilde{H}$, then q_t^{det} would not be defined for all $t \in [0, 2(b-a)]$. Indeed, by (3.6) there is $t < 2(b-a)$ such that $2a + t - q_t^{\text{det}} = b$ (the chain breaks), at which time $\tilde{H}$ is undefined. This is the main reason why we chose to work with H.

We can now define the deviation process $y_t := q_t - q_t^{\text{det}}$ on the interval $[0, \tau]$, since $\tau \leqslant 2(b-a)$. By using a Taylor expansion, we see that y solves

$$\mathrm{d}y_t = \frac{1}{\varepsilon}\left[A(t)y_t + b(y_t, t)\right]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t\ , \qquad y_0 = 0\ , \tag{3.7}$$

where

$$A(t) = -U''(q_t^{\text{det}}) - U''(2a + t - q_t^{\text{det}})$$

and $b(y,t)$ contains the remainder terms. Furthermore, by (U1) there is a constant $M > 0$ such that $|b(y,t)| \leqslant My^2$ for all pairs $(y,t) \in \mathcal{D}$, where $\mathcal{D}$ is given in (3.11).

By (U4), we can also find constants $A_+ > A_- > 0$ such that

$$-A_+ \leqslant A(t) \leqslant -A_- \tag{3.8}$$

for all $t \in [0, 2(b-a)]$. This decomposition into linear and nonlinear parts is standard and also used in [BG06].

For the chain to be unbroken, y_t must satisfy

$$2a + t - b - q_t^{\text{det}} < y_t < b - q_t^{\text{det}} ,$$

which we write as

$$d_-(t) < y_t < d_+(t) , \tag{3.9}$$

where, using (3.6), we have for times $t \gg \varepsilon|\ln \varepsilon|$ that

$$d_+(t) = b - q_t^{\text{det}} = b - a - t/2 + \frac{\varepsilon}{4U''(a + t/2)} + \mathcal{O}(\varepsilon^2) \tag{3.10}$$

and

$$d_-(t) = 2a + t - b - q_t^{\text{det}} = a - b + t/2 + \frac{\varepsilon}{4U''(a + t/2)} + \mathcal{O}(\varepsilon^2) .$$

The problem is then to study the first exit of the process (y_t, t) from the space-time domain $\mathcal{D} = \mathcal{D}(\varepsilon)$, given by

$$\mathcal{D} = \{(y, t) : d_-(t) < y < d_+(t), 0 \leqslant t \leqslant 2(b-a)\} . \tag{3.11}$$

See Figure 3.2 below for an illustration of these two curves. The stopping time τ given in (3.4) can be written

$$\tau = \inf\{t \geqslant 0 : (y_t, t) \notin \mathcal{D}\} . \tag{3.12}$$

Then $y_\tau = d_-(\tau)$ corresponds to $q_\tau = 2a + \tau - b$, that is, the right bond breaking, and $y_\tau = d_+(\tau)$ corresponds to the left bond breaking.

Since $q_t^{\text{det}} \leqslant a + t/2$, it follows that $d_+(t) \geqslant -d_-(t)$ for all $t \in [0, 2(b-a)]$ and so the curve $d_-(t)$ crosses zero before $d_+(t)$. This corresponds to the fact that in the deterministic case, when $y \equiv 0$, the curve $d_-(t)$ is hit before $d_+(t)$ and the right bond breaks.

We can then state Theorem 3.1.1 as follows.

Theorem 3.2.1 (Alternative version of Theorem 3.1.1)**.** *Let y_t solve (3.7) and define τ as in (3.12).*

1. *There exists a constant $k > 0$ such that if $k\sigma|\ln\sigma|^{1/2} \leqslant \varepsilon(\sigma) \ll 1$ then*

$$\lim_{\sigma\downarrow 0} \mathbb{P}(y_\tau = d_-(\tau)) = 1 \ .$$

2. *If*

$$\exp\left(-\frac{r(\sigma)}{\sigma^{2/3}}\right) \ll \varepsilon(\sigma) \ll \sigma|\ln\sigma|^{-1/2}$$

for some $\sigma^{2/3} \ll r(\sigma) \ll 1$, then

$$\lim_{\sigma\downarrow 0} \mathbb{P}(y_\tau = d_+(\tau)) = \lim_{\sigma\downarrow 0} \mathbb{P}(y_\tau = d_-(\tau)) = 1/2 \ .$$

3.2.2 Linearisation of y_t

Solving (3.7), we find that the process y_t is given by

$$y_t = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon} \, \mathrm{d}W_s + \frac{1}{\varepsilon} \int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon} \, b(y_s, s) \, \mathrm{d}s =: y_t^0 + R(t) \ ,$$

where $\alpha(t,s) = \int_s^t A(u) \, \mathrm{d}u$ satisfies $-A_+(t-s) \leqslant \alpha(t,s) \leqslant -A_-(t-s)$ as a consequence of (3.8). We will also write $\alpha(t) = \alpha(t,0)$.

As long as y_t^0 does not become too large, then $R(t)$ can be bounded using that $|b(y,t)| \leqslant My^2$:

Lemma 3.2.2. *If $|y_t^0| < D(1 - MD/A_-)$ for all $0 \leqslant t \leqslant 2(b-a)$, then for all $0 \leqslant t \leqslant \tau$, $|y_t| < D$ and $|R(t)| < MD^2/A_-$.*

Proof. For all $t \leqslant \tau$, we have

$$\begin{aligned}
|y_t - y_t^0| = |R(t)| &\leqslant \frac{1}{\varepsilon} \int_0^t |b(y_s, s)| \, \mathrm{e}^{\alpha(t,s)/\varepsilon} \, \mathrm{d}s \\
&\leqslant \frac{M}{\varepsilon} \sup_{0 \leqslant s \leqslant t} |y_s|^2 \int_0^t \mathrm{e}^{-A_-(t-s)/\varepsilon} \, \mathrm{d}s \qquad (3.13) \\
&\leqslant \frac{M}{A_-} \sup_{0 \leqslant s \leqslant t} |y_s|^2 \ .
\end{aligned}$$

Let $\tau(D) = \inf\{t \geqslant 0 : |y_t| \geqslant D\}$ and suppose that $\tau(D) \leqslant \tau$. Then, since

$\tau \leqslant 2(b-a)$, we have

$$\begin{aligned} D = |y_{\tau(D)}| &\leqslant \sup_{0 \leqslant t \leqslant 2(b-a)} |y_t^0| + MD^2/A_- \\ &< D(1 - MD/A_-) + MD^2/A_- \\ &= D \;. \end{aligned}$$

So $|y_t|$ can never become D and $\tau(D) > \tau$. The bound on $R(t)$ follows by (3.13). $\square$

Motivated by this lemma, we introduce the event E given by

$$E = E(D) = \left\{ \sup_{0 \leqslant t \leqslant 2(b-a)} |y_t^0| \geqslant D(1 - MD/A_-) \right\} \;.$$

The following lemma will be used to show the values of D such that $E(D)$ is unlikely. It is proved in a similar way to Proposition 3.1.5 from [BG06].

Lemma 3.2.3. *There exist constants c_1, $c_2 > 0$, depending on U, such that for any $t \in [0, 2(b-a)]$ and any $H > 0$,*

$$\mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} |y_s^0| \geqslant H \right) \leqslant \left(\frac{c_1 t}{\varepsilon} + 2 \right) \exp\left(-\frac{c_2 H^2}{\sigma^2} \right) \;.$$

Proof. We have

$$\mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} |y_s^0| \geqslant H \right) \leqslant \sum_{j=0}^{N-1} P_j \;,$$

where

$$P_j = \mathbb{P}\left(\sup_{s_j \leqslant s < s_{j+1}} |y_s^0| \geqslant H \right)$$

and $0 = s_0 < s_1 < \ldots < s_N = t$ is a partition of $[0,t]$, chosen according to $\alpha(s_{j+1}, s_j) = -\varepsilon$ for $0 \leqslant j \leqslant N-2$ and $N = \lceil |\alpha(t)|/\varepsilon \rceil \leqslant A_+ t/\varepsilon + 1$. Using that for $s_j \leqslant s < s_{j+1}$,

$$|y_s^0| = \left| \frac{\sigma}{\sqrt{\varepsilon}} \int_0^s \mathrm{e}^{\alpha(s,u)/\varepsilon} \, \mathrm{d}W_u \right| \leqslant \mathrm{e}^{\alpha(s_j)/\varepsilon} \left| \frac{\sigma}{\sqrt{\varepsilon}} \int_0^s \mathrm{e}^{-\alpha(u)/\varepsilon} \, \mathrm{d}W_u \right| ,$$

we have for all $0 \leqslant j \leqslant N-1$,

$$\begin{aligned}
P_j &\leqslant \mathbb{P}\left(\sup_{0 \leqslant s<s_{j+1}} \left|\int_0^s \mathrm{e}^{-\alpha(u)/\varepsilon} \,\mathrm{d}W_u\right| \geqslant \frac{\sqrt{\varepsilon}}{\sigma} \mathrm{e}^{-\alpha(s_j)/\varepsilon} H\right) \\
&\leqslant 2\exp\left(-\frac{\varepsilon}{2\sigma^2} \frac{\mathrm{e}^{-2} H^2}{\int_0^{s_{j+1}} \mathrm{e}^{2\alpha(s_{j+1},u)/\varepsilon} \,\mathrm{d}s}\right) \\
&\leqslant 2\exp\left(-\frac{A_- \mathrm{e}^{-2} H^2}{\sigma^2}\right) ,
\end{aligned}$$

where the second line follows by the inequality

$$\mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} \left|\int_0^s \varphi(u) \,\mathrm{d}W_u\right| \geqslant \delta\right) \leqslant 2\exp\left(-\frac{\delta^2}{2\int_0^t \varphi(u)^2 \,\mathrm{d}u}\right), \tag{3.14}$$

which holds for all deterministic, Borel-measurable functions $\varphi : [0,t] \to \mathbb{R}$ (see Lemma B.1.3 from [BG06]). Summing over the partition gives the result with $c_1 = 2A_+$ and $c_2 = A_- \mathrm{e}^{-2}$. □

Applying this lemma with $H = D(1 - MD/A_-)$ gives

$$\mathbb{P}(E(D)) \leqslant \frac{C}{\varepsilon} \exp\left(-\frac{c_2 D^2}{\sigma^2}(1 - MD/A_-)^2\right) . \tag{3.15}$$

Remark 3.2.4. *Requiring the right-hand side above to be small leads to the lower bound on ε in Theorem 3.2.1(2). In the case that U is quadratic, there is no non-linear term and so such a bound is not required.*

We are now ready to treat the fast and slow pulling regimes from Theorem 3.2.1.

3.2.3 Fast Pulling

From our expression for the curve $d_+(t)$ given in (3.10), it follows that there exists $c_1 > 0$ such that $d_+(t) \geqslant c_1\varepsilon$ for all $0 \leqslant t \leqslant 2(b-a)$ and ε sufficiently small. Therefore, if we show that $|y_t| < c_1\varepsilon$ for all $0 \leqslant t \leqslant \tau$ then we will be done. By Lemma 3.2.2, we just need to show that $\mathbb{P}(E(c_1\varepsilon)) \to 0$. The bound (3.15) tells us that if $\varepsilon \geqslant k\sigma|\ln\sigma|^{1/2}$ for large enough $k > 0$, the probability tends to zero as $\sigma \downarrow 0$.

3.2.4 Slow Pulling

In this section, we consider the slow pulling regime from Theorem 3.2.1. That is, we suppose there is $\sigma^{2/3} \ll r(\sigma) \ll 1$ such that

$$\exp\left(-\frac{r(\sigma)}{\sigma^{2/3}}\right) \ll \varepsilon(\sigma) \ll \sigma|\ln\sigma|^{-1/2} . \tag{3.16}$$

We now outline the strategy. Writing $y_t = y_t^0 + R(t)$, we can express (3.9) as

$$d_-(t) - R(t) \leqslant y_t^0 \leqslant d_+(t) - R(t) , \tag{3.17}$$

where it is enough in this slow pulling case to know that for $0 \leqslant t \leqslant 2(b-a)$,

$$\begin{aligned} d_+(t) &= b - a - t/2 + \mathcal{O}(\varepsilon) , \\ d_-(t) &= -(b - a - t/2) + \mathcal{O}(\varepsilon) , \end{aligned} \tag{3.18}$$

The reason we do not need more information about the $\mathcal{O}(\varepsilon)$ terms is that the noise will dominate them and so their particular form is unimportant. We shall choose $D = D(\sigma)$ such that $\mathbb{P}(E(D)) \to 0$ (see Proposition 3.2.6 below) and condition on $E^c = E(D)^c$. Given such a D, combining (3.18) with Lemma 3.2.2 tells us that there are constants C_1, $C_2 > 0$ such that for all $t \leqslant \tau$,

$$\begin{aligned} d_+(t) - R(t) &> b - a - t/2 - C_1 D^2 - C_2\varepsilon , \\ d_-(t) - R(t) &< -(b - a - t/2) + C_1 D^2 + C_2\varepsilon . \end{aligned}$$

Therefore, if we let

$$d(t) = b - a - t/2 - C_1 D^2 - C_2\varepsilon \tag{3.19}$$

and

$$\nu = \inf\{t \geqslant 0 : |y_t^0| \geqslant d(t)\} ,$$

then $\nu < \tau$ when we condition on E^c. By choosing D small, we guarantee that y_t is well-approximated by the linear process y_t^0 and we expect $\nu \approx \tau$. By symmetry, y_t^0 has an equal chance to hit $d(t)$ or $-d(t)$. We shall use this to show that y_t has an equal chance to hit $d_+(t)$ or $d_-(t)$ in the limit as $\sigma \downarrow 0$ (see Fig. 3.2 below).

Before proceeding further, we give a result about the distribution of $d(\nu)$. The following lemma, which we prove after Proposition 3.2.7 below, shows that under suitable conditions, $\mathbb{P}(d(\nu) < \sigma^2/D) \to 0$.

Lemma 3.2.5. *There exists a constant $c > 0$ such that whenever $D = D(\sigma)$ satisfies*

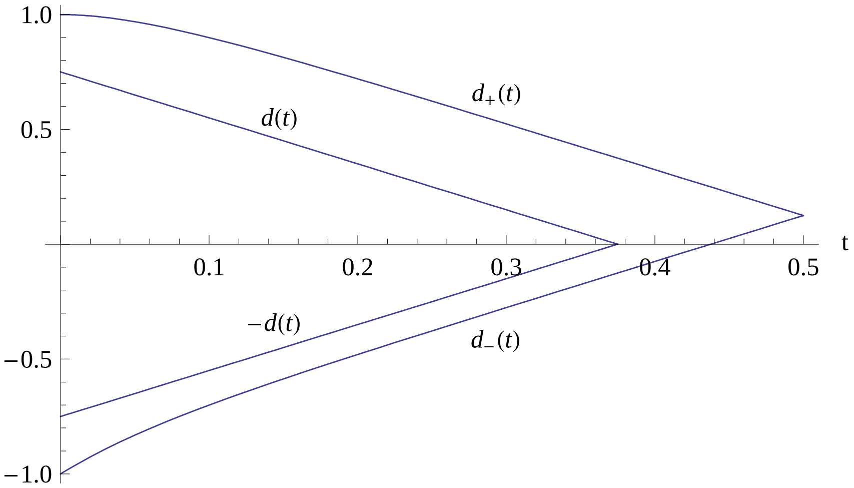


Figure 3.2: In the slow pulling regime, we show that the process y_t hits $d_+(t)$ or $d_-(t)$ soon after the linear process y_t^0 hits $d(t)$ or $-d(t)$, respectively.

$\sigma \ll D \ll 1$ *and* σ *is sufficiently small,*

$$\mathbb{P}(d(\nu) < \sigma^2/D) \leqslant \frac{c\,\sigma}{D} .$$

This lemma shows that the event $\{d(\nu) < \sigma^2/D\}$ is unlikely when $\sigma \ll D \ll 1$, which we shall assume below. Therefore, instead of conditioning on just E^c, we shall condition on the event $\mathcal{E}^c$, given by

$$\mathcal{E}^c = E^c \cap \{d(\nu) \geqslant \sigma^2/D\} ,$$

and write $\mathbb{Q}(\,\cdot\,) = \mathbb{P}(\,\cdot\,|\,\mathcal{E}^c)$. As $\lim_{\sigma\downarrow 0} \mathbb{Q}(A) = \lim_{\sigma\downarrow 0} \mathbb{P}(A)$ for all events A when the limit exists, it is sufficient for our purposes to consider just $\mathbb{Q}$.

We now explain our choice of D. The choice is made in such a way that the right-hand side of (3.15) tends to zero and Lemma 3.2.5 above applies, as well as all bounds used in the proof of Proposition 3.2.7 below.

Proposition 3.2.6. *We may choose* $D(\sigma)$ *according to one of the following two cases:*

1) If $\varepsilon = \varepsilon(\sigma)$ *satisfies*

$$\exp\left(-\frac{r(\sigma)}{\sigma^{2/3}}\right) \ll \varepsilon \ll \sigma^{4/3} |\ln \sigma|^{-1/2}$$

for some $\sigma^{2/3} \ll r(\sigma) \ll 1$, then there exists $D = D(\sigma)$ such that

$$\max\{\sigma, \varepsilon^{1/2}\} \ll D \ll \sigma^{2/3}$$

and $\mathbb{P}(E(D)) \to 0$ as $\sigma \downarrow 0$;

2) If $\sigma^{4/3}|\ln\sigma|^{-1/2} \lesssim \varepsilon \ll \sigma|\ln\sigma|^{-1/2}$ then there exists $D = D(\sigma)$ such that

$$\sigma \ll D \ll \min\{\sigma^2/\varepsilon, \varepsilon^{1/2}\}$$

and $\mathbb{P}(E(D)) \to 0$ as $\sigma \downarrow 0$.

Proof. 1) By (3.15), we know that $\mathbb{P}(E(D)) \to 0$ if $\mathrm{e}^{-cD^2/\sigma^2} \ll \varepsilon$ for some constant $c > 0$ and sufficiently small D. If $\varepsilon \geqslant \sigma^2$, we may choose $D = c^{-1/2}\sigma^{2/3}|\ln\sigma|^{-1/4}$. If $\varepsilon < \sigma^2$, we may take $D = c^{-1/2}\sigma^{2/3}\sqrt{r(\sigma)}$.

2) We may choose $D = k\sigma|\ln\sigma|^{1/2}$ with $k > 0$ large enough. □

The following proposition has two parts and will complete the proof of Theorem 3.1.1. We know by symmetry that y^0_t has an equal chance to hit $\pm d(t)$. Conditioning on $\mathcal{E}^c$ does not affect this symmetry, so that

$$\mathbb{Q}(y^0_\nu = d(\nu)) = \mathbb{Q}(y^0_\nu = -d(\nu)) = 1/2 \; .$$

Both cases are similar, so we treat only the case $y^0_\nu = d(\nu)$. The first part of the proposition shows that, given $y^0_\nu = d(\nu)$, there exists a small interval $[\nu, \nu+\Delta]$ such that $\tau \in [\nu, \nu+\Delta]$. The second part shows that, in this interval, y_t does not hit $d_-(t)$. We write $\mathbb{Q}^{\nu,d(\nu)}(\,\cdot\,) = \mathbb{Q}(\,\cdot\,|\, y^0_\nu = d(\nu))$ and $\mathbb{P}^{\nu,d(\nu)}(\,\cdot\,) = \mathbb{P}(\,\cdot\,|\, y^0_\nu = d(\nu))$.

Proposition 3.2.7. *Given $\varepsilon = \varepsilon(\sigma)$ satisfying (3.16), pick $D = D(\sigma)$ according to Proposition 3.2.6. Then there exists $\Delta = \Delta(\sigma, D, \varepsilon) > 0$ such that*

1.

$$\lim_{\sigma\downarrow 0} \mathbb{Q}\left(\tau > \nu + \Delta,\, y^0_\nu = d(\nu)\right) = 0 \; ,$$

2.

$$\lim_{\sigma\downarrow 0} \mathbb{Q}\left(y_t > d_-(t) \text{ for all } t \in [\nu, (\nu+\Delta)\wedge\tau],\, y^0_\nu = d(\nu)\right) = 1/2 \; .$$

Proof. We begin by choosing Δ. When $\varepsilon \ll \sigma^{4/3}|\ln\sigma|^{-1/2}$ and D satisfies the conditions in part (1) of Proposition 3.2.6, we may choose Δ so that

$$D/\sigma \ll \sqrt{\varepsilon/\Delta} \ll \sigma/D^2 \; . \tag{3.20}$$

If $\varepsilon \gtrsim \sigma^{4/3}|\ln \sigma|^{-1/2}$ and D satisfies the conditions in part (2) of Proposition 3.2.6, we may choose Δ so that

$$D/\sigma \ll \sqrt{\varepsilon/\Delta} \ll \sigma/\varepsilon \ . \tag{3.21}$$

Now we are ready to prove each part of the proposition. In doing so, we will use Lemma 3.2.9, which is stated and proved in Section 3.2.5 below.

1) If $\tau > \nu + \Delta$, then $y_t < d_+(t)$ for all $t \in [\nu, \nu + \Delta]$, which can be written as $y_t^0 < d_+(t) - R(t)$. On $\mathcal{E}^c$ and for $t \leqslant \tau$, we have by Lemma 3.2.2 that $d_+(t) - R(t) \leqslant d_+(t) + MD^2/A_-$. Therefore, it is enough to show that

$$\lim_{\sigma \downarrow 0} \mathbb{Q}^{\nu, d(\nu)} \left(y_t^0 < d_+(t) + MD^2/A_- \text{ for all } t \in [\nu, \nu + \Delta] \right) = 0 \ .$$

Call the event on the left-hand side above $A_1[\nu, \nu + \Delta]$. By definition of $\mathbb{Q}$ it follows that

$$\begin{aligned} \mathbb{Q}^{\nu,d(\nu)}(A_1[\nu, \nu + \Delta]) &= \mathbb{Q}^{\nu,d(\nu)}(A_1[\nu, \nu + \Delta], \, \sigma^2/D \ \leqslant \ d(\nu) \ \leqslant \ D) \\ &\leqslant \sup_{\nu_- \,\leqslant\, \nu_0 \,\leqslant\, \nu_+} \mathbb{Q}^{\nu_0, d(\nu_0)}(A_1[\nu_0, \nu_0 + \Delta]) \ . \end{aligned}$$

where ν_-, ν_+ are deterministic times such that $d(\nu_-) = D^2$, $d(\nu_+) = \sigma^2/D$, and ν_0 represents a non-random initial time. Furthermore, as $\mathbb{P}(\mathcal{E}) \to 0$, we only need to consider $\mathbb{P}^{\nu_0, d(\nu_0)}(A_1[\nu_0, \nu_0 + \Delta])$ when letting $\sigma \downarrow 0$. For $t \ \geqslant \ \nu_0$,

$$y_t^0 = d(\nu_0) \, \mathrm{e}^{\alpha(t,\nu_0)/\varepsilon} + \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_{\nu_0}^t \mathrm{e}^{\alpha(t,s)/\varepsilon} \, \mathrm{d}W(s)}_{=:\, \eta(t,\nu_0)} \ . \tag{3.22}$$

Letting $A_2 = A_2[\nu_0, \nu_0 + \Delta]$ be given by

$$A_2 = \{\eta(t, \nu_0) < d_+(t) + MD^2/A_- - d(\nu_0) \, \mathrm{e}^{\alpha(t,\nu_0)/\varepsilon} \ \text{ for all } t \in [\nu_0, \nu_0 + \Delta]\} \ ,$$

it follows that $\mathbb{P}^{\nu_0, d(\nu_0)}(A_1[\nu_0, \nu_0 + \Delta]) = \mathbb{P}(A_2)$. For $\nu_0 \in [\nu_-, \nu_+]$ and $t \in [\nu_0, \nu_0 + \Delta]$,

$$\begin{aligned} d_+(t) - d(\nu_0) \, \mathrm{e}^{\alpha(t,\nu_0)/\varepsilon} \ &\leqslant \ d(\nu_0)(1 - \ \mathrm{e}^{\alpha(t,\nu_0)/\varepsilon} \,) + \mathcal{O}(\varepsilon) + \mathcal{O}(D^2) \\ &\leqslant \ A_+ D\Delta/\varepsilon + \mathcal{O}(\varepsilon) + \mathcal{O}(D^2) \ . \end{aligned}$$

Therefore,

$$\sup_{\nu_- \,\leqslant\, \nu_0 \,\leqslant\, \nu_+} \mathbb{P}(A_2) \ \leqslant \ \sup_{\nu_- \,\leqslant\, \nu_0 \,\leqslant\, \nu_+} \mathbb{P}(\tilde{A}_2) \ ,$$

where

$$\tilde{A}_2 = \left\{ \sup_{t\in[\nu_0,\nu_0+\Delta]} \eta(t,\nu_0) < A_+ D\Delta/\varepsilon + \mathcal{O}(\varepsilon) + \mathcal{O}(D^2) \right\} .$$

By Lemma 3.2.9, the right-hand side tends to zero as $\sigma \downarrow 0$.

2) As we are conditioning on $\mathcal{E}^c$, we have by definition of $d(t)$ that for $t \leqslant \tau$, $y_t > d_-(t)$ as long as $y_t^0 > -d(t)$. Therefore, if we show that for all $t \in [\nu, \nu+\Delta]$, both $y_t^0 \geqslant 0$ and $-d(t) < 0$, then we will be done. Write

$$A_3[\nu, \nu+\Delta] = \left\{ \inf_{t\in[\nu,\nu+\Delta]} y_t^0 < 0 \right\} .$$

Similarly to above, it follows by definition of $\mathbb{Q}$ that

$$\mathbb{Q}^{\nu,d(\nu)}(A_3[\nu,\nu+\Delta]) \leqslant \sup_{\nu_- \leqslant \nu_0 \leqslant \nu_+} \mathbb{Q}^{\nu_0,d(\nu_0)}(A_3[\nu_0,\nu_0+\Delta]) .$$

It is again sufficient to consider just $\mathbb{P}^{\nu_0,d(\nu_0)}(A_3[\nu_0,\nu_0+\Delta])$ when letting $\sigma \downarrow 0$. As the distribution of y_t^0, when started from zero, is symmetric about the origin, we can apply a reflection principle (see Lemma B.4.1 from Appendix of [BG06]) to give

$$\begin{aligned} \mathbb{P}^{\nu_0,d(\nu_0)}(A_3[\nu_0,\nu_0+\Delta]) &= 2\mathbb{P}^{\nu_0,d(\nu_0)}\left(y^0_{\nu_0+\Delta} < 0\right) \\ &= 2\mathbb{P}\left(\eta(\nu_0+\Delta,\nu_0) < -d(\nu_0)\, \mathrm{e}^{\alpha(\nu_0+\Delta,\nu_0)/\varepsilon}\right) . \end{aligned}$$

For $\sigma^2/D \leqslant d(\nu_0) \leqslant D$, the probability above is biggest when $d(\nu_0) = \sigma^2/D$. Using that $\mathrm{Var}(\eta(\nu_0+\Delta,\nu_0)) \asymp \sigma^2\Delta/\varepsilon$ and $\mathrm{e}^{\alpha(\nu_0+\Delta,\nu_0)/\varepsilon} \asymp 1$ for $\nu_- \leqslant \nu_0 \leqslant \nu_+$, it follows by evaluating the corresponding Gaussian integral that the probability tends to zero as $\sigma \downarrow 0$.

To see that $-d(t) < 0$ for all $t \in [\nu, \nu+\Delta]$, we have simply that for $\mathbb{P}$-almost all paths in $\mathcal{E}^c$,

$$-d(t) \leqslant -d(\nu) + \mathcal{O}(\Delta) \leqslant -\sigma^2/D + \mathcal{O}(\Delta) .$$

As $\Delta \ll \sigma^2/D$, which is a consequence of (3.20) and (3.21), the result follows.

□

3.2.5 Auxillary Results and Proofs

We begin this section by proving Lemma 3.2.5, which is restated for convenience.

Lemma 3.2.8 (Lemma 3.2.5 restated)**.** *There exists a constant $c > 0$ such that* *whenever $D = D(\sigma)$ satisfies $\sigma \ll D \ll 1$ and σ is sufficiently small,*

$$\mathbb{P}(d(\nu) < \sigma^2/D) \;\leqslant\; \frac{c\,\sigma}{D}\;.$$

Proof. Let ν_+ be the deterministic time such that $d(\nu_+) = \sigma^2/D$. Then

$$\begin{aligned}\mathbb{P}\left(d(\nu) < \sigma^2/D\right) \;&\leqslant\; \mathbb{P}\left(|y^0_{\nu_+}| < \sigma^2/D\right)\\ &\leqslant\; 1 - 2\Phi\left(-\frac{\sigma^2/D}{\sqrt{\mathrm{Var}(y^0_{\nu_+})}}\right)\;.\end{aligned}$$

By (3.19), it follows that ν_+ is bounded away from zero and so $\mathrm{Var}(y^0_{\nu_+}) \asymp \sigma^2$. The result follows using the simple inequality $\Phi(-x) \;\geqslant\; 1/2 - x/(2\pi)$, valid for all $x \;\geqslant\; 0$. $\square$

We also used the following lemma in the proof of Proposition 3.2.7.

Lemma 3.2.9. *For $t_0 \;\geqslant\; 0$, let*

$$\eta(t,t_0) = \frac{\sigma}{\sqrt{\varepsilon}}\int_{t_0}^{t} \mathrm{e}^{\alpha(t,s)/\varepsilon}\,\mathrm{d}W_s\;,$$

where $\alpha(t,s) = \int_s^t A(u)\,\mathrm{d}u$ for some continuous function $A : [0,\infty) \to \mathbb{R}$ with $A(u) \asymp -1$ uniformly for all $u \;\geqslant\; 0$.

1. *If $L = L(\sigma) > 0$, $\varepsilon = \varepsilon(\sigma) > 0$ and $\Delta = \Delta(\sigma) > 0$ satisfy*

$$L \ll \sigma\sqrt{\Delta/\varepsilon}, \qquad \Delta \ll \varepsilon$$

then

$$\lim_{\sigma\downarrow 0}\mathbb{P}\left(\sup_{t\in[t_0,t_0+\Delta]}\eta(t,t_0) \;\geqslant\; L\right) = 1\;. \tag{3.23}$$

2. *On the other hand, if L, ε and Δ satisfy*

$$L \gg \sigma\sqrt{\Delta/\varepsilon}, \qquad \Delta \ll \varepsilon$$

then

$$\lim_{\sigma\downarrow 0}\mathbb{P}\left(\sup_{t\in[t_0,t_0+\Delta]}\eta(t,t_0) \;\geqslant\; L\right) = 0\;. \tag{3.24}$$

Proof. 1) Let A_1 be the event on the left-hand side of (3.23) and note that

$$A_1 = \left\{ \mathrm{e}^{-\alpha(t)/\varepsilon}\, \eta(t,t_0) \;\geqslant\; L\,\mathrm{e}^{-\alpha(t)/\varepsilon} \text{ for some } t \in [t_0, t_0+\Delta] \right\} .$$

where $\alpha(t) = \alpha(t,0)$. On this interval, $\mathrm{e}^{-\alpha(t)/\varepsilon} \;\leqslant\; \mathrm{e}^{-\alpha(t_0+\Delta)/\varepsilon}$, so that

$$A_1 \supset A_2 \;:=\; \left\{ \sup_{t\in[t_0,t_0+\Delta]} \mathrm{e}^{-\alpha(t)/\varepsilon}\, \eta(t,t_0) \;\geqslant\; L\,\mathrm{e}^{-\alpha(t_0+\Delta)/\varepsilon} \right\} .$$

The process

$$\mathrm{e}^{-\alpha(t)/\varepsilon}\, \eta(t,t_0) = \frac{\sigma}{\sqrt{\varepsilon}} \int_{t_0}^{t} \mathrm{e}^{-\alpha(s)/\varepsilon}\, \mathrm{d}W_s$$

is a Gaussian martingale and so satisfies a reflection principle. Therefore,

$$\begin{aligned} \mathbb{P}(A_2) &= 2\mathbb{P}\left(\mathrm{e}^{-\alpha(t_0+\Delta)/\varepsilon}\, \eta(t_0+\Delta,t_0) \;\geqslant\; L\,\mathrm{e}^{-\alpha(t_0+\Delta)/\varepsilon} \right) \\ &= 2\Phi\left(-\frac{L}{\sqrt{\mathrm{Var}(\eta(t_0+\Delta,t_0))}} \right) \\ &\geqslant\; 1 - \frac{2L}{\sqrt{2\pi \mathrm{Var}(\eta(t_0+\Delta,t_0))}} . \end{aligned} \tag{3.25}$$

Since $A(u) \asymp -1$, it follows that there is $c > 0$ such that

$$\begin{aligned} \mathrm{Var}(\eta(t_0+\Delta,t_0)) &= \frac{\sigma^2}{\varepsilon} \int_{t_0}^{t_0+\Delta} \mathrm{e}^{2\alpha(t_0+\Delta,s)/\varepsilon}\, \mathrm{d}s \\ &\geqslant\; \frac{\sigma^2}{\varepsilon} \int_{t_0}^{t_0+\Delta} \mathrm{e}^{-c(t_0+\Delta-s)/\varepsilon}\, \mathrm{d}s \\ &= \frac{\sigma^2}{c}(1 - \mathrm{e}^{-c\Delta/\varepsilon}) . \end{aligned}$$

Therefore, $\mathrm{Var}(\eta(t_0+\Delta,t_0)) \;\geqslant\; \sigma^2\Delta/(2\varepsilon)$ for Δ/ε small enough and the right-hand side of (3.25) tends to one as $\sigma \downarrow 0$.

2) For A_1 as above, we find that $A_1 \subset A_3$, where

$$A_3 = \left\{ \sup_{t\in[t_0,t_0+\Delta]} \mathrm{e}^{-\alpha(t)/\varepsilon}\, \eta(t,t_0) \;\geqslant\; L\,\mathrm{e}^{-\alpha(t_0)/\varepsilon} \right\} .$$

We can again apply the reflection principle to obtain

$$\begin{aligned}\mathbb{P}(A_3) &= 2\mathbb{P}\left(\mathrm{e}^{-\alpha(t_0+\Delta)/\varepsilon}\,\eta(t_0+\Delta,t_0) \;\geqslant\; L\,\mathrm{e}^{-\alpha(t_0)/\varepsilon}\right)\\ &= 2\Phi\left(-\frac{L\,\mathrm{e}^{\alpha(t_0+\Delta,t_0)}}{\sqrt{\mathrm{Var}(\eta(t_0+\Delta,t_0))}}\right)\\ &\leqslant C\exp\left(-\frac{L^2\,\mathrm{e}^{2\alpha(t_0+\Delta,t_0)}}{2\mathrm{Var}(\eta(t_0+\Delta,t_0))}\right)\;.\end{aligned}\tag{3.26}$$

Similarly to above, we find that $\mathrm{Var}(\eta(t_0+\Delta,t_0)) \leqslant \sigma^2\Delta/\varepsilon$ for Δ/ε small enough, while $\mathrm{e}^{2\alpha(t_0+\Delta,t_0)/\varepsilon} \geqslant \mathrm{e}^{-c\Delta/\varepsilon} \geqslant 1/2$ for some constant $c > 0$. Then the right-hand side of (3.26) tends to zero. □

3.3 Three Breakable Bonds

3.3.1 The Model and Main Result

Having established the behaviour of a chain of three particles, we now wish to extend this analysis to longer chains. As a first step in this direction, we now consider a chain with four particles, $q_L < q_1 < q_2 < q_R$, interacting with each other via the same pairwise potential $\tilde{U}$ as given in (3.1). As before, $q_L \equiv 0$, but now $q_R(s) = 3a + \varepsilon s$. Initially, only nearest neighbours are interacting and the time-dependent potential energy of the configuration $(0, q_1, q_2, 3a + \varepsilon s)$ is given by

$$\tilde{H}(q_1, q_2, \varepsilon s) = \tilde{U}(q_1) + \tilde{U}(q_2 - q_1) + \tilde{U}(3a + \varepsilon s - q_2)\;.$$

Although it is possible for next-to-nearest neighbour interactions to occur, we shall again assume a certain boundedness property of solutions (see (3.42) and Lemma 3.4.2), in which case $\tilde{H}$ is always as given above. Similarly to the previous section, we can equivalently work with the function

$$H(q_1, q_2, \varepsilon s) = U(q_1) + U(q_2 - q_1) + U(3a + \varepsilon s - q_2)\;,$$

as we shall only consider the behaviour of the chain for times such that $H = \tilde{H}$. Then $q(s) = (q_1(s), q_2(s))$ evolves according to

$$\mathrm{d}q_i(s) = -\frac{\partial H}{\partial q_i}(q(s), \varepsilon s)\,\mathrm{d}s + \sigma\,\mathrm{d}W_i(s)\;,\quad q_i(0) = ia\;,$$

where W_1, W_2 are independent, standard Brownian motions. We introduce the same time change as before, $t = \varepsilon s$, to yield

$$\mathrm{d}q_i(t) = -\frac{1}{\varepsilon}\frac{\partial H}{\partial q_i}(q(t), t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_i(t)\ , \quad q_i(0) = ia\ . \tag{3.27}$$

As in the previous section, we consider the chain to break as soon as there is no longer any interaction between two neighbouring particles. This breaking time is bounded above by $3(b-a)$, which is when the total length of the chain becomes $3b$. Writing $\mathbb{R}^2_* = \{(q_1, q_2) \in \mathbb{R}^2 : q_1 \leqslant q_2\}$, we define the space-time domain

$$\mathcal{D}_q = \{(q_1, q_2, t) \in \mathbb{R}^2_* \times [0, 3(b-a)] : q_1 < b,\ q_2 - q_1 < b,\ 3a + t - q_2 < b\}\ . \tag{3.28}$$

Then the breaking time of the chain is given by

$$\tau = \inf\{t \geqslant 0 : (q(t), t) \notin \mathcal{D}_q\}\ . \tag{3.29}$$

We say the left bond breaks if $q_1(\tau) = b$, the middle if $q_2(\tau) - q_1(\tau) = b$, and the right if $3a + \tau - q_2(\tau) = b$. The following theorem is analogous to Theorem 3.1.1.

Theorem 3.3.1. *Let q_t solve (3.27) and define τ as in (3.29).*

1. *(Fast pulling) There is $k > 0$ such that if $k\sigma|\ln\sigma|^{1/2} \leqslant \varepsilon(\sigma) \ll 1$ then*

$$\lim_{\sigma\downarrow 0} \mathbb{P}(\textit{Right bond breaks}) = 1\ .$$

2. *(Slow pulling) Suppose that*

$$0 < \liminf_{\sigma\downarrow 0} \mathbb{P}(\textit{Middle bond breaks}) \leqslant \limsup_{\sigma\downarrow 0} \mathbb{P}(\textit{Middle bond breaks}) < 1\ . \tag{3.30}$$

If

$$\exp\left(-\frac{r(\sigma)}{\sigma^{2/3}}\right) \ll \varepsilon(\sigma) \ll \sigma|\ln\sigma|^{-1/2}$$

for some $\sigma^{2/3} \ll r(\sigma) \ll 1$, then

$$0 < \lim_{\sigma\downarrow 0} \mathbb{P}(\textit{Left bond breaks}) = \lim_{\sigma\downarrow 0} \mathbb{P}(\textit{Right bond breaks}) < 1\ .$$

Remark 3.3.2. • *Although we have made the assumption that (3.30) holds, we do not expect this to be an assumption at all, i.e. we believe this can be proved to hold. We will actually make a slightly different assumption from*

(3.30) when proving (2), but which turns out to be equivalent (see (3.52) and Proposition 3.4.4).

- *In the slow pulling regime, we do not know what the limit is equal to. However, in the case of a harmonic potential and no pulling, we saw in Chapter 2 that the limit is* $1/4$ *for the left and right bonds and* $1/2$ *for the middle bond (see (2.7)).*

- *As in the case of two breakable bonds, we will see in the proof of (1) that there are constants* $c_1, c_2 > 0$ *such that for* σ *sufficiently small,*

$$\mathbb{P}(\textit{Right bond breaks}) > 1 - c_1\varepsilon^{-1}\,\mathrm{e}^{-c_2\varepsilon^2/\sigma^2}\ .$$

3.4 Proof of Theorem 3.3.1

We follow the same approach as in the case of two breakable bonds: in Section 3.4.1 we reformulate the theorem above in terms of a more convenient process $z(t)$, giving us Theorem 3.4.1. Then in Section 3.4.2, we linearise $z(t)$ and show that the nonlinear part stays small. In Sections 3.4.3 and 3.4.4 we prove Theorem 3.4.1 (1) and (2). Finally, in Section 3.4.5 we prove some auxillary results.

3.4.1 An Alternative Formulation

Let $q^{\mathrm{det}}(t) = (q_1^{\mathrm{det}}(t), q_2^{\mathrm{det}}(t))$ be the solution of the ODE obtained from (3.27) by setting $\sigma = 0$:

$$\dot{q}_i^{\mathrm{det}}(t) = -\frac{1}{\varepsilon}\frac{\partial H}{\partial q_i}(q^{\mathrm{det}}(t), t)\ , \quad q_i^{\mathrm{det}}(0) = ia\ . \tag{3.31}$$

Letting $q^*(t) = (a + t/3, 2a + 2t/3)$, we see that for all $t \geqslant 0$, $\nabla_q H(q^*(t), t) = 0$ and so $q^*(t)$ is a slow manifold, corresponding to the configuration of equally spaced particles. By Tihonov's Theorem (Theorem B.0.2), $\|q^{\mathrm{det}}(t) - q^*(t)\| = \mathcal{O}(\varepsilon)$ uniformly in time. In addition, it follows from (U4) that $q_i^{\mathrm{det}}(t) \leqslant i(a + t/3)$: $(q_i^*)'(t) > 0$ and if at some time $q_i^{\mathrm{det}}(t) = i(a + t/3)$, then we can check that $\dot{q}_i^{\mathrm{det}}(t) \leqslant 0$.

The system (3.31) has an adiabatic manifold $\bar{q}(t)$ given by

$$\bar{q}(t) = q^*(t) - \frac{\varepsilon}{3U''(a + t/3)}A^{-1}\begin{pmatrix}1\\2\end{pmatrix} + \mathcal{O}(\varepsilon^2)$$

for all $0 \leqslant t \leqslant 3(b-a)$, where

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} . \tag{3.32}$$

This expansion of $\bar{q}(t)$ was obtained using (B.3). By the uniform asymptotic stability of $q^*(t)$, $\bar{q}(t)$ is locally attractive and we have

$$\begin{aligned} \|q^{\text{det}}(t) - \bar{q}(t)\| &\leqslant \|q^{\text{det}}(0) - \bar{q}(0)\| \, \mathrm{e}^{-\kappa t/\varepsilon} \\ &\leqslant C \, \mathrm{e}^{-\kappa t/\varepsilon} \end{aligned}$$

for some C, $\kappa > 0$. After calculating A^{-1}, it follows that for times $t \gg \varepsilon |\ln \varepsilon|$,

$$q^{\text{det}}(t) = q^*(t) - \frac{\varepsilon}{9U''(a+t/3)} \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \mathcal{O}(\varepsilon^2) .$$

As in the previous section, we now introduce the deviation process

$$y(t) = (y_1(t), y_2(t)) := q(t) - q^{\text{det}}(t) .$$

Using Taylor expansions, we find that y solves

$$\mathrm{d}y(t) = \frac{1}{\varepsilon}[A(t)y(t) + b(y(t),t)] \, \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \, \mathrm{d}W_t , \tag{3.33}$$

where $W_t = (W_1(t), W_2(t))$, $A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$ is given by

$$\begin{aligned} a_{11}(t) &= -U''(q_1^{\text{det}}(t)) - U''(q_2^{\text{det}}(t) - q_1^{\text{det}}(t)) , \\ a_{22}(t) &= -U''(q_2^{\text{det}}(t) - q_1^{\text{det}}(t)) - U''(3a + t - q_2^{\text{det}}(t)) , \\ a_{1,2}(t) &= a_{2,1}(t) = U''(q_2^{\text{det}}(t) - q_1^{\text{det}}(t)) , \end{aligned}$$

and $b(y(t),t) \in \mathbb{R}^2$ contains the remainder terms. By (U1), there is $M_1 > 0$ such that $\|b(y,t)\| \leqslant M_1 \|y\|^2$ for all pairs (y,t) such that $(y + q^{\text{det}}(t), t) \in \mathcal{D}_q$, where $\mathcal{D}_q$ was given in (3.28). Since $q^{\text{det}}(t)$ is close to $q^*(t)$, we expect all the $U''(\,\cdot\,)$ terms above to be almost equal. Indeed, by Taylor's Theorem, there exists $C > 0$ such that for all $j \in \{0,1,2\}$ and all $0 \leqslant t \leqslant 3(b-a)$,

$$|U''(q_{j+1}^{\text{det}}(t) - q_j^{\text{det}}(t)) - U''(a + t/3)| \leqslant C\varepsilon ,$$

where we take $q_0^{\det} \equiv 0$ and $q_3^{\det}(t) = 3a + t$. Therefore, we can decompose $A(t)$ in (3.33) as

$$A(t) = -u(t)A + \varepsilon A^1(t) ,$$

where A is given in (3.32), $u(t) := U''(a + t/3)$ and there exists $M_2 > 0$ such that $\|A^1(t)\| \leqslant M_2$ for all $0 \leqslant t \leqslant 3(b - a)$. Then

$$\mathrm{d}y(t) = \frac{1}{\varepsilon}[-u(t)Ay(t) + \varepsilon A^1(t)y(t) + b(y(t), t)]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t .$$

In terms of $y(t)$ and $q^{\det}(t)$, the conditions for the chain to be unbroken can be written as follows:

- The left bond is unbroken if

$$y_1(t) < b - q_1^{\det}(t) ;$$

- The middle bond is unbroken if

$$y_2(t) - y_1(t) < b + q_1^{\det}(t) - q_2^{\det}(t) ;$$

- The right bond is unbroken if

$$y_2(t) > 3a + t - q_2^{\det}(t) - b .$$

Now we introduce a more convenient set of coordinates. Letting

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = P^{-1}$$

gives

$$P^{-1}AP = D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} .$$

If $z = P^{-1}y$, then τ can be expressed as

$$\tau = \inf\{t \geqslant 0 : (Pz(t) + q^{\det}(t), t) \notin \mathcal{D}_q\} ,$$

where the process $z(t) = (z_1(t), z_2(t))$ solves

$$\mathrm{d}z(t) = \frac{1}{\varepsilon}[-u(t)Dz(t) + \varepsilon P^{-1}A^1(t)Pz(t) + P^{-1}b(Pz(t), t)]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t . \quad (3.34)$$

As P^{-1} is orthogonal, $P^{-1}W_t$ is again a standard, two-dimensional Brownian motion, and so we have omitted P^{-1} from the noise term.

The conditions for the chain to be unbroken can be written in terms of $z_1(t)$ and $z_2(t)$ as follows:

- The left bond is unbroken if

$$z_1(t) < \sqrt{2}(b - q_1^{\mathrm{det}}(t)) - z_2(t) \ =: \ d_+(t) \ ; \tag{3.35}$$

- The middle bond is unbroken if

$$z_2(t) > -\frac{1}{\sqrt{2}}(b + q_1^{\mathrm{det}}(t) - q_2^{\mathrm{det}}(t)) \ =: \ d_M(t) \ ; \tag{3.36}$$

- The right bond is unbroken if

$$z_1(t) > \sqrt{2}(3a + t - q_2^{\mathrm{det}}(t) - b) + z_2(t) \ =: \ d_-(t) \ . \tag{3.37}$$

The curves $d_+(t)$ and $d_-(t)$ are analogous to those of the previous section. However, they now involve the process $z_2(t)$, so are random. By the expression for $q^{\mathrm{det}}(t)$, we know that for $\varepsilon|\ln\varepsilon| \ll t \leqslant 3(b-a)$,

$$\begin{aligned} d_+(t) &= \sqrt{2}(b - a - t/3) + \left(\frac{4\sqrt{2}}{9u(t)}\right)\varepsilon + \mathcal{O}(\varepsilon^2) - z_2(t) \ , \\ d_M(t) &= -\frac{1}{\sqrt{2}}(b - a - t/3) - \left(\frac{1}{9\sqrt{2}u(t)}\right)\varepsilon + \mathcal{O}(\varepsilon^2) \ , \\ d_-(t) &= -\sqrt{2}(b - a - t/3) + \left(\frac{5\sqrt{2}}{9u(t)}\right)\varepsilon + \mathcal{O}(\varepsilon^2) + z_2(t) \ . \end{aligned} \tag{3.38}$$

Define the space-time set $\mathcal{D}_z$ by

$$\mathcal{D}_z = \{(z_1, z_2, t) \in \mathbb{R}^2 \times [0, 3(b-a)] : d_-(t) < z_1 < d_+(t), \ z_2 > d_M(t)\} \ .$$

Then τ can be expressed as

$$\tau = \inf\{t \ \geqslant \ 0 : (z(t), t) \notin \mathcal{D}_z\} \ . \tag{3.39}$$

We can then state Theorem 3.3.1 as follows.

Theorem 3.4.1 (Alternative version of Theorem 3.3.1)**.** *Let $z(t)$ solve (3.34) and define τ as in (3.39).*

1. *(Fast pulling) There is $k > 0$ such that if $k\sigma|\ln\sigma|^{1/2} \leqslant \varepsilon(\sigma) \ll 1$ then*

$$\lim_{\sigma\downarrow 0} \mathbb{P}(z_1(\tau) = d_-(\tau)) = 1 .$$

2. *(Slow pulling) Suppose that*

$$0 < \liminf_{\sigma\downarrow 0} \mathbb{P}(z_2(\tau) = d_M(\tau)) \leqslant \limsup_{\sigma\downarrow 0} \mathbb{P}(z_2(\tau) = d_M(\tau)) < 1 . \tag{3.40}$$

If

$$\exp\left(-\frac{r(\sigma)}{\sigma^{2/3}}\right) \ll \varepsilon(\sigma) \ll \sigma|\ln\sigma|^{-1/2}$$

for some $\sigma^{2/3} \ll r(\sigma) \ll 1$, then

$$0 < \lim_{\sigma\downarrow 0} \mathbb{P}(z_1(\tau) = d_-(\tau)) = \lim_{\sigma\downarrow 0} \mathbb{P}(z_1(\tau) = d_+(\tau)) < 1 .$$

3.4.2 Linearisation of $z(t)$

The solution of (3.34) is given by

$$z(t) = z^0(t) - \int_0^t \mathrm{e}^{\alpha(t,s)D/\varepsilon} P^{-1}A^1(s)Pz(s)\,\mathrm{d}s - \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\alpha(t,s)D/\varepsilon} P^{-1}b(Pz(s),s)\,\mathrm{d}s ,$$

where $\alpha(t,s) = -\int_s^t u(w)\,\mathrm{d}w \asymp -(t-s)$ uniformly by (U4), and $z^0(t) = (z_1^0(t), z_2^0(t))$ is given by

$$z_1^0(t) = \frac{\sigma}{\sqrt{\varepsilon}}\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\,\mathrm{d}W_1(s) , \quad z_2^0(t) = \frac{\sigma}{\sqrt{\varepsilon}}\int_0^t \mathrm{e}^{3\alpha(t,s)/\varepsilon}\,\mathrm{d}W_2(s) .$$

We will use later that $z_1^0(t)$ and $z_2^0(t)$ are independent. By the boundedness of $A^1(t)$, $P^{-1}b(Pz,t)$, and τ, there are constants C_1, $C_2 > 0$ such that for all $t \leqslant \tau$,

$$\|z(t) - z^0(t)\| \leqslant C_1\varepsilon \sup_{0\leqslant s\leqslant t}\|z(s)\| + C_2 \sup_{0\leqslant s\leqslant t}\|z(s)\|^2 . \tag{3.41}$$

Similarly to the previous section, we can write $z_i(t) = z_i^0(t) + R_i(t)$ and use (3.41) to show that if $z_1^0(t)$ and $z_2^0(t)$ do not become too large then they will be the dominant terms. The following lemma is analogous to Lemma 3.2.2 from the previous section.

Lemma 3.4.2. *Suppose that $|z_i^0(t)| < D(1 - C_1\varepsilon - C_2D)/\sqrt{2}$ for all $0 \leqslant t \leqslant 3(b-a)$ and each $i \in \{1,2\}$, where C_1, $C_2 > 0$ are as in (3.41). Then for all $0 \leqslant t \leqslant \tau$, $\|z(t)\| < D$ and so $|R_i(t)| < C_1\varepsilon D + C_2D^2$.*

Proof. Let $\tau(D) = \inf\{t \geqslant 0 : \|z(t)\| \geqslant D\}$ and suppose that $\tau(D) < \tau$. Then, since $\tau \leqslant 3(b-a)$, we have

$$\begin{aligned} D = \|z_{\tau(D)}\| &\leqslant \sup_{0 \leqslant t \leqslant 3(b-a)} \|z^0(t)\| + C_1\varepsilon D + C_2 D^2 \\ &< D(1 - C_1\varepsilon - C_2 D) + C_1\varepsilon D + C_2 D^2 \\ &= D \; . \end{aligned}$$

So $\|z(t)\|$ can never become D and $\tau(D) > \tau$. The bound on $R_i(t)$ follows by (3.41). $\square$

Similarly to the case of two breakable bonds, we introduce the event E given by

$$E = E(D, \varepsilon) = \bigcup_{i=1,2} \left\{ \sup_{0 \leqslant t \leqslant 3(b-a)} |z_i^0(t)| \geqslant \frac{D}{\sqrt{2}}(1 - C_1\varepsilon - C_2 D) \right\} \; , \tag{3.42}$$

where C_1, C_2 are as in (3.41). To obtain a bound for the probability of E, we can apply Lemma 3.2.3 to each process $z_1^0(t)$ and $z_2^0(t)$. We find constants c_1, $c_2 > 0$ such that

$$\mathbb{P}(E) \leqslant \frac{c_1}{\varepsilon} \exp\left(-\frac{c_2 D^2}{\sigma^2} (1 - C_1\varepsilon - C_2 D)^2 \right) \; . \tag{3.43}$$

We are now ready to prove parts (1) and (2) of Theorem 3.4.1. In each case, we will choose a suitable D such that the right-hand side of (3.43) tends to zero and consider events contained within E^c.

3.4.3 Fast Pulling

By (3.38), there exists a constant $c_1 > 0$ such that for all $0 \leqslant t \leqslant \tau$ and ε sufficiently small,

$$\begin{aligned} d_+(t) &\geqslant c_1\varepsilon - z_2(t) \; , \\ d_M(t) &\leqslant -c_1\varepsilon \; . \end{aligned}$$

Let $D = c_2\varepsilon$, where $0 < c_2 < c_1/2$. Then on the event E^c, these inequalities tell us $z_1(t) < d_+(t)$ and $z_2(t) > d_M(t)$ for all $t \leqslant \tau$, in which case it follows that $z_1(\tau) = d_-(\tau)$. So we just need to show that when $D = c_2\varepsilon$, $\mathbb{P}(E) \to 0$. By (3.43), this holds by taking $\varepsilon \geqslant k\sigma|\ln\sigma|^{1/2}$ for $k > 0$ large enough.

3.4.4 Slow Pulling

We now consider the case

$$\exp\left(-\frac{r(\sigma)}{\sigma^{2/3}}\right) \ll \varepsilon \ll \sigma|\ln\sigma|^{-1/2} \tag{3.44}$$

for some $\sigma^{2/3} \ll r(\sigma) \ll 1$. As we expect the noise to dominate the dynamics, it will be enough for our purposes to know that for all $0 \leqslant t \leqslant \tau$,

$$\begin{aligned} d_+(t) &= \sqrt{2}(b-a-t/3) + \mathcal{O}(\varepsilon) - z_2(t) \; , \\ d_M(t) &= -\frac{1}{\sqrt{2}}(b-a-t/3) + \mathcal{O}(\varepsilon) \; , \\ d_-(t) &= -\sqrt{2}(b-a-t/3) + \mathcal{O}(\varepsilon) + z_2(t) \; . \end{aligned} \tag{3.45}$$

Recalling the decomposition $z_i(t) = z_i^0(t) + R_i(t)$, we can express conditions (3.35), (3.36) and (3.37), respectively, as:

- The left bond is unbroken if

$$z_1^0(t) < \sqrt{2}(b-a-t/3) + \mathcal{O}(\varepsilon) - (R_1(t) + R_2(t)) - z_2^0(t) \; ; \tag{3.46}$$

- The middle bond is unbroken if

$$z_2^0(t) > -\frac{1}{\sqrt{2}}(b-a-t/3) + \mathcal{O}(\varepsilon) - R_2(t) \; ; \tag{3.47}$$

- The right bond is unbroken if

$$z_1^0(t) > -\sqrt{2}(b-a-t/3) + \mathcal{O}(\varepsilon) + (R_2(t) - R_1(t)) + z_2^0(t) \; . \tag{3.48}$$

We will again choose D according to Proposition 3.2.6 so that $\mathbb{P}(E) \to 0$ and $D \gg \varepsilon$. Then it is sufficient to consider events conditioned on E^c when taking the limit $\sigma \downarrow 0$. In this case, we have for all $t \leqslant \tau$ that $|R_i(t)| \leqslant C_1\varepsilon D + C_2 D^2 = \mathcal{O}(D^2)$ for all $t \leqslant \tau$ and so there is a curve $d(t)$ of the form

$$d(t) = \sqrt{2}(b-a-t/3) - z_2^0(t) - \mathcal{O}(D^2) - \mathcal{O}(\varepsilon) \; , \tag{3.49}$$

where the $\mathcal{O}(\cdot)$ terms are positive and independent of time, such that if $z_1^0(t) \leqslant d(t)$ then (3.46) holds and if $z_1^0(t) \geqslant -d(t)$ then (3.48) holds. Similarly, there is a curve

$d_M^0(t)$ of the form

$$d_M^0(t) = -\frac{1}{\sqrt{2}}(b - a - t/3) + \mathcal{O}(\varepsilon) + \mathcal{O}(D^2) ,$$

again with the $\mathcal{O}(\cdot)$ terms positive and time-independent, such that if $z_2^0(t) > d_M^0(t)$ then (3.47) holds.

Define the stopping time ν by

$$\nu = \inf\{t \geqslant 0 : |z_1^0(t)| \geqslant d(t)\} . \tag{3.50}$$

Note that ν only involves the Gaussian processes $z_1^0(t)$ and $z_2^0(t)$. Note also that ν may be smaller or larger than τ, depending on the behaviour of $z_2(t)$. But if $z_2(t) > d_M(t)$ for all $t \leqslant \nu \wedge \tau$ and E^c holds, then $\nu \leqslant \tau$. This follows since $\tau \leqslant 3(b-a)$ and if $\tau < \nu$, but $z_2(t) > d_M(t)$ for all $t \leqslant \tau$, then either $z_1(\tau) = d_+(\tau)$ or $z_1(\tau) = d_-(\tau)$ holds. But on E^c, this means that $z_1^0(t)$ hits $\pm d(t)$ for some $t \leqslant \tau$, so that $\nu \leqslant \tau$.

Let

$$B_M(\nu) = \{z_2(t) > d_M(t) \text{ for all } 0 \leqslant t \leqslant \nu \wedge \tau\}$$

be the event that the chain does not break in the middle before time $\nu \wedge \tau$. By the above discussion, $B_M(\nu) \cap E^c \subset \{\nu \leqslant \tau\}$. As we want to consider the breaking properties of the left and right bonds, we would like to condition on the event $B_M(\nu) \cap E^c$. However, in order to use the fact that $z_1^0(t)$ and $z_2^0(t)$ are independent, it is more convenient to instead condition on the set $B_M^0(\nu) \cap E^c$, where

$$B_M^0(\nu) = \{z_2^0(t) > d_M^0(t) \text{ for all } 0 \leqslant t \leqslant \nu\} . \tag{3.51}$$

It is clear that any path in $B_M^0(\nu) \cap E^c$ must be in $B_M(\nu) \cap E^c$, by definition of $d_M^0(t)$. To see that $B_M^0(\nu) \cap E^c \subsetneq B_M(\nu) \cap E^c$, observe that $z_2^0(t)$ may hit $d_M^0(t)$ without $z_2(t)$ hitting $d_M(t)$. However, we show later in Proposition 3.4.5 that

$$\lim_{\sigma \downarrow 0} \mathbb{P}((B_M(\nu) \cap E^c) \setminus (B_M^0(\nu) \cap E^c)) = 0 .$$

Therefore, we may equivalently condition on either event when considering the limit $\sigma \downarrow 0$. Note that $B_M^0(\nu) \cap E^c \subset \{\nu \leqslant \tau\}$, which follows from $B_M^0(\nu) \cap E^c \subset B_M(\nu) \cap E^c$.

We make the following assumption, which we will see is equivalent to (3.40):

$$0 < \liminf_{\sigma\downarrow 0} \mathbb{P}(B_M^0(\nu)) \leqslant \limsup_{\sigma\downarrow 0} \mathbb{P}(B_M^0(\nu)) < 1 . \tag{3.52}$$

We expect this assumption to be true in all cases and so not an assumption at all. This is because $z_1^0(t)$ and $z_2^0(t)$ are roughly Ornstein-Uhlenbeck processes of the same order, and the curves $d(t)$ and $d_M^0(t)$ are also of the same order for $t < \nu$, so that both curves should have non-vanishing probability of being hit first.

Assumption (3.52) tells us that if $\mathbb{P}(E) \to 0$ then $\mathbb{P}(\,\cdot \mid B_M^0(\nu) \cap E^c) \leqslant p_0\,\mathbb{P}(\,\cdot\,)$ for some $p_0 > 0$ and all σ sufficiently small. We make use of this inequality later and is the reason why this assumption is required.

Let $\mathbb{Q}(\,\cdot\,) = \mathbb{P}(\,\cdot \mid B_M^0(\nu) \cap E^c)$. We shall now show that $\mathbb{Q}(z_1(\tau) = d_+(\tau)) = \mathbb{Q}(z_1(\tau) = d_-(\tau)) = 1/2$ as $\sigma \downarrow 0$, which will complete the proof of Theorem 3.4.1.

First, we give a result about some properties of ν. Consider the event F given by

$$F = F(\nu, D) = \{d(\nu) < \sigma^2/D\} \cup \{z_2^0(\nu) < d_M^0(\nu) + \sigma^2/D\} . \tag{3.53}$$

The following lemma, which is proved in Section 3.4.5 below, shows that under suitable hypotheses, $\mathbb{Q}(F) \to 0$.

Lemma 3.4.3. *Given $\varepsilon = \varepsilon(\sigma)$ satisfying (3.44), let $D = D(\sigma)$ be chosen according to Proposition 3.2.6. Let ν and $B_M^0(\nu)$ be defined as in (3.50) and (3.51), respectively, and assume (3.52). Then there exists $c > 0$ such that for σ sufficiently small,*

$$\mathbb{Q}(F) \leqslant \frac{c\,\sigma}{D} .$$

For any event A, it is sufficient, as a consequence of this lemma, to consider $\mathbb{Q}(A \cap F^c)$ when taking the limit $\sigma \downarrow 0$.

By definition of the relevant quantities, conditioning on $B_M^0(\nu) \cap E^c$ does not affect the symmetry of the distribution of $z_1^0(\nu)$, so that

$$\mathbb{Q}(z_1^0(\nu) = d(\nu)) = \mathbb{Q}(z_1^0(\nu) = -d(\nu)) = 1/2 .$$

We will consider the case $z_1^0(\nu) = d(\nu)$. The case $z_1^0(\nu) = -d(\nu)$ is similar. Let

$$\mathbb{Q}^{\nu,d(\nu)}(\,\cdot\,) = \mathbb{Q}(\,\cdot \mid z_1^0(\nu) = d(\nu)) = \mathbb{P}(\,\cdot \mid z_1^0(\nu) = d(\nu),\, B_M^0(\nu) \cap E^c)$$

and

$$\mathbb{P}^{\nu,d(\nu)}(\,\cdot\,) = \mathbb{P}(\,\cdot \mid z_1^0(\nu) = d(\nu)) \; .$$

Assuming (3.52) and using that $\mathbb{Q}(z_1^0(\nu) = d(\nu)) = \mathbb{P}(z_1^0(\nu) = d(\nu)) = 1/2$, we find there is a constant $p_0 > 0$ such that for σ sufficiently small,

$$\mathbb{Q}^{\nu,d(\nu)}(\,\cdot\,) \;\leqslant\; p_0\, \mathbb{P}^{\nu,d(\nu)}(\,\cdot\,) \; . \tag{3.54}$$

We will make use of this inequality below. The following proposition is an analogue of Proposition 3.2.7 and will complete the proof of Theorem 3.3.1. We show that there is a small interval $[\nu, \nu+\Delta]$ such that $\tau \in [\nu, \nu+\Delta]$, and in which $z_1(t)$ does not hit $d_-(t)$ and $z_2(t)$ does not hit $d_M(t)$. In the case that $z_1^0(\nu) = -d(\nu)$, the same proposition holds with $d_+(t)$ and $d_-(t)$ switching places.

Proposition 3.4.4. *Given $\varepsilon = \varepsilon(\sigma)$ satisfying (3.44), pick $D = D(\sigma)$ according to Proposition 3.2.6, and assume that (3.52) holds. Then there exists $\Delta = \Delta(\sigma, D, \varepsilon) > 0$ such that*

1.

$$\lim_{\sigma \downarrow 0} \mathbb{Q}\left(\tau > \nu + \Delta,\; z_1^0(\nu) = d(\nu)\right) = 0 \; ;$$

2.

$$\lim_{\sigma \downarrow 0} \mathbb{Q}\left(z_1(t) > d_-(t) \textit{ for all } t \in [\nu, (\nu+\Delta)\wedge\tau],\; z_1^0(\nu) = d(\nu)\right) = 1/2 \; ;$$

3.

$$\lim_{\sigma \downarrow 0} \mathbb{Q}\left(z_2(t) > d_M(t) \textit{ for all } t \in [\nu, (\nu+\Delta)\wedge\tau]\right) = 1 \; .$$

Proof. We choose a similar Δ to that in the proof of Proposition 3.2.7, but with the extra requirement that $\Delta|\ln\sigma| \ll \varepsilon$. That is, when $\varepsilon \ll \sigma^{4/3}|\ln\sigma|^{-1/2}$ and D satisfies the conditions in part (1) of Proposition 3.2.6, we may choose Δ so that

$$D/\sigma \ll \sqrt{\varepsilon/\Delta} \ll \sigma/D^2 \; , \qquad \Delta|\ln\sigma| \ll \varepsilon \; .$$

If $\varepsilon \lesssim \sigma^{4/3}|\ln\sigma|^{-1/2}$ and D satisfies the conditions in part (2) of Proposition 3.2.6, we may choose Δ so that

$$D/\sigma \ll \sqrt{\varepsilon/\Delta} \ll \sigma/\varepsilon \; , \qquad \Delta|\ln\sigma| \ll \varepsilon \; .$$

1) If $\tau > \nu + \Delta$, then $z_1(t) < d_+(t)$ for all $t \in [\nu, \nu+\Delta]$, which is the same as $z_1^0(t) < d_+(t) - R_1(t)$. As we have conditioned on E^c, there is a $\mathcal{O}(D^2)$ term,

independent of time, such that

$$d_+(t) - R_1(t) \leqslant \sqrt{2}(b-a-t/3) + \mathcal{O}(\varepsilon) + \mathcal{O}(D^2) - z_2^0(t) \;=:\; d_+^0(t) \,,$$

where we may also assume the $\mathcal{O}(\varepsilon)$ term is independent of time. Therefore, it is enough to show $\mathbb{Q}^{\nu,d(\nu)}(A_1) \to 0$, where

$$A_1 = \{z_1^0(t) < d_+^0(t) \text{ for all } t \in [\nu, \nu+\Delta]\} \,.$$

On the event $B_M^0(\nu) \cap E^c$, we recall that $\nu \leqslant \tau \leqslant 3(b-a)$. Therefore, we have by the boundedness of $z_1^0(t)$ and $z_2^0(t)$ on $[0, 3(b-a)]$ that at time ν, $|z_2^0(\nu)| < D$ and $d(\nu) < D$. Let $A_2 = \{d(\nu) < D\}$, $A_3 = \{z_2^0(\nu) < D\}$. It follows that $\mathbb{Q}^{\nu,d(\nu)}(A_1) = \mathbb{Q}^{\nu,d(\nu)}(A_1 \cap A_2 \cap A_3)$, and by (3.54),

$$\mathbb{Q}^{\nu,d(\nu)}(A_1 \cap A_2 \cap A_3) \leqslant p_0 \, \mathbb{P}^{\nu,d(\nu)}(A_1 \cap A_2 \cap A_3) \,. \tag{3.55}$$

For $t \geqslant \nu$,

$$z_1^0(t) = d(\nu)\,\mathrm{e}^{\alpha(t,\nu)/\varepsilon} + \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_\nu^t \mathrm{e}^{\alpha(t,s)/\varepsilon} \,\mathrm{d}W_1(s)}_{=:\, \eta_1(t,\nu)} \tag{3.56}$$

and

$$z_2^0(t) = z_2^0(\nu)\,\mathrm{e}^{3\alpha(t,\nu)/\varepsilon} + \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_\nu^t \mathrm{e}^{3\alpha(t,s)/\varepsilon} \,\mathrm{d}W_2(s)}_{=:\, \eta_2(t,\nu)} \,. \tag{3.57}$$

Then, given $z_1^0(\nu) = d(\nu)$, we may rewrite A_1 as

$$A_1 = \{\eta_1(t,\nu) < d_+^0(t) - d(\nu)\,\mathrm{e}^{\alpha(t,\nu)/\varepsilon} \text{ for all } t \in [\nu, \nu+\Delta]\} \,.$$

Note that for $t \geqslant \nu$, $d_+^0(t) \leqslant d_+^0(\nu) - (z_2^0(t) - z_2^0(\nu))$, and for all $t \geqslant 0$, $d_+^0(t) = d(t) + \mathcal{O}(D^2) + \mathcal{O}(\varepsilon)$. Then using (3.57) and that $d(\nu) < D$ and $|z_2^0(\nu)| < D$, we have

$$d_+^0(t) - d(\nu)\,\mathrm{e}^{\alpha(t,\nu)/\varepsilon} \leqslant c\, D\Delta/\varepsilon + \mathcal{O}(D^2) + \mathcal{O}(\varepsilon) - \eta_2(t,\nu) \,,$$

where $c > 0$ is some constant. This tells us that

$$\mathbb{P}^{\nu,d(\nu)}(A_1 \cap A_2 \cap A_3) \leqslant \mathbb{P}(A_4) \,,$$

where

$$A_4 = \left\{ \sup_{t\in[\nu,\nu+\Delta]} (\eta_1(t,\nu) + \eta_2(t,\nu)) \;\leqslant\; c\,D\Delta/\varepsilon + \mathcal{O}(D^2) + \mathcal{O}(\varepsilon) \right\} .$$

By Lemma 3.4.8, this has zero probability as $\sigma \downarrow 0$.

2) The inequality $z_1(t) > d_-(t)$ is the same as $z_1^0(t) > d_-(t) - R_1(t)$. As we have conditioned on $B_M^0(\nu) \cap E^c$, there is a $\mathcal{O}(D^2)$ term, independent of time, such that for all $t \in [\nu, (\nu+\Delta) \wedge \tau]$,

$$d_-(t) - R_1(t) \;\leqslant\; -\sqrt{2}(b - a - t/3) + \mathcal{O}(D^2) + \mathcal{O}(\varepsilon) + z_2^0(t) \;:=\; d_-^0(t) ,$$

where we may also take the $\mathcal{O}(\varepsilon)$ term independent of time. Therefore, if $z_1^0(t) > d_-^0(t)$ for all $t \in [\nu, \nu+\Delta]$, then $z_1(t) > d_-(t)$ for all $t \in [\nu, (\nu+\Delta) \wedge \tau]$. This first inequality holds if, for all $t \in [\nu, \nu+\Delta]$, $d_-^0(t) < 0$ and $z_1^0(t) \;\geqslant\; 0$, which we now show to be the case.

Firstly, we show that $\lim_{\sigma\downarrow 0} \mathbb{Q}(A_5) = 0$, where

$$A_5 = \{d_-^0(t) \;\geqslant\; 0 \text{ for some } t \in [\nu, \nu+\Delta]\} .$$

By Lemma 3.4.3 and the discussion thereafter, it is sufficient to consider $\mathbb{Q}(A_5 \cap F^c)$ when taking the limit $\sigma \downarrow 0$. Clearly, $d_-^0(t) = -d(t) + \mathcal{O}(D^2) + \mathcal{O}(\varepsilon)$. We then have for all $t \in [\nu, \nu+\Delta]$,

$$\begin{aligned}
d_-^0(t) &= d_-^0(\nu) + \mathcal{O}(\Delta) + z_2^0(t) - z_2^0(\nu) \\
&= -d(\nu) + \mathcal{O}(D^2) + \mathcal{O}(\varepsilon) - z_2^0(\nu)(1 - \mathrm{e}^{3\alpha(t,\nu)/\varepsilon}) + \eta_2(t,\nu) \\
&= -d(\nu) + \mathcal{O}(D^2) + \mathcal{O}(\varepsilon) + \mathcal{O}(D\Delta/\varepsilon) + \eta_2(t,\nu) \\
&\leqslant\; -\sigma^2/D + \mathcal{O}(D^2) + \mathcal{O}(\varepsilon) + \mathcal{O}(D\Delta/\varepsilon) + \eta_2(t,\nu) \\
&\leqslant\; -\sigma^2/(2D) + \eta_2(t,\nu) ,
\end{aligned}$$

where the final inequality holds by taking σ sufficiently small and the penultimate inequality uses the definition of F^c. Therefore, $\mathbb{Q}(A_5) \to 0$ if

$$\lim_{\sigma\downarrow 0} \mathbb{P} \left\{ \sup_{t\in[\nu,\nu+\Delta]} \eta_2(t,\nu) \;\geqslant\; \sigma^2/(2D) \right\} = 0 ,$$

which holds by Lemma 3.2.9.

Now we will show that $\lim_{\sigma\downarrow 0}\mathbb{Q}^{\nu,d(\nu)}(A_6[\nu,\nu+\Delta])=0$, where

$$A_6[\nu,\nu+\Delta]=\left\{\inf_{t\in[\nu,\nu+\Delta]} z_1^0(t)<0\right\} .$$

By Lemma 3.4.3 and (3.54), it is enough to show that $\mathbb{P}^{\nu,d(\nu)}(A_6[\nu,\nu+\Delta]\cap F^c)\to 0$. Then

$$\mathbb{P}^{\nu,d(\nu)}(A_6[\nu,\nu+\Delta]\cap F^c)\ \leqslant \sup_{y\,\geqslant\,\sigma^2/D,\,\nu_0\,\geqslant\,0}\mathbb{P}^{\nu_0,y}(A_6[\nu_0,\nu_0+\Delta]) .$$

As the distribution of $z_1^0(t)$, when started from zero, is symmetric about zero, we have by the reflection principle (see Lemma B.4.1 from [BG06]) that

$$\begin{aligned}\mathbb{P}^{\nu_0,y}(A_6[\nu_0,\nu_0+\Delta]) &= 2\mathbb{P}^{\nu_0,y}\big(z_1^0(\nu_0+\Delta)<0\big)\\ &= 2\Phi\left(-\frac{\mathrm{e}^{\alpha(\nu_0+\Delta,\nu_0)/\varepsilon}\,y}{\sqrt{\mathrm{Var}(\eta_1(\nu_0+\Delta))}}\right) .\end{aligned}$$

Then for $y\ \geqslant\ \sigma^2/D$ and σ sufficiently small,

$$y\,\mathrm{e}^{\alpha(\nu_0+\Delta,\nu_0)/\varepsilon}\ \geqslant\ \frac{\sigma^2}{D}(1-\mathcal{O}(\Delta/\varepsilon))\ \geqslant\ \frac{\sigma^2}{2D} ,$$

while $\mathrm{Var}(\eta_1(\nu_0+\Delta,\nu_0))\asymp\sigma^2\Delta/\varepsilon$ uniformly for all $\nu_0\ \geqslant\ 0$. Therefore,

$$\frac{y\,\mathrm{e}^{-\alpha(\nu_0+\Delta,\nu_0)/\varepsilon}}{\sqrt{\mathrm{Var}(\eta(\nu_0+\Delta,\nu_0))}}\ \geqslant\ \frac{\sigma}{2D}\sqrt{\frac{\varepsilon}{\Delta}} .$$

The right-hand side tends to infinity and, therefore, the probability tends to zero as $\sigma\downarrow 0$.

3) As we have conditioned on E^c, it follows by definition of $d_M^0(t)$ that if $z_2^0(t)\ \geqslant\ d_M^0(t)$ for all $t\in[\nu,\nu+\Delta]$, then $z_2(t)>d_M(t)$ for all $t\in[\nu,(\nu+\Delta)\wedge\tau]$. Clearly, $d_M^0(t)\ \leqslant\ d_M^0(\nu)+\mathcal{O}(\Delta)$ for all $t\in[\nu,\nu+\Delta]$. So letting

$$A_7=\{z_2^0(t)<d_M^0(\nu)+\mathcal{O}(\Delta)\text{ for some }t\in[\nu,\nu+\Delta]\} ,$$

we would like to show that $\mathbb{Q}(A_7)\to 0$. We again use Lemma 3.4.3 and consider $\mathbb{Q}(A_7\cap F^c)$. By definition of $\mathbb{Q}$, it also follows that

$$\mathbb{Q}(A_7\cap F^c)=\mathbb{Q}(A_7\cap F^c,\,z_2^0(\nu)<D) .$$

Again, it is enough by assumption (3.52) to consider just $\mathbb{P}$. Let

$$A_8^{\nu_0,y_1,y_2} = \{\eta_2(t,\nu_0) \leqslant y_1 + \mathcal{O}(\Delta) - y_2\,\mathrm{e}^{3\alpha(t,\nu_0)/\varepsilon} \text{ for some } t \in [\nu_0,\nu_0+\Delta]\}\ ,$$

where ν_0 represents a fixed initial time, and y_1, y_2 are constants representing the values of $d_M^0(\nu)$ and $z_2^0(\nu)$, respectively. Then we have

$$\mathbb{P}(A_7 \cap F^c,\, z_2^0(\nu) < D) \leqslant \sup_{\substack{y_1+\sigma^2/D \leqslant y_2 \leqslant D,\\ \nu_0 \geqslant 0}} \mathbb{P}(A_8^{\nu_0,y_1,y_2})\ .$$

For any $\nu_0 \geqslant 0$, all $t \in [\nu_0,\nu_0+\Delta]$ and $y_1 + \sigma^2/D \leqslant y_2 \leqslant D$, we have

$$y_1 + \mathcal{O}(\Delta) - y_2\,\mathrm{e}^{3\alpha(t,\nu_0)/\varepsilon} \leqslant -\sigma^2/D + \mathcal{O}(D\Delta/\varepsilon) \leqslant -\sigma^2/(2D)\ .$$

Therefore,

$$\sup_{\substack{y_1+\sigma^2/D \leqslant y_2 \leqslant D,\\ \nu_0 \geqslant 0}} \mathbb{P}(A_8^{\nu_0,y_1,y_2}) \leqslant \sup_{\nu_0 \geqslant 0} \mathbb{P}\left\{\inf_{t\in[\nu_0,\nu_0+\Delta]} \eta_2(t,\nu_0) \leqslant -\sigma^2/(2D)\right\}\ .$$

By Lemma 3.2.9, the right-hand side tends to zero as $\sigma \downarrow 0$.

□

3.4.5 Auxillary Results and Proofs

In the previous section, we used the following proposition to show that conditioning on $B_M^0(\nu) \cap E^c$ is equivalent to conditioning on $B_M(\nu) \cap E^c$ in the limit as $\sigma \downarrow 0$.

Proposition 3.4.5. *Given $\varepsilon = \varepsilon(\sigma)$ satisfying (3.44), let $D = D(\sigma)$ be chosen according to Proposition 3.2.6 and assume that (3.52) holds. Then*

$$\lim_{\sigma\downarrow 0} \mathbb{P}(B_M^0(\nu)^c \cap B_M(\nu) \cap E^c) = 0\ . \tag{3.58}$$

Proof. As we have seen, $B_M(\nu) \cap E^c \subset \{\nu \leqslant \tau\}$. On E^c and for $t \leqslant \tau$, $z_2(t) > d_M(t)$ implies that $z_2^0(t) > d_M^0(t) - \mathcal{O}(\varepsilon) - \mathcal{O}(D^2)$ for some positive $\mathcal{O}(\cdot)$ terms independent of time. Therefore, $B_M(\nu) \cap E^c \subset A_1 \cap E^c$, where

$$A_1 = \{z_2^0(t) > d_M^0(t) - \mathcal{O}(\varepsilon) - \mathcal{O}(D^2) \text{ for all } 0 \leqslant t \leqslant \nu\}\ .$$

So the probability in (3.58) is bounded above by $\mathbb{P}(B_M^0(\nu)^c \cap A_1 \cap E^c)$. We now show that $\mathbb{P}(A_1 \cap E^c \,|\, B_M^0(\nu)^c) \to 0$, where $\liminf_{\sigma\downarrow 0} \mathbb{P}(B_M^0(\nu)^c) > 0$ by (3.52).

Let

$$\tilde{\nu} = \inf\{t \geqslant 0 : z_2^0(t) \leqslant d_M^0(t)\} , \tag{3.59}$$

and note that

$$B_M^0(\nu)^c = \{\tilde{\nu} \leqslant \nu\} = \{|z_1^0(t)| < d(t) \text{ for all } 0 \leqslant t < \tilde{\nu}\} . \tag{3.60}$$

Define

$$\tilde{\nu}_+ := 3(b-a) - \sigma^2/D \tag{3.61}$$

and let

$$\mathcal{E}^c = E^c \cap \{|z_1^0(\tilde{\nu})| \leqslant d(\tilde{\nu}) - \tilde{c}\,\sigma^2/D\} \cap \{\tilde{\nu} \leqslant \tilde{\nu}_+\} ,$$

where $\tilde{c} > 0$ is a constant chosen small enough so that $d(\tilde{\nu}) - \tilde{c}\,\sigma^2/D > 0$ whenever $\tilde{\nu} \leqslant \tilde{\nu}_+$. By Lemma 3.4.6 below, it is sufficient to consider just $\mathbb{P}(A_1 \cap \mathcal{E}^c \,|\, B_M^0(\nu)^c)$ when taking the limit as $\sigma \downarrow 0$.

In a similar way to Proposition 3.4.4, we now show there is a small interval $[\tilde{\nu}, \tilde{\nu}+\Delta]$ such that $\nu \notin [\tilde{\nu}, \tilde{\nu}+\Delta]$ and in which $z_2^0(t)$ hits $d_M^0(t) - \mathcal{O}(\varepsilon) - \mathcal{O}(D^2)$. Choose $\Delta = \Delta(\sigma) > 0$ so that

$$D/\sigma \ll \sqrt{\varepsilon/\Delta} \ll \min(\sqrt{D/\varepsilon}, 1/\sqrt{D}) .$$

For $t \geqslant \tilde{\nu}$, we have $z_1^0(t) = z_1^0(\tilde{\nu})\,\mathrm{e}^{\alpha(t,\tilde{\nu})/\varepsilon} + \eta_1(t,\tilde{\nu})$ and $z_2^0(t) = d_M^0(\tilde{\nu})\,\mathrm{e}^{3\alpha(t,\tilde{\nu})/\varepsilon} + \eta_2(t,\tilde{\nu})$, where the processes η_i were introduced in (3.56) and (3.57). Let

$$\begin{aligned} A_2 &= \{z_2^0(t) > d_M^0(t) - \mathcal{O}(\varepsilon) - \mathcal{O}(D^2) \text{ for all } t \in [\tilde{\nu}, \tilde{\nu}+\Delta]\} \\ &= \{\eta_2(t,\tilde{\nu}) > d_M^0(t) - d_M^0(\tilde{\nu})\,\mathrm{e}^{3\alpha(t,\tilde{\nu})/\varepsilon} - \mathcal{O}(\varepsilon) - \mathcal{O}(D^2) \text{ for all } t \in [\tilde{\nu}, \tilde{\nu}+\Delta]\} . \end{aligned}$$

Then

$$\mathbb{P}(A_1 \cap \mathcal{E}^c, \, \tilde{\nu}+\Delta < \nu \,|\, B_M^0(\nu)^c) \leqslant \mathbb{P}(A_2 \cap \mathcal{E}^c \,|\, B_M^0(\nu)^c) \leqslant p_0\,\mathbb{P}(A_2 \cap \mathcal{E}^c) ,$$

where we have used (3.52) for the final inequality. We will show that the right-hand side tends to zero. Note that for $t \geqslant \tilde{\nu}$, $d_M^0(t) \geqslant d_M^0(\tilde{\nu})$, and on $\mathcal{E}^c$, we have by the boundedness of $z_2^0(t)$ that $d_M^0(\tilde{\nu}) \geqslant -D$. Then for all $t \in [\tilde{\nu}, \tilde{\nu}+\Delta]$,

$$\begin{aligned} d_M^0(t) - d_M^0(\tilde{\nu})\,\mathrm{e}^{3\alpha(t,\tilde{\nu})/\varepsilon} &\geqslant d_M^0(\tilde{\nu})(1 - \mathrm{e}^{3\alpha(t,\tilde{\nu})/\varepsilon}) \\ &\geqslant -c\,D\Delta/\varepsilon , \end{aligned}$$

where $c > 0$ is a constant. Letting

$$A_3 = \left\{ \inf_{t\in[\tilde\nu,\tilde\nu+\Delta]} \eta_2(t,\tilde\nu) > -c\,D\Delta/\varepsilon - \mathcal{O}(\varepsilon) - \mathcal{O}(D^2) \right\} ,$$

we have $\mathbb{P}(A_2 \cap \mathcal{E}^c) \leqslant \mathbb{P}(A_3)$. By Lemma 3.2.9, $\mathbb{P}(A_3) \to 0$ as $\sigma \downarrow 0$.

Now we must show that $\mathbb{P}(\mathcal{E}^c,\, \tilde\nu + \Delta \geqslant \nu \,|\, B_M^0(\nu)^c) \to 0$. In other words, show that $z_1^0(t)$ does not hit $\pm d(t)$ in the interval $[\tilde\nu, \tilde\nu + \Delta]$. We will just show that z_1^0 does not hit $+d(t)$, the other case being similar. Let

$$\begin{aligned} A_4 &= \{z_1^0(t) \geqslant d(t) \text{ for some } t \in [\tilde\nu, \tilde\nu+\Delta]\} \\ &= \{\eta_1(t,\tilde\nu) \geqslant d(t) - z_1^0(\tilde\nu)\,\mathrm{e}^{\alpha(t,\tilde\nu)/\varepsilon} \text{ for some } t \in [\tilde\nu, \tilde\nu+\Delta]\} . \end{aligned}$$

On $\mathcal{E}^c$, we have for $t \in [\tilde\nu, \tilde\nu+\Delta]$ that

$$\begin{aligned} d(t) &= d(\tilde\nu) - \mathcal{O}(\Delta) + d_M^0(\tilde\nu)(1 - \mathrm{e}^{3\alpha(t,\tilde\nu)/\varepsilon}) - \eta_2(t,\tilde\nu) \\ &\geqslant d(\tilde\nu) - \mathcal{O}(D\Delta/\varepsilon) - \eta_2(t,\tilde\nu) , \end{aligned}$$

while by the definition of $\mathcal{E}^c$,

$$z_1^0(\tilde\nu)\,\mathrm{e}^{\alpha(t,\tilde\nu)/\varepsilon} \leqslant d(\tilde\nu) - \tilde c\,\sigma^2/D .$$

Combining these two inequalities gives

$$d(t) - z_1^0(\tilde\nu)\,\mathrm{e}^{\alpha(t,\tilde\nu)/\varepsilon} \geqslant \tilde c\,\sigma^2/D - \mathcal{O}(D\Delta/\varepsilon) - \eta_2(t,\tilde\nu) .$$

Then $A_4 \cap \mathcal{E}^c \subset A_5 \cap \mathcal{E}^c \subset A_5$, where

$$A_5 = \left\{ \sup_{t\in[\tilde\nu,\tilde\nu+\Delta]} (\eta_1(t,\tilde\nu) + \eta_2(t,\tilde\nu)) \geqslant \tilde c\,\sigma^2/D - \mathcal{O}(D\Delta/\varepsilon) \right\} .$$

The event A_5 implies that either $\eta_1(t,\tilde\nu)$ or $\eta_2(t,\tilde\nu)$ exceeds $\tilde c\,\sigma^2/(2D) - \mathcal{O}(D\Delta/\varepsilon)$. For $i \in \{1,2\}$, we have by Lemma 3.2.9 that

$$\lim_{\sigma\downarrow 0} \mathbb{P}\left\{ \sup_{t\in[\tilde\nu,\tilde\nu+\Delta]} \eta_i(t,\tilde\nu) \geqslant \tilde c\,\sigma^2/(2D) - \mathcal{O}(D\Delta/\varepsilon) \right\} = 0 .$$

□

The following lemma was used in the above proposition.

Lemma 3.4.6. *Let $\tilde{\nu}$ and $\tilde{\nu}_+$ be defined as in (3.59) and (3.61), respectively. Under the conditions of Proposition 3.4.5, there exist constants $c_1, c_2 > 0$ such that for σ sufficiently small,*

$$\mathbb{P}(\tilde{\nu} > \tilde{\nu}_+,\, E^c \,|\, B_M^0(\nu)^c) \leqslant \frac{c_1\sigma}{D}$$

and

$$\mathbb{P}\left(|z_1^0(\tilde{\nu})| \geqslant d(\tilde{\nu}) - \tilde{c}\,\sigma^2/D,\, \tilde{\nu} \leqslant \tilde{\nu}_+,\, E^c \,|\, B_M^0(\nu)^c\right) \leqslant \frac{c_2\sigma}{D}\,.$$

where $\tilde{c} > 0$ is any constant such that $d(\tilde{\nu}) - \tilde{c}\,\sigma^2/D > 0$ whenever $\tilde{\nu} \leqslant \tilde{\nu}_+$.

Proof. For $t \leqslant \nu$, we have $z_2^0(t) \leqslant h(t)$ where

$$h(t) = \sqrt{2}(b - a - t/3) + \mathcal{O}(D^2) + \mathcal{O}(\varepsilon)$$

is the curve such that $d(t) = 0$ when $z_2^0(t) = h(t)$. Then

$$\begin{aligned}
\mathbb{P}(\tilde{\nu} > \tilde{\nu}_+,\, E^c \,|\, B_M^0(\nu)^c) &\leqslant \mathbb{P}(\tilde{\nu} > \tilde{\nu}_+ \,|\, B_M^0(\nu)^c) \\
&= \mathbb{P}(\nu \geqslant \tilde{\nu} > \tilde{\nu}_+ \,|\, B_M^0(\nu)^c) \\
&\leqslant \mathbb{P}(z_2^0(\tilde{\nu}_+) > d_M^0(\tilde{\nu}_+),\, \tilde{\nu}_+ \leqslant \nu \,|\, B_M^0(\nu)^c) \\
&\leqslant \mathbb{P}(d_M^0(\tilde{\nu}_+) < z_2^0(\tilde{\nu}_+) \leqslant h(\tilde{\nu}_+) \,|\, B_M^0(\nu)^c) \\
&\leqslant p_0\, \mathbb{P}(d_M^0(\tilde{\nu}_+) < z_2^0(\tilde{\nu}_+) \leqslant h(\tilde{\nu}_+))\,.
\end{aligned}$$

Evaluating the corresponding Gaussian integral yields

$$\mathbb{P}(d_M^0(\tilde{\nu}_+) < z_2^0(\tilde{\nu}_+) \leqslant h(\tilde{\nu}_+)) \leqslant \frac{c\,\sigma}{D}$$

for some constant $c > 0$ and σ sufficiently small.

Using that $B_M^0(\nu)^c \subset \{|z_1^0(\tilde{\nu})| \leqslant d(\tilde{\nu})\}$, we have

$$\begin{aligned}
&\mathbb{P}\left(|z_1^0(\tilde{\nu})| > d(\tilde{\nu}) - \tilde{c}\,\sigma^2/D,\, \tilde{\nu} \leqslant \tilde{\nu}_+,\, E^c \,|\, B_M^0(\nu)^c\right) \\
&\qquad \leqslant p_0\, \mathbb{P}\left(|z_1^0(\tilde{\nu})| > d(\tilde{\nu}) - \tilde{c}\,\sigma^2/D,\, \tilde{\nu} \leqslant \tilde{\nu}_+,\, E^c,\, |z_1^0(\tilde{\nu})| \leqslant d(\tilde{\nu})\right)\,,
\end{aligned}$$

where

$$d(\tilde{\nu}) = \frac{3\sqrt{2}}{2}(b - a - \tilde{\nu}/3) - \mathcal{O}(\varepsilon) - \mathcal{O}(D^2)\,. \tag{3.62}$$

Note too that on E^c, we have by the boundedness of $z_2^0(t)$ that $\tilde{\nu} \geqslant \tilde{\nu}_-$ for some constant $\tilde{\nu}_- > 0$. Then by the independence of z_1^0 and $\tilde{\nu}$, it follows that

$$\begin{aligned}
&\mathbb{P}\left(|z_1^0(\tilde{\nu})| > d(\tilde{\nu}) - \tilde{c}\,\sigma^2/D,\, \tilde{\nu} \leqslant \tilde{\nu}_+,\, E^c,\, |z_1^0(\tilde{\nu})| \leqslant d(\tilde{\nu})\right) \\
&\qquad \leqslant \sup_{\tilde{\nu}_- \leqslant \nu_0 \leqslant \tilde{\nu}_+} \mathbb{P}\left(d(\nu_0) - \tilde{c}\,\sigma^2/D < |z_1^0(\nu_0)| \leqslant d(\nu_0)\right) ,
\end{aligned}$$

where on the right-hand side, $d(\nu_0)$ means (3.62) evaluated at $\tilde{\nu} = \nu_0$ and $z_1^0(\nu_0)$ is the usual z_1^0 process evaluated at the fixed time ν_0. For $\tilde{\nu}_- \leqslant \nu_0 \leqslant \tilde{\nu}_+$, we have

$$\begin{aligned}
&\frac{1}{2}\mathbb{P}\left(d(\nu_0) - \tilde{c}\,\sigma^2/D < |z_1^0(\nu_0)| \leqslant d(\nu_0)\right) \\
&\quad = \Phi\left(\frac{d(\nu_0)}{\sqrt{\mathrm{Var}(z_1^0(\nu_0))}}\right) - \Phi\left(\frac{d(\nu_0) - \tilde{c}\,\sigma^2/D}{\sqrt{\mathrm{Var}(z_1^0(\nu_0))}}\right) \\
&\quad \leqslant \frac{\tilde{c}\,\sigma^2/D}{\sqrt{2\pi \mathrm{Var}(z_1^0(\nu_0))}} \\
&\quad \leqslant \frac{c_2\sigma}{D} ,
\end{aligned}$$

where in the final line we used that $\mathrm{Var}(z_1^0(\nu_0)) \asymp \sigma^2$ for such ν_0. $\square$

We now give the proof of Lemma 3.4.3, which is restated for convenience.

Lemma 3.4.7 (Lemma 3.4.3 restated)**.** *Given $\varepsilon = \varepsilon(\sigma)$ satisfying (3.44), let $D = D(\sigma)$ be chosen according to Proposition 3.2.6. Let ν and $B_M^0(\nu)$ be defined as in (3.50) and (3.51), respectively, and assume (3.52). Then there exists $c > 0$ such that for sufficiently small σ,*

$$\mathbb{Q}(F) \leqslant \frac{c\,\sigma}{D} .$$

Proof. Write $F = F_1 \cup F_2$, where $F_1 = \{d(\nu) < \sigma^2/D\}$ and $F_2 = \{z_2^0(\nu) < d_M^0(\nu) + \sigma^2/D\}$. We will deal with $\mathbb{Q}(F_1)$ and $\mathbb{Q}(F_2)$ separately.

Recall that $B_M^0(\nu) \cap E^c \subset \{\nu \leqslant \tau\}$ and that $\tau \leqslant 3(b-a)$. Therefore,

$$\begin{aligned}
\mathbb{Q}(F_1) &= \mathbb{Q}(F_1,\, \nu \leqslant 3(b-a)) \\
&\leqslant p_0\,\mathbb{P}(F_1,\, \nu \leqslant 3(b-a),\, E^c) .
\end{aligned}$$

Denote the event on the right-hand side by A_1. Let $\nu^* = \inf\{t \geqslant 0 : d(t) < \sigma/D^2\}$, which depends on z_2^0 but is independent of z_1^0. Since $F_1 \subset \{|z_1^0(\nu^*)| < \sigma^2/D,\, \nu > \nu^*\}$, we have

$$F_1 \cap \{\nu \leqslant 3(b-a)\} \subset \{|z_1^0(\nu^*)| < \sigma^2/D,\, \nu^* < 3(b-a)\} .$$

Furthermore, on E^c we have for σ sufficiently small that $\nu^* \geqslant \nu_-^* > 0$, where ν_-^* is a constant independent of σ. Therefore,

$$\begin{aligned}
\mathbb{P}(A_1) &\leqslant \mathbb{P}\left(|z_1^0(\nu^*)| < \sigma^2/D,\, \nu_-^* \leqslant \nu^* \leqslant 3(b-a)\right) \\
&\leqslant \sup_{\nu_-^* \leqslant \nu_0 \leqslant 3(b-a)} \mathbb{P}(|z_1^0(\nu_0)| < \sigma^2/D) \\
&\leqslant \sup_{\nu_-^* \leqslant \nu_0 \leqslant 3(b-a)} \left(1 - 2\Phi\left(-\frac{\sigma^2/D}{\sqrt{\operatorname{Var}(z_1^0(\nu_0))}}\right)\right) .
\end{aligned}$$

We know $\operatorname{Var}(z_1^0(\nu_0)) \asymp \sigma^2$ for such ν_0, and so using the simple estimate $1/2 - \Phi(-x) \leqslant x$, valid for all $x \geqslant 0$, tells us that $\mathbb{P}(A_1) \leqslant c\,\sigma/D$ for some constant $c > 0$.

Now we turn to $\mathbb{Q}(F_2)$. First note that given $B_M^0(\nu)$, we have for all $t \leqslant \nu$ that

$$\begin{aligned}
d(t) &= \sqrt{2}(b-a-t/3) - z_2^0(t) - \mathcal{O}(D^2) - \mathcal{O}(\varepsilon) \\
&< \sqrt{2}(b-a-t/3) - d_M^0(t) - \mathcal{O}(D^2) - \mathcal{O}(\varepsilon) \\
&< \frac{3\sqrt{2}}{2}(b-a-t/3) .
\end{aligned}$$

Therefore, $d(t) - 2\sigma^2/D < g(t)$, where

$$g(t) = \frac{3\sqrt{2}}{2}(b-a-t/3) - 2\sigma^2/D .$$

The lines $d(t)$ and $g(t)$ coincide when $z_2^0(t) = z^*(t)$, where

$$z^*(t) = -\frac{1}{\sqrt{2}}(b-a-t/3) + 2\sigma^2/D - \mathcal{O}(D^2) - \mathcal{O}(\varepsilon) .$$

It follows that if $z_2^0(t) \leqslant z^*(t)$, then $d(t) \geqslant g(t)$, while if $z_2^0(t) > z^*(t)$ then $d(t) < g(t)$. Now let

$$\hat{\nu} = \inf\{t \geqslant 0 : |z_1^0(t)| \geqslant g(t)\} ,$$

and note that $\hat{\nu}$ is independent of z_2^0. It is clear that $\hat{\nu}$ is bounded above by the deterministic time $\hat{\nu}_+ = \inf\{t \geqslant 0 : g(t) = 0\} = 3(b-a) - 2\sqrt{2}\,\sigma^2/D$. If $z_2^0(\nu) \leqslant z^*(\nu)$, then we must have $\hat{\nu} \leqslant \nu$ and $z_2^0(\hat{\nu}) \leqslant z^*(\hat{\nu})$. We also know that on E^c, $\hat{\nu} \geqslant \hat{\nu}_- > 0$ for all σ sufficiently small, where $\hat{\nu}_-$ is a constant independent of σ.

For σ sufficiently small, we have

$$z^*(t) \geqslant d_M^0(t) + \sigma^2/D .$$

Therefore,

$$\begin{aligned}
\mathbb{Q}(F_2) &\leqslant \mathbb{Q}(z_2^0(\nu) \leqslant z^*(\nu)) \\
&\leqslant \mathbb{Q}(z_2^0(\hat\nu) \leqslant z^*(\hat\nu),\, \hat\nu \leqslant \nu) \\
&\leqslant p_0\, \mathbb{P}(z_2^0(\hat\nu) \leqslant z^*(\hat\nu),\, \hat\nu \leqslant \nu,\, B_M^0(\nu),\, E^c)\;.
\end{aligned}$$

Call the event on the right-hand side above A_2. Now observe that

$$\{\hat\nu \leqslant \nu\} \cap B_M^0(\nu) \subset \{z_2^0(\hat\nu) \geqslant d_M^0(\hat\nu)\}\;.$$

Therefore,

$$\begin{aligned}
\mathbb{P}(A_2) &\leqslant \mathbb{P}\left(d_M^0(\hat\nu) \leqslant z_2^0(\hat\nu) \leqslant z^*(\hat\nu),\, \hat\nu_- \leqslant \hat\nu \leqslant \hat\nu_+\right) \\
&\leqslant \sup_{\hat\nu_- \leqslant \nu_0 \leqslant \hat\nu_+} \mathbb{P}\left(d_M^0(\nu_0) \leqslant z_2^0(\nu_0) \leqslant z^*(\nu_0)\right)\;,
\end{aligned}$$

where in the second line, $z_2^0(\nu_0)$ is the usual z_2^0 process at a given time ν_0, which follows by the independence of $\hat\nu$ and z_2^0. We then have for all $\hat\nu_- \leqslant \nu_0 \leqslant \hat\nu_+$,

$$\begin{aligned}
\mathbb{P}(d_M^0(\nu_0) \leqslant z_2^0(\nu_0) \leqslant z^*(\nu_0)) &= \Phi\left(\frac{z^*(\nu_0)}{\sqrt{\mathrm{Var}(z_2^0(\nu_0))}}\right) - \Phi\left(\frac{d_M^0(\nu_0)}{\sqrt{\mathrm{Var}(z_2^0(\nu_0))}}\right) \\
&\leqslant \frac{z^*(\nu_0) - d_M^0(\nu_0)}{\sqrt{2\pi\mathrm{Var}(z_2^0(\nu_0))}} \\
&\leqslant \frac{c}{\sigma}(z^*(\nu_0) - d_M^0(\nu_0))\;,
\end{aligned}$$

where in the final line we used that $\mathrm{Var}(z_2^0(\nu_0)) \asymp \sigma^2$ for such ν_0. The result now follows from the definitions of $z^*(\nu_0)$ and $d_M^0(\nu_0)$. □

The following lemma was used in the proof of Proposition 3.4.4(1).

Lemma 3.4.8. *For $t_0 \geqslant 0$, define the processes $\eta_1(t,t_0)$, $\eta_2(t,t_0)$ as in (3.56) and (3.57). Suppose that $L = L(\sigma)$, $\varepsilon = \varepsilon(\sigma)$ and $\Delta = \Delta(\sigma)$ satisfy*

$$L \ll \sigma\sqrt{\Delta/\varepsilon}\;, \qquad \Delta|\ln\sigma| \ll \varepsilon\;.$$

Then

$$\lim_{\sigma\downarrow 0} \mathbb{P}\left(\sup_{t_0 \leqslant t \leqslant t_0+\Delta} (\eta_1(t,t_0) + \eta_2(t,t_0)) > L\right) = 1\;. \tag{3.63}$$

Proof. Letting $\tilde{W}_i(s) = W_i(t_0+s) - W_i(t_0)$, for $s \in [0, \Delta]$, we have

$$\eta_i(t,t_0) = \frac{\sigma}{\sqrt{\varepsilon}} \int_{t_0}^{t} \mathrm{e}^{\alpha(t,s)/\varepsilon} \, \mathrm{d}W_i(s) = \frac{\sigma}{\sqrt{\varepsilon}} \int_{0}^{t-t_0} \mathrm{e}^{\alpha(t-t_0,s)/\varepsilon} \, \mathrm{d}\tilde{W}_i(s) \;,$$

where $t - t_0 \in [0, \Delta]$, so that we can consider just the case $t_0 = 0$ in (3.63). We will write $\eta_i(t) = \eta_i(t, 0)$.

Let A be the event on the left-hand side of (3.63) with $t_0 = 0$. Using Ito's integration by parts formula, we have

$$\eta_1(t) + \eta_2(t) = \frac{\sigma}{\sqrt{\varepsilon}}(W_1(t) + W_2(t)) - C(t) \;,$$

where

$$C(t) = \frac{\sigma}{\varepsilon^{3/2}} \int_0^t u(s)(\,\mathrm{e}^{\alpha(t,s)/\varepsilon} W_1(s) + 3\,\mathrm{e}^{3\alpha(t,s)/\varepsilon} W_2(s))\,\mathrm{d}s \;.$$

Let $\mathcal{W} = \mathcal{W}(\Delta)$ be the set

$$\mathcal{W}(\Delta) = \bigcup_{i=1,2} \left\{ \sup_{0\,\leqslant\, t\,\leqslant\,\Delta} |W_i(t)| > \Delta^{1/2} |\ln\sigma| \right\} \;.$$

It is straightforward to see that $\mathbb{P}(\mathcal{W}(\Delta)) \to 0$. Indeed, by the reflection principle,

$$\begin{aligned} \mathbb{P}\left(\sup_{0\,\leqslant\, t\,\leqslant\,\Delta} |W_i(t)| > \Delta^{1/2}|\ln\sigma| \right) &= 2\,\mathbb{P}\left(|W_i(\Delta)| > \Delta^{1/2}|\ln\sigma| \right) \\ &= 4\Phi\left(-|\ln\sigma|\right) \;. \end{aligned}$$

So we only need to consider $A \cap \mathcal{W}^c$. On the event $\mathcal{W}^c$,

$$\sup_{0\,\leqslant\, t\,\leqslant\,\Delta} |C(t)| \;\leqslant\; \frac{C\,\sigma}{\varepsilon^{3/2}} \Delta^{3/2} |\ln\sigma|$$

for some constant $C > 0$ so that

$$\eta_1(t) + \eta_2(t) > \frac{\sigma}{\sqrt{\varepsilon}}(W_1(t) + W_2(t)) - \frac{C\,\sigma}{\varepsilon^{3/2}} \Delta^{3/2} |\ln\sigma| \;.$$

Then $A \cap \mathcal{W}^c \supset B \cap \mathcal{W}^c$, where

$$B = \left\{ \sup_{0\,\leqslant\, t\,\leqslant\,\Delta} \frac{\sigma}{\sqrt{\varepsilon}}(W_1(t) + W_2(t)) > L + \frac{C\,\sigma}{\varepsilon^{3/2}} \Delta^{3/2} |\ln\sigma| \right\}$$

and so

$$\mathbb{P}(A \cap \mathcal{W}^c) \;\geqslant\; \mathbb{P}(B \cap \mathcal{W}^c) \;\geqslant\; \mathbb{P}(B) - \mathbb{P}(\mathcal{W}) \;.$$

We just need to check that $\mathbb{P}(B) \to 1$. We know that $W_1(t) + W_2(t)$ has the same law as $\sqrt{2}W(t)$, where W is a standard Brownian motion starting from zero, in which case we can use the reflection principle. Indeed, we have

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}\left(\sup_{0 \leqslant t \leqslant \Delta} \frac{\sigma\sqrt{2}}{\sqrt{\varepsilon}} W(t) > L + \frac{C\,\sigma}{\varepsilon^{3/2}}\Delta^{3/2}|\ln\sigma|\right) \\
&= 2\,\mathbb{P}\left(W(\Delta) > \frac{\sqrt{\varepsilon}}{\sigma\sqrt{2}}(L + \frac{C\,\sigma}{\varepsilon^{3/2}}\Delta^{3/2}|\ln\sigma|)\right) \\
&= 2\Phi\left(-\sqrt{\frac{\varepsilon}{2\Delta}}\left(L/\sigma + \frac{C}{\varepsilon^{3/2}}\Delta^{3/2}|\ln\sigma|\right)\right) \\
&\geqslant 1 - \sqrt{\frac{2\varepsilon}{\Delta}}\left(L/\sigma + \frac{C}{\varepsilon^{3/2}}\Delta^{3/2}|\ln\sigma|\right)
\end{aligned}$$

and the right-hand side tends to one as $\sigma \downarrow 0$. □

3.5 $N + 1$ Breakable Bonds

We now discuss how the methods used so far in this chapter may be extended to deal with chains of $N + 1$ breakable bonds. In principle, the same strategy should work and some parts of the argument generalise easily. The main difficulty is that the domain of the process is much more complicated. We shall now outline these points in more detail.

Consider a chain with $N + 2$ particles, $q_L < q_1 < \ldots < q_N < q_R$, interacting with each other via the same pairwise potential $\tilde{U}$ as given in (3.1). As before, $q_L \equiv 0$, but now $q_R(s) = (N + 1)a + \varepsilon s$. Initially, only nearest neighbours are interacting. Although it is possible for next-to-nearest neighbour interactions to occur, we can again assume a certain boundedness property of solutions such that we do not need to consider this situation. After making the time change $t = \varepsilon s$, the process $q(t) = (q_1(t), \ldots, q_N(t))$ moves according to the SDE

$$\mathrm{d}q_i(t) = -\frac{1}{\varepsilon}\frac{\partial H}{\partial q_i}(q(t), t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_i(t), \quad q_i(0) = ia\ ,$$

for $1 \leqslant i \leqslant N$, where $W(t) = (W_1(t), \ldots, W_N(t))$ is a standard N-dimensional Brownian motion and H is the time-dependent potential energy of the chain configuration $(0, q_1, \ldots, q_N, (N + 1)a + t)$, given by

$$H(q, t) = U(q_1) + U(q_2 - q_1) + \ldots + U(q_N - q_{N-1}) + U((N + 1)a + t - q_N)\ .$$

Note that for our purposes we can again work in terms of U instead of $\tilde{U}$. Following the same steps as in the previous cases, we find the deterministic solution $q^{\det}(t)$, obtained when $\sigma = 0$, is close to $q^*(t)$, given by

$$q^*(t) = (a + t/(N+1), 2(a + t/(N+1)), \dots, N(a + t/(N+1))).$$

In fact, for times $\varepsilon|\ln \varepsilon| \ll t \leqslant (N+1)(b-a)$, we have

$$q_t^{\det} = q^*(t) - \frac{\varepsilon}{(N+1)u(t)} A^{-1} \begin{pmatrix} 1 \\ 2 \\ \vdots \\ N \end{pmatrix} + \mathcal{O}(\varepsilon^2) \,,$$

where $u(t) = U''(a + t/(N+1))$ and

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} .$$

We again introduce the deviation process $y(t) = q(t) - q^{\det}(t)$, which solves

$$\mathrm{d}y(t) = \frac{1}{\varepsilon} \left[A(t)y(t) + b(y(t), t)\right] \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \mathrm{d}W_t \,,$$

where $A(t) = (a_{ij}(t))_{i,j=1}^N$ is given by

$$a_{ij}(t) = \begin{cases} -U''(q_i^{\det}(t) - q_{i-1}^{\det}(t)) - U''(q_{i+1}^{\det}(t) - q_i^{\det}(t)) & j = i \\ U''(q_{i+1}^{\det}(t) - q_i^{\det}(t)) & j = i-1,\, i+1 \\ 0 & \text{otherwise} \end{cases}$$

and $b(y_t, t)$, containing the remainder terms, satisfies $\|b(y,t)\| \leqslant M_1 \|y\|^2$ for all pairs $(y, t) \in \mathcal{D}$, where $\mathcal{D}$ is the space-time domain in which the chain is unbroken. We can decompose $A(t)$ as

$$A(t) = -u(t)A + \varepsilon A^1(t) \,,$$

with $\|A^1(t)\| \leqslant M_2$ for all $0 \leqslant t \leqslant (N+1)(b-a)$ so that

$$\mathrm{d}y(t) = \frac{1}{\varepsilon}\left[-u(t)Ay(t) + \varepsilon A^1(t)y(t) + b(y(t),t)\right] \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \,.$$

We again change coordinates using the orthogonal transformation P such that $P^{-1}AP = D$ (see Section 2.2), where $D = \mathrm{diag}(\lambda_1,\dots,\lambda_N)$ with

$$\lambda_i = 2\left[1-\cos\left(\frac{\pi i}{N+1}\right)\right] \,.$$

The matrix P has the vectors $v_i = (v_{i1},\dots,v_{iN})$ as columns, where

$$v_{ij} = \sqrt{\frac{2}{N+1}}\sin\left(\frac{\pi ij}{N+1}\right) \,.$$

Letting $z = P^{-1}y$, the equation becomes

$$\mathrm{d}z(t) = \frac{1}{\varepsilon}\left[-u(t)Dz(t) + \varepsilon P^{-1}A^1(t)Pz(t) + P^{-1}b(P^{-1}z(t),t)\right] \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \,. \tag{3.64}$$

As in the case of three breakable bonds, we have omitted P^{-1}, which is an orthogonal matrix, from the noise term. Then the left bond is unbroken if $q_1(t) < b$, which is the same as

$$v_{11}z_1(t) + \ldots + v_{1N}z_N(t) < b - q_1^{\mathrm{det}}(t) \,, \tag{3.65}$$

while the right bond is unbroken if $q_R(t) - q_N(t) < b$, which means

$$v_{N1}z_1(t) + \ldots + v_{NN}z_N(t) > (N+1)a + t - q_N^{\mathrm{det}}(t) - b \,. \tag{3.66}$$

Letting

$$Z_e(t) = \sum_{j\ \mathrm{even}} v_{1j}z_j(t) \,, \quad Z_o(t) = \sum_{j\ \mathrm{odd}} v_{1j}z_j(t) \,,$$

and using that $v_{1i} = (-1)^{i+1}v_{Ni}$, we can write (3.65) and (3.66), respectively, as

$$Z_o(t) < b - q_1^{\mathrm{det}}(t) - Z_e(t) \ =: \ d_+(t)$$

and

$$Z_o(t) > (N+1)a + t - q_N^{\mathrm{det}}(t) - b + Z_e(t) \ =: \ d_-(t) \,.$$

Therefore, the left and right bonds are unbroken as long as $d_-(t) < Z_o(t) < d_+(t)$. This is analogous to the case of three breakable bonds considered in the previous section (see (3.35) and (3.37)).

Solving (3.64), we find

$$z(t) = z^0(t) - \int_0^t \mathrm{e}^{\alpha(t,s)D/\varepsilon} P^{-1} A^1(s) P z(s) \,\mathrm{d}s - \frac{1}{\varepsilon} \int_0^t \mathrm{e}^{\alpha(t,s)D/\varepsilon} P^{-1} B(Pz(s), s) \,\mathrm{d}s \;,$$

where $\alpha(t,s) = -\int_s^t u(w)\,\mathrm{d}w$ satisfies $\alpha(t,s) \asymp -(t-s)$ for all $0 \leqslant s \leqslant t \leqslant N(b-a)$ and $z^0(t) = (z_1^0(t), \dots, z_N^0(t))$ is given by

$$z_i^0(t) = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{\lambda_i \alpha(t,s)/\varepsilon} \,\mathrm{d}W_i(s) \;.$$

By the boundedness of $A^1(t)$, $P^{-1}b(Pz,t)$ and the breaking time τ, there are constants C_1, $C_2 > 0$, depending on N, such that for all $t \leqslant \tau$,

$$\|z(t) - z^0(t)\| \leqslant C_1 \varepsilon \sup_{0 \leqslant s \leqslant t} \|z(s)\| + C_2 \sup_{0 \leqslant s \leqslant t} \|z(s)\|^2 \;. \tag{3.67}$$

Then we obtain a result as in Lemma 3.4.2 and can work with just $z^0(t)$ thereafter. We define a curve analogous to $d(t)$ given in (3.49) and consider the first time ν that $Z_o(t)$ hits $\pm d(t)$. This part of the argument follows along the same lines as the case of three breakable bonds. The main difficulty extending the result is conditioning on the other $N-1$ bonds not breaking before time ν and estimating the distribution of $Z_o(\nu)$. In the previous section, we just had to condition on one bond not breaking, which just meant $z_2(t)$ not hitting $d_M(t)$ (see (3.36)). Now the analogue is $(N-1)$-dimensional. Although the general approach of the previous section should work, it does not appear to generalise immediately and requires a careful inspection of the domain.

Chapter 4

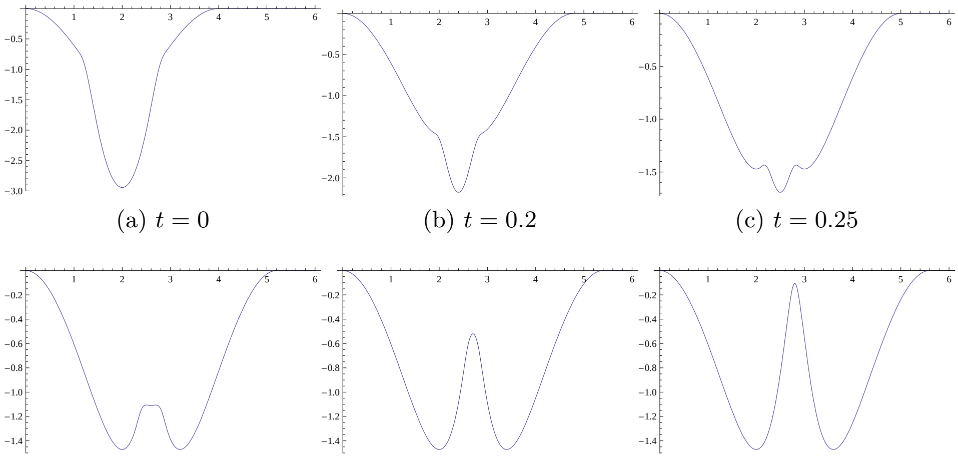
Smooth Potentials and Full Langevin Dynamics

In this chapter, we treat a similar model as in Chapter 3, but with the pairwise potential going smoothly to zero. Furthermore, we do not assume the dynamics to be overdamped from the outset. Indeed, we begin with the full Langevin equation. We then consider two simplifications, including the overdamped equation, before showing that for particle mass small enough, the overdamped approximation is accurate. In Section 4.1, we introduce our model and deduce equation (4.2), which is our main object of study. Our results are then stated in Theorems 4.1.1, 4.1.2, 4.1.3 and 4.1.4. In Sections 4.2, 4.3 and 4.4, we give the proofs.

4.1 The Model and Main Results

Three particles $q_L < q < q_R$ in $\mathbb{R}$ interact with each other via a pairwise potential U. We assume that U is smooth with finite range $b > 0$ and a unique minimum at $0 < a < b$, with $U''(a) > 0$. We also assume that there is a unique $c_0 \in (a, b)$ such that $U''(c_0) = 0$. The particle q_L is fixed at the origin and the position of q_R at time $s \geqslant 0$ is given by $q_R(s) = 2a(1 + \varepsilon s)$, where $\varepsilon > 0$ is a small parameter. We study the behaviour of the middle particle, with position at time s given by q_s. Initially, it has position $q_0 = a$ so that the distance between neighbouring particles is a. The middle particle evolves according to

$$\begin{aligned} \mathrm{d}q_s &= p_s \, \mathrm{d}s \,, \\ \varepsilon^{\beta-1} \mathrm{d}p_s &= -p_s \, \mathrm{d}s - \frac{\partial H}{\partial q}(p_s, q_s, \varepsilon s) \, \mathrm{d}s + \sigma \, \mathrm{d}W_s \,, \end{aligned}$$



(a) $t = 0$ (b) $t = 0.2$ (c) $t = 0.25$

(d) $t = 0.3$ (e) $t = 0.35$ (f) $t = 0.4$

Figure 4.1: $U(q) + U(2a(1+t) - q)$ plotted against q at various times for a smooth potential $U(q) = -q^2\, \mathrm{e}^{-1/(3-q)}$. The central well becomes unstable as t increases.

where W_s is a standard one-dimensional Brownian motion with $W_0 = 0$, $\sigma > 0$ is the noise intensity, $\beta \in \mathbb{R}$ and $H(p, q, \varepsilon s)$ is given by

$$H(p, q, \varepsilon s) = \frac{p^2}{2} + U(q) + U(2a(1+\varepsilon s) - q) \;.$$

Rescaling time as $t = \varepsilon s$, this is the same in law as solving

$$\begin{aligned} \mathrm{d}q_t &= \frac{1}{\varepsilon} p_t \,\mathrm{d}t \;, \\ \varepsilon^{\beta-1} \mathrm{d}p_t &= -\frac{1}{\varepsilon} p_t \,\mathrm{d}t - \frac{1}{\varepsilon}\frac{\partial H}{\partial q}(p_t, q_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \;. \end{aligned} \tag{4.1}$$

We easily check that the configuration of equally spaced particles satisfies $\partial_q H = 0$, $\partial_q^2 H > 0$ until time t_0, where $a(1+t_0) = c_0$. Until this time, it is a stable configuration and so we expect $q_{t_0} \approx a(1+t_0)$. For $t > t_0$, this configuration becomes unstable and new minima emerge (see Fig. 4.1). Therefore, we expect q_t to quickly move away from the chain midpoint and towards one of these newly formed minima. Note that as a function of q, H is symmetric about $q = a(1+t)$, but its time-dependence introduces asymmetry, as we shall see below. Once q_t has approached one of these new minima, we expect it to stay there as the energy barrier to escape becomes higher. The evolution of the chain, therefore, is determined by its behaviour around the bifurcation of H at $t = t_0$, which we shall now consider.

Letting $z_t = a(1+t) - q_t$, we express the term $\partial H/\partial q$ appearing in (4.1) in terms of z. By a Taylor expansion in space, we find

$$\begin{aligned}\frac{\partial H}{\partial q}(p_t, q_t, t) &= U'(q_t) - U'(2a(1+t) - q_t) \\ &= U'(a(1+t) - z_t) - U'(a(1+t) + z_t) \\ &\approx -2U''(a(1+t))z_t - \frac{1}{3}U^{(4)}(a(1+t))z_t^3 \ .\end{aligned}$$

Assuming that there is $0 < T < t_0$ such that for $t \in [t_0 - T, t_0 + T]$, $U^{(4)}(a(1+t))$ is negative and bounded away from zero (see comment below), we have by a Taylor expansion in time,

$$-2U''(a(1+t))z_t - \frac{1}{3}U^{(4)}(a(1+t))z_t^3 \approx 2a(t-t_0)z_t - Cz_t^3 \ .$$

We remark that this assumption about $U^{(4)}(a(1+t))$ should not have much effect. At most, the right-hand side above would have a $+Cz_t^3$ term appearing, but in either case this term is very small for z and $t - t_0$ close to zero, which is where most of our analysis will take place.

Making the space and time transformations $q = a(1+t) - q$, $p = a - p/\varepsilon$ and $t = t - t_0$, as well as normalising constants to one, we arrive at the SDE

$$\begin{aligned}\mathrm{d}q_t &= p_t \,\mathrm{d}t \ , \\ \varepsilon^{\beta}\mathrm{d}p_t &= -p_t \,\mathrm{d}t + \frac{1}{\varepsilon}(tq_t - q_t^3 + \varepsilon)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \ .\end{aligned} \tag{4.2}$$

This will be our main equation for the rest of this chapter. By the above discussion, understanding how its solution behaves will be a good indication of the behaviour of the original chain. Equation (4.2) represents the motion of the particle q in the potential $(1/\varepsilon)V(q,t) := (1/\varepsilon)(-\frac{1}{2}tq^2 + \frac{1}{4}q^4)$ with an additional $+1$ force giving the particle a small bias towards the right. This force comes from pulling the chain (q_L, q, q_R) and corresponds to the fact that in the absence of noise, q does not just stay at the chain midpoint $a(1+t)$, but lags behind by a small amount.

Rephrasing the discussion after (4.1), we see that the function V represents the energy of a given chain configuration. For negative times, the origin, corresponding to equally spaced particles, minimises V. When $t = 0$, V undergoes a symmetric pitchfork bifurcation at the origin. For positive times, V has two minima located at $\pm\sqrt{t}$. For $t > 0$ large enough these minima at $\pm\sqrt{t}$ correspond to the configurations where q is a distance a from q_L or q_R, respectively, and more than b from the other. In terms of (4.2), the aim of this chapter can be roughly stated as to investigate how the speed of pulling determines whether q_t moves towards $+\sqrt{t}$ or $-\sqrt{t}$ as t becomes positive.

In Chapter 3, we defined the chain to break as soon as the distance between q and one of its neighbours exceeded the range of the pairwise potential $\tilde{U}$. Either the left or right bond could break, depending on whether $q_R - q > b$ or $q - q_L > b$, respectively. Now that we are dealing with U smoothly going to zero, we may consider the dynamics beyond this point and alternatively consider the chain to break as soon as the chain configuration reaches a neighbourhood of one of the energy minima that emerges after the bifurcation. In the above formalism, this means the process q_t reaching a neighbourhood of $\pm\sqrt{t}$. To avoid defining what it means for the chain to break, we shall instead give a precise description of the behaviour of q_t that contains more information than either of these possible definitions.

Equation (4.2) in full is not something we can treat. But there are two obvious simplifications: the first is to omit the q_t^3 term in the equation for p, leading to a linear equation that can be solved explicitly; the second is to set $\varepsilon^\beta \mathrm{d}p = 0$ and consider the resulting one-dimensional equation for q, the so-called overdamped approximation in which inertial effects due to mass are neglected. We treat both of these and obtain satisfactory results.

Taking $\sigma = \varepsilon^{\alpha+1/2}$ for $\alpha > -1/2$, we firstly consider the linear SDE

$$\begin{aligned} \mathrm{d}q_t^0 &= p_t^0 \,\mathrm{d}t \;, \\ \varepsilon^\beta \mathrm{d}p_t^0 &= -p_t^0 \,\mathrm{d}t + \frac{1}{\varepsilon}(tq_t^0 + \varepsilon)\,\mathrm{d}t + \varepsilon^\alpha \,\mathrm{d}W_t \;. \end{aligned} \tag{4.3}$$

Having removed σ from this equation, we shall take the limit as $\varepsilon \downarrow 0$. Denoting by $\mathbb{P}^s$ the law of the solution with vanishing initial condition at time $s < 0$, we have the following two theorems:

Theorem 4.1.1. *Let q_t^0 be the solution of (4.3) with any $\beta \in \mathbb{R}$. If $\alpha > 1/4$ then*

$$\lim_{\varepsilon\downarrow 0} \liminf_{s\to-\infty} \mathbb{P}^s\left(\lim_{t\to\infty} q_t^0 = +\infty\right) = 1 \;,$$

while if $\alpha < 1/4$ then

$$\lim_{\varepsilon\downarrow 0} \liminf_{s\to-\infty} \mathbb{P}^s\left(\lim_{t\to\infty} q_t^0 = +\infty\right) = \lim_{\varepsilon\downarrow 0} \liminf_{s\to-\infty} \mathbb{P}^s\left(\lim_{t\to\infty} q_t^0 = -\infty\right) = 1/2 \;.$$

Theorem 4.1.2. *Let q_t^0 be the solution of (4.3) with $\beta > -1$ and let $s < 0$ be the* *starting time. If* $\alpha > (1+\min\{\beta, 0\})/4$ *then*

$$\lim_{\varepsilon\downarrow 0} \mathbb{P}^s \left(\lim_{t\to\infty} q_t^0 = +\infty \right) = 1 \ ,$$

while if $\alpha < (1+\min\{\beta, 0\})/4$ *then*

$$\lim_{\varepsilon\downarrow 0} \mathbb{P}^s \left(\lim_{t\to\infty} q_t^0 = +\infty \right) = \lim_{\varepsilon\downarrow 0} \mathbb{P}^s \left(\lim_{t\to\infty} q_t^0 = -\infty \right) = 1/2 \ .$$

Theorem 4.1.1 shows that the threshold between fast and slow pulling regimes, when the starting time is sent to minus infinity, is given by $\alpha = 1/4$ and is independent of β. However, Theorem 4.1.2 shows that if we start the processes at a finite negative time $s < 0$, then neglecting mass does have an effect, but only for $\beta < 0$. These results have some clear limitations: for $t > 0$, q_t^0 shoots off quickly to $\pm\infty$ as the drift becomes ever more repelling, while the solution of (4.2) is prevented from doing this by the nonlinear term and so is more likely to return to the origin.

Secondly, we neglect the mass term $\varepsilon^\beta \mathrm{d}p$ in (4.2) and consider the one-dimensional overdamped equation

$$\mathrm{d}q_t = \frac{1}{\varepsilon}(tq_t - q_t^3 + \varepsilon)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \ . \tag{4.4}$$

Again letting $\mathbb{P}^s$ denote the law of the solution with vanishing initial condition at time $s < 0$, we have

Theorem 4.1.3. *Let* q_t *solve (4.4). There exist constants* $c_1, \gamma > 0$ *such that if* $t_1 = c_1\sqrt{\varepsilon|\ln\sigma|}$ *then*

1. *(Fast Pulling) for any* $\sigma^{4/3}|\ln\sigma|^{2/3} \ll \varepsilon(\sigma) \ll 1$,

$$\lim_{\sigma\downarrow 0} \liminf_{s\to-\infty} \mathbb{P}^s \left(\inf_{t_1 \leqslant t} \frac{q_t}{\sqrt{t}} > \gamma \right) = 1 \ .$$

2. *(Slow Pulling) for any* $\sigma^2|\ln\sigma|^3 \lesssim \varepsilon(\sigma) \ll \sigma^{4/3}|\ln\sigma|^{-13/6}$,

$$\lim_{\sigma\downarrow 0} \limsup_{s\to-\infty} \mathbb{P}^s \left(\inf_{t_1 \leqslant t} \frac{q_t}{\sqrt{t}} > \gamma \right) = 1/2$$

and

$$\lim_{\sigma\downarrow 0} \limsup_{s\to-\infty} \mathbb{P}^s \left(\sup_{t_1 \leqslant t} \frac{q_t}{\sqrt{t}} < -\gamma \right) = 1/2 \ .$$

Letting $\sigma = \varepsilon^{\alpha+1/2}$ above gives $\alpha = 1/4$ as the threshold between the different regimes, as found in Theorem 4.1.1. We also remark that the threshold between fast and slow pulling regimes here differs from that in Chapter 3, where it was roughly $\varepsilon = \sigma$. This difference can be attributed to the bifurcation of V: for t near zero, V is almost flat and so the additional $+1$ force requires slower pulling, or stronger noise, to be counteracted than it did in Chapter 3, where the potential had positive curvature bounded away from zero.

Having considered two simplifications of (4.2), we finally return to the full solution itself. Intuitively, by taking β large, the effect of the mass term ε^{β} should become small and the solution should behave like that of the overdamped equation (4.4). So if we show that for suitably large β, the difference between the two solutions stays small, we can use Theorem 4.1.3 to tell us about (4.2). This leads us to the following result.

Theorem 4.1.4. *Let q_t solve (4.2) with $\beta > 2$. There exist constants $c_1, \gamma > 0$, independent of β and σ, such that for $t_1 = c_1\sqrt{\varepsilon|\ln\sigma|}$ and any $t_2 > t_1$,*

1. *(Fast Pulling) if $\sigma^{4/3}|\ln\sigma|^{2/3} \ll \varepsilon(\sigma) \ll 1$ then*

$$\lim_{\sigma\downarrow 0}\liminf_{s\to-\infty} \mathbb{P}^s\left(\inf_{t_1 \leqslant t \leqslant t_2} \frac{q_t}{\sqrt{t}} > \gamma\right) = 1\ ,$$

2. *(Slow Pulling) if $0 < \delta < \beta/2 - 1$ and $\sigma^{2/(1+2\delta)} \ll \varepsilon(\sigma) \ll \sigma^{4/3}|\ln\sigma|^{-13/6}$ then*

$$\lim_{\sigma\downarrow 0}\limsup_{s\to-\infty} \mathbb{P}^s\left(\inf_{t_1 \leqslant t \leqslant t_2} \frac{q_t}{\sqrt{t}} > \gamma\right) = 1/2$$

and

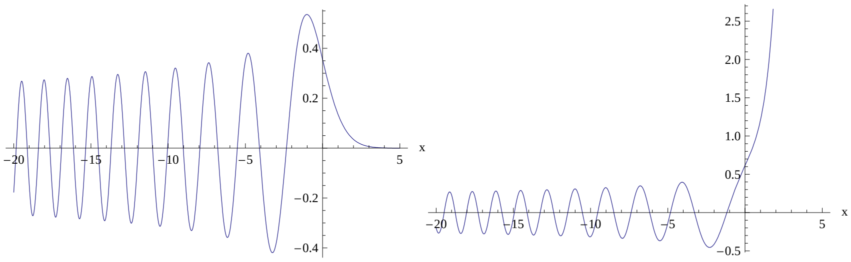
$$\lim_{\sigma\downarrow 0}\limsup_{s\to-\infty} \mathbb{P}^s\left(\sup_{t_1 \leqslant t \leqslant t_2} \frac{q_t}{\sqrt{t}} < -\gamma\right) = 1/2\ .$$

Note that we only consider finite time intervals here and that there is a slight difference between the lower bound on ε in (2) above and in Theorem 4.1.3(2). This second point is related to the fact that the mass is of the form ε^{β}. However, it does not affect the threshold between the two regimes.

We finally note that (4.2) is not suitable for considering the chain 'breaking' due to a large deviation event. In that case, our expansion of the potential is not valid.

4.2 The Linear Model

In this section, we prove Theorems 4.1.1 and 4.1.2. After solving (4.3) and performing some preliminary calculations, we then complete the proofs separately in



(a) Ai(x) (b) Bi(x)

Figure 4.2: The Airy functions Ai and Bi.

Sections 4.2.1 and 4.2.2.

Equation (4.3) is simple enough to have an explicit solution in terms of Airy functions $\mathrm{Ai}(x)$, $\mathrm{Bi}(x)$ (see Fig. 4.2 and [dlm07]): using that $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are linearly independent solutions of the equation $w''(x) = xw(x)$, and that the Wronskian $\mathrm{Ai}(x)\mathrm{Bi}'(x) - \mathrm{Bi}(x)\mathrm{Ai}'(x) = 1/\pi$, we find that the process

$$
\begin{aligned}
q_t^0 = \pi\varepsilon^{(1-2\beta)/3} \bigg(&-\mathrm{Ai}(t(\varepsilon,\beta)) \int_{-T}^{t} \mathrm{e}^{-(t-s)\varepsilon^{-\beta}/2}\, \mathrm{Bi}(s(\varepsilon,\beta))(\mathrm{d}s + \varepsilon^{\alpha}\mathrm{d}W_s) \\
&+ \mathrm{Bi}(t(\varepsilon,\beta)) \int_{-T}^{t} \mathrm{e}^{-(t-s)\varepsilon^{-\beta}/2}\, \mathrm{Ai}(s(\varepsilon,\beta))(\mathrm{d}s + \varepsilon^{\alpha}\mathrm{d}W_s) \bigg)
\end{aligned} \tag{4.5}
$$

is the (almost surely unique) solution of (4.3) with zero initial conditions and starting time $-T < 0$, where we set $s(\varepsilon,\beta) = \varepsilon^{-(1+\beta)/3}(s + \varepsilon^{1-\beta}/4)$.

The functions Ai and Bi admit the asymptotic expansions

$$\mathrm{Ai}(s) = \frac{1}{2\sqrt{\pi}} s^{-1/4}\, \mathrm{e}^{-2s^{3/2}/3} \left(1 + \mathcal{O}(s^{-2/3})\right), \qquad s > 1 \tag{4.6}$$

$$\mathrm{Bi}(s) = \frac{1}{\sqrt{\pi}} s^{-1/4}\, \mathrm{e}^{2s^{3/2}/3} \left(1 + \mathcal{O}(s^{-2/3})\right), \qquad s > 1 \tag{4.7}$$

$$\mathrm{Ai}(s) = \frac{1}{\sqrt{\pi}} |s|^{-1/4} \cos\left(2|s|^{3/2}/3 - \pi/4\right) \left(1 + \mathcal{O}(|s|^{-3})\right) + \mathcal{O}(|s|^{-7/4}), \qquad s < -1 \tag{4.8}$$

$$\mathrm{Bi}(s) = \frac{-1}{\sqrt{\pi}} |s|^{-1/4} \sin\left(2|s|^{3/2}/3 - \pi/4\right) \left(1 + \mathcal{O}(|s|^{-3})\right) + \mathcal{O}(|s|^{-7/4}), \qquad s < -1 \tag{4.9}$$

The behaviour of q_t^0 as $t \to \infty$ can be described in a straightforward way by considering the 'renormalised process'

$$\tilde{q}_t = \frac{1}{\pi\varepsilon^{(1-2\beta)/3}} \frac{\mathrm{e}^{t\varepsilon^{-\beta}/2}}{\mathrm{Bi}(t(\varepsilon,\beta))} q_t^0 \ .$$

Indeed, we have the following result.

Proposition 4.2.1. *Almost surely, $\tilde{q}_\infty := \lim_{t\to\infty} \tilde{q}_t$ exists and is a Gaussian random variable with mean*

$$m^\varepsilon(\beta,T) = \varepsilon^{(1+\beta)/3}\,\mathrm{e}^{-\varepsilon^{1-2\beta}/8} \int_{T_-(\varepsilon,\beta)}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}/2}\,\mathrm{Ai}(s)\,\mathrm{d}s \tag{4.10}$$

and variance

$$v^\varepsilon(\alpha,\beta,T) = \varepsilon^{2\alpha+(1+\beta)/3}\,\mathrm{e}^{-\varepsilon^{1-2\beta}/4} \int_{T_-(\varepsilon,\beta)}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}}\,\mathrm{Ai}^2(s)\,\mathrm{d}s \ , \tag{4.11}$$

where $T_-(\varepsilon,\beta) = \varepsilon^{-(1+\beta)/3}(-T+\varepsilon^{1-\beta}/4)$.

Proof. Let us first investigate the deterministic integrals in (4.5). Since $\mathrm{Bi}(s) < \mathrm{Bi}(t)$ for $s < t$ and $t \geqslant 0$, we have that the deterministic integral in the first line of (4.5), after renormalisation, is bounded by $\mathrm{Ai}(t(\varepsilon,\beta)) \int_{-\infty}^{t} \mathrm{e}^{s\varepsilon^{-\beta}/2}\,\mathrm{d}s$ for large enough t, and thus converges to zero as $t\to\infty$ due to (4.6). The limit of the corresponding renormalised integral in the second line of (4.5) is given by

$$\int_{-T}^{\infty} \mathrm{e}^{s\varepsilon^{-\beta}/2}\,\mathrm{Ai}(s(\varepsilon,\beta))\,\mathrm{d}s = \varepsilon^{(1+\beta)/3}\,\mathrm{e}^{-\varepsilon^{1-2\beta}/8} \int_{T_-(\varepsilon,\beta)}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}/2}\,\mathrm{Ai}(s)\,\mathrm{d}s \ . \tag{4.12}$$

This is also the limit of $\mathbb{E}(\tilde{q}_t)$ as $t\to\infty$.

The stochastic integrals in $\tilde{q}_t$ are given by

$$J_1(t) = \varepsilon^\alpha h(t) \int_{-T}^{t} f_1(s)\,\mathrm{d}W_s \ , \qquad J_2(t) = \varepsilon^\alpha \int_{-T}^{t} f_2(s)\,\mathrm{d}W_s \ ,$$

with

$$h(t) = \frac{\mathrm{Ai}(t(\varepsilon,\beta))}{\mathrm{Bi}(t(\varepsilon,\beta))} \ , \quad f_1(s) = \mathrm{Bi}(s(\varepsilon,\beta))\,\mathrm{e}^{s\varepsilon^{-\beta}/2} \ , \quad f_2(s) = \mathrm{Ai}(s(\varepsilon,\beta))\,\mathrm{e}^{s\varepsilon^{-\beta}/2} \ .$$

By the time change $\tau_1(t) = \varepsilon^{2\alpha}\int_{-T}^{t} f_1^2(s)\,\mathrm{d}s$, $J_1(t)$ equals $h(t)B_{\tau_1(t)}$ in distribution, where B_s is a standard Brownian motion. By the law of the iterated logarithm,

$$\limsup_{t\to\infty} \frac{|J_1(t)|}{h(t)\sqrt{2\tau_1(t)\ln\ln\tau_1(t)}} = 1 \tag{4.13}$$

almost surely. Again using $\mathrm{Bi}(s) < \mathrm{Bi}(t)$ for $s < t$ and $t \geqslant 0$, we find

$$h(t)\sqrt{\tau_1(t)} \leqslant \mathrm{Ai}(t(\varepsilon,\beta))\varepsilon^{\alpha}\left(\int_{-T}^{t} \mathrm{e}^{s\varepsilon^{\beta}}\,\mathrm{d}s\right)^{1/2} ,$$

which converges to zero superexponentially fast by (4.6). By (4.7) it is easy to see that $\ln\ln\tau_1(t)$ grows only proportionally to $\ln t$, and thus the denominator on the left-hand side of (4.13) converges to zero. It follows that $J_1(t) \to 0$ as $t \to \infty$, almost surely. On the other hand, J_2 is a square-integrable martingale, and thus converges almost surely. Each $J_2(t)$ is Gaussian, and thus so is the limit. It has mean zero and variance

$$v^{\varepsilon}(\alpha,\beta,T) = \varepsilon^{2\alpha}\int_{-T}^{\infty} f_2^2(s)\,\mathrm{d}s = \varepsilon^{2\alpha}\int_{-T}^{\infty} \mathrm{e}^{s\varepsilon^{-\beta}}\,\mathrm{Ai}^2(s(\varepsilon,\beta))\,\mathrm{d}s .$$

The same change of variable that was employed (4.12) yields (4.11). □

Since $\mathrm{Bi}(t)\,\mathrm{e}^{-t}$ diverges as $t \to \infty$, Proposition 4.2.1 tells us that $\lim_{t\to\infty}|q_t^0| = \infty$ almost surely. Whether the divergence is to plus or minus infinity is determined by the sign of $\tilde{q}_\infty$. As $\tilde{q}_\infty$ is a Gaussian random variable, it follows that if

$$\lim_{\varepsilon\to 0}\lim_{T\to+\infty}\frac{m^{\varepsilon}(\beta,T)}{\sqrt{v^{\varepsilon}(\alpha,\beta,T)}} = +\infty ,$$

then

$$\lim_{\varepsilon\to 0}\lim_{T\to+\infty}\mathbb{P}^{-T}\left(\lim_{t\to\infty} q_t^0 = +\infty\right) = 1 .$$

On the other hand, if

$$\lim_{\varepsilon\to 0}\lim_{T\to+\infty}\frac{m^{\varepsilon}(\beta,T)}{\sqrt{v^{\varepsilon}(\alpha,\beta,T)}} = 0 ,$$

then

$$\lim_{\varepsilon\to 0}\lim_{T\to+\infty}\mathbb{P}^{-T}\left(\lim_{t\to\infty} q_t^0 = +\infty\right) = 1/2 .$$

The same idea applies if we only let $\varepsilon \to 0$ and keep T fixed. We deal with each of these cases in the following two sections.

4.2.1 Starting Time Sent to Minus Infinity

The aim of this section is to prove Theorem 4.1.1. When we let $T \to \infty$, $m^\varepsilon(\beta, T) \to m^\varepsilon(\beta)$, where

$$\begin{aligned} m^\varepsilon(\beta) &= \varepsilon^{(1+\beta)/3} \, \mathrm{e}^{-\varepsilon^{1-2\beta}/8} \int_{-\infty}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}/2} \, \mathrm{Ai}(s) \, \mathrm{d}s \\ &= \varepsilon^{(1+\beta)/3} \, \mathrm{e}^{-\varepsilon^{1-2\beta}/12} \, . \end{aligned}$$

The final line follows by the standard result [dlm07] that $\int_{-\infty}^{\infty} \mathrm{e}^{ps} \mathrm{Ai}(s) \, \mathrm{d}s = \mathrm{e}^{p^3/3}$ for all $p > 0$. We also have $v^\varepsilon(\alpha, \beta, T) \to v^\varepsilon(\alpha, \beta)$, where

$$v^\varepsilon(\alpha, \beta) = \varepsilon^{2\alpha + (1+\beta)/3} \, \mathrm{e}^{-\varepsilon^{1-2\beta}/4} \int_{-\infty}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}} \, \mathrm{Ai}^2(s) \, \mathrm{d}s \, .$$

To see how this behaves, we consider the function $J(p)$, defined by

$$J(p) = \int_{-\infty}^{\infty} \mathrm{e}^{2ps} \, \mathrm{Ai}^2(s) \, \mathrm{d}s \, ,$$

which behaves as follows.

Lemma 4.2.2. *There exist positive constants c_1 and c_2 such that*

(i) $\lim_{p\to\infty} p^{1/2} \, \mathrm{e}^{-2p^3/3} \, J(p) = c_1$,

(ii) $\lim_{p\to 0} p^{1/2} \, \mathrm{e}^{-2p^3/3} \, J(p) = c_2$.

Proof. Consider first the case $p \to \infty$. Then,

$$\int_{-\infty}^{1} \mathrm{e}^{2ps} \, \mathrm{Ai}^2(s) \, \mathrm{d}s \; p^{1/2} \, \mathrm{e}^{-2p^3/3} \;\leqslant\; C \, \mathrm{e}^{-2p^3/3 + 2p} \, p^{-1/2} \to 0$$

as $p \to \infty$. For $s > 1$ we use (4.6) to find

$$\begin{aligned} \int_{1}^{\infty} \mathrm{e}^{2ps} \, \mathrm{Ai}^2(s) \, \mathrm{d}s &= \frac{1}{4\pi} \int_{1}^{\infty} \mathrm{e}^{-4s^{3/2}/3 + 2ps} \, s^{-1/2} (1 + \mathcal{O}(s^{-3/2})) \, \mathrm{d}s \\ &= \frac{p}{4\pi} \int_{1/p^2}^{\infty} \mathrm{e}^{-p^3(4t^{3/2}/3 - 2t)} \, t^{-1/2} (1 + \mathcal{O}(p^{-3} t^{-3/2})) \, \mathrm{d}t \, , \end{aligned}$$

where we used the substitution $s = p^2 t$. Decompose the final integral as $\int_{1/p^2}^{\infty} = \int_{1/p^2}^{1/4} + \int_{1/4}^{\infty}$. The first of these is bounded by $C \, \mathrm{e}^{p^3/3}$ and can be ignored. For the second, we have $\mathcal{O}(p^{-3} t^{-3/2}) = \mathcal{O}(p^{-3})$ and can take this outside the integral. Then by the Laplace method,

$$\int_{1/4}^{\infty} \mathrm{e}^{-p^3(4t^{3/2}/3-2t)}\, t^{-1/2}\, \mathrm{d}t = \mathrm{e}^{2p^3/3} \int_{1/4}^{\infty} \mathrm{e}^{-p^3[(t-1)^2+\mathcal{O}(t-1)^3]}\, t^{-1/2}\, \mathrm{d}t$$
$$= \mathrm{e}^{2p^3/3}\, p^{-3/2}\sqrt{2\pi}(1+\mathcal{O}(1/p^3))\ .$$

Thus (i) holds with $c_1 = 2^{-3/2}\pi^{-1/2}$. For (ii), we use that

$$\int_{-1}^{\infty} \mathrm{e}^{2ps}\, \mathrm{Ai}^2(s)\, \mathrm{d}s \to \int_{-1}^{\infty} \mathrm{Ai}^2(s)\, \mathrm{d}s = \text{const.}$$

as $p \to 0$. Using (4.8) and ignoring the $\mathcal{O}(|s|^{-7/4})$ term, which does not change the resulting expression below, we then get

$$\int_{-\infty}^{-1} \mathrm{e}^{2ps}\, \mathrm{Ai}^2(s)\, \mathrm{d}s = \frac{1}{\pi}\int_{-\infty}^{-1} \mathrm{e}^{2ps}\, |s|^{-1/2} \cos^2\Big(2|s|^{3/2}/3 - \pi/4\Big)\, \big(1+\mathcal{O}(|s|^{-3})\big)\, \mathrm{d}s$$
$$= \frac{1}{\pi\sqrt{p}}\int_{p}^{\infty} \mathrm{e}^{-2t}\, t^{-1/2} \cos^2\Big(2p^{-3/2}t^{3/2}/3 - \pi/4\Big)\, \mathrm{d}t + \mathcal{O}(1)\ ,$$

where in the last line we used the substitution $t = -ps$. As $p \to 0$, the integral in the last line above converges to

$$\frac{1}{2}\int_{0}^{\infty} t^{-1/2}\, \mathrm{e}^{-2t}\, \mathrm{d}t = \frac{\sqrt{\pi}}{2\sqrt{2}}\ ,$$

which proves (ii) with $c_2 = 2^{-3/2}\pi^{-1/2}$. □

When substituting $p = \varepsilon^{(1-2\beta)/3}/2$ into the last lemma, we find that as ε becomes small,

$$v^{\varepsilon}(\alpha,\beta) \asymp \varepsilon^{2\alpha+(1+\beta)/3}\, \mathrm{e}^{-\varepsilon^{1-2\beta}/4}\, \varepsilon^{(1-2\beta)/6}\, \mathrm{e}^{\varepsilon^{1-2\beta}/12} = \varepsilon^{2\alpha+2\beta/3+1/6}\, \mathrm{e}^{-\varepsilon^{1-2\beta}/6}\ .$$

Thus, $m^{\varepsilon}(\beta)/\sqrt{v^{\varepsilon}(\alpha,\beta)} \asymp \varepsilon^{-\alpha+1/4}$, which completes the proof of Theorem 4.1.1.

4.2.2 Fixed Starting Time

The aim of this section is to prove Theorem 4.1.2. We begin by stating four lemmas that will be proved in Section 4.5. These will enable us to analyse the behaviour of $m^{\varepsilon}(\beta,T)/\sqrt{v^{\varepsilon}(\alpha,\beta,T)}$ as $\varepsilon \to 0$.

Lemma 4.2.3. *Suppose that for all $p > 1$, $1 \leqslant a(p) \leqslant p^2$. Then there is a* *constant $C > 0$ such that for all sufficiently large p,*

$$C\,\mathrm{e}^{p^3/3} \;\leqslant\; \int_a^\infty \mathrm{e}^{ps}\,\mathrm{Ai}(s)\,\mathrm{d}s \;\leqslant\; \mathrm{e}^{p^3/3}\ .$$

Lemma 4.2.4. *Suppose that for all $p > 1$, $1 \leqslant a(p) \leqslant p^2$. Then there are constants C_1, $C_2 > 0$ such that for all sufficiently large p,*

$$C_1 p^{-1/2}\,\mathrm{e}^{2p^3/3} \;\leqslant\; \int_a^\infty \mathrm{e}^{2ps}\,\mathrm{Ai}^2(s)\,\mathrm{d}s \;\leqslant\; C_2 p^{-1/2}\,\mathrm{e}^{2p^3/3}\ .$$

Lemma 4.2.5. *There is a constant $C > 0$ such that for all $a \geqslant 0$ and $p > 0$,*

$$\left|\int_{-a}^\infty \mathrm{e}^{ps}\,\mathrm{Ai}(s)\,\mathrm{d}s - \mathrm{e}^{p^3/3}\right| < Cp^{-1}\,\mathrm{e}^{-pa}\ .$$

Lemma 4.2.6. *There exist constants $C_1, C_2 > 0$ such that for all sufficiently large $a > 0$,*

$$\int_{-a}^\infty \mathrm{Ai}^2(s)\,\mathrm{d}s \;\geqslant\; C_1\,a^{1/2} - C_2\,a^{-1}\ .$$

Proof of Theorem 4.1.2. We will split the analysis into four parts according to the value of β. The four lemmas above will be applied with $p = \varepsilon^{(1-2\beta)/3}/2$ and $a = T_-(\varepsilon,\beta)$, unless otherwise stated.

Case One: $\beta > 1$

We have $T_-(\varepsilon,\beta) \to \infty$ and $\varepsilon^{(1-2\beta)/3} \to \infty$ as $\varepsilon \to 0$. Applying Lemmas 4.2.3 and 4.2.4, we find that for sufficiently small ε,

$$2^{-3/2}\,\mathrm{e}^{\varepsilon^{1-2\beta}/24} \;\leqslant\; \int_{T_-(\varepsilon,\beta)}^\infty \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}/2}\,\mathrm{Ai}(s)\,\mathrm{d}s \;\leqslant\; \mathrm{e}^{\varepsilon^{1-2\beta}/24}$$

and

$$C_1\varepsilon^{(2\beta-1)/6}\,\mathrm{e}^{\varepsilon^{1-2\beta}/12} \;\leqslant\; \int_{T_-(\varepsilon,\beta)}^\infty \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}}\,\mathrm{Ai}^2(s)\,\mathrm{d}s \;\leqslant\; C_2\varepsilon^{(2\beta-1)/6}\,\mathrm{e}^{\varepsilon^{1-2\beta}/12}\ .$$

It follows that

$$\frac{m^\varepsilon(\beta,T)}{\sqrt{v^\varepsilon(\alpha,\beta,T)}} \asymp \varepsilon^{-\alpha}\varepsilon^{(1+\beta)/6}\varepsilon^{(1-2\beta)/12} = \varepsilon^{-\alpha+1/4}\ .$$

Case Two: $1/2 \leqslant \beta \leqslant 1$

We will show the steps for $1/2 < \beta < 1$. In this case, $T_-(\varepsilon,\beta) \to -\infty$ as $\varepsilon \to 0$. This also holds if $\beta = 1$ and $T > 1/4$. If $\beta = 1$ and $0 < T \leqslant 1/4$, then we can follow the steps in Case One. If $\beta = 1/2$, then $p = \varepsilon^{(1-2\beta)/3}/2 = 1/2$ and the calculations are much easier. Otherwise, $p \to \infty$.

Letting $1/2 < \beta < 1$, we have by Lemma 4.2.5 that

$$\left| \int_{T_-(\varepsilon,\beta)}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}/2} \,\mathrm{Ai}(s)\,\mathrm{d}s - \mathrm{e}^{\varepsilon^{1-2\beta}/24} \right| < C\varepsilon^{(2\beta-1)/3}\,\mathrm{e}^{-\varepsilon^{(1-2\beta)/3}|T_-(\varepsilon,\beta)|/2} \,.$$

The term on the right-hand side behaves like $\varepsilon^{(2\beta-1)/3}\,\mathrm{e}^{-\varepsilon^{-\beta}} \to 0\,.$

Decompose the integral in (4.11) into the sum $\int_{T_-(\varepsilon,\beta)}^{\infty} = \int_{T_-(\varepsilon,\beta)}^{1} + \int_1^{\infty}$. By the boundedness of Ai, we have

$$\int_{T_-(\varepsilon,\beta)}^{1} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}}\,\mathrm{Ai}^2(s)\,\mathrm{d}s \;\leqslant\; C\varepsilon^{(2\beta-1)/3}\,\mathrm{e}^{\varepsilon^{(1-2\beta)/3}} \,.$$

Lemma 4.2.4 gives

$$\int_{1}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}}\,\mathrm{Ai}^2(s)\,\mathrm{d}s \;\leqslant\; C\varepsilon^{(1-2\beta)/6}\,\mathrm{e}^{\varepsilon^{1-2\beta}/12} \,.$$

For a lower bound, we use Lemma 4.2.4 to give

$$\begin{aligned} \int_{T_-(\varepsilon,\beta)}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}}\,\mathrm{Ai}^2(s)\,\mathrm{d}s \;&\geqslant\; \int_{1}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}}\,\mathrm{Ai}^2(s)\,\mathrm{d}s \\ &\geqslant\; C\varepsilon^{(2\beta-1)/6}\,\mathrm{e}^{\varepsilon^{1-2\beta}/12} \,. \end{aligned}$$

Therefore,

$$\frac{m^{\varepsilon}(\beta,T)}{\sqrt{v^{\varepsilon}(\alpha,\beta,T)}} \asymp \varepsilon^{-\alpha}\varepsilon^{(1+\beta)/6}\varepsilon^{(1-2\beta)/12} = \varepsilon^{-\alpha+1/4} \,.$$

Case Three: $0 \leqslant \beta < 1/2$

Again $T_-(\varepsilon,\beta) \to -\infty$, but now $\varepsilon^{(1-2\beta)/3} \to 0\,.$ For $m^{\varepsilon}(\beta,T)$, we can apply Lemma 4.2.5. The term $\mathrm{e}^{p^3/3}$ tends to one as $\varepsilon \to 0$, and we find $m^{\varepsilon}(\beta,T) \asymp \varepsilon^{(1+\beta)/3}$.

To obtain an upper bound for the integral in (4.11), we first decompose it into the sum

$$\int_{T_-(\varepsilon,\beta)}^{\infty} = \int_{T_-(\varepsilon,\beta)}^{-\varepsilon^{(2\beta-1)/3}} + \int_{-\varepsilon^{(2\beta-1)/3}}^{-\varepsilon^{(2\beta-1)/12}} + \int_{-\varepsilon^{(2\beta-1)/12}}^{1} + \int_{1}^{\infty} \,. \tag{4.14}$$

If $\beta = 0$ then instead of the first two integrals on the right-hand side we just write $\int_{T_-(\varepsilon,\beta)}^{-1}$. For $s < 0$ with $|s|$ sufficiently large, it follows from (4.8) that $\mathrm{Ai}^2(s) \leqslant$

$C|s|^{-1/2}$. Therefore,

$$\begin{aligned}\int_{T_-(\varepsilon,\beta)}^{-\varepsilon^{(2\beta-1)/3}} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}} \operatorname{Ai}^2(s)\,\mathrm{d}s &\leqslant C \int_{T_-(\varepsilon,\beta)}^{-\varepsilon^{(2\beta-1)/3}} |s|^{-1/2}\, \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}}\, \mathrm{d}s \\ &\leqslant C\varepsilon^{(1-2\beta)/6} \int_{T_-(\varepsilon,\beta)}^{-\varepsilon^{(2\beta-1)/3}} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}}\, \mathrm{d}s \\ &\leqslant C\varepsilon^{(2\beta-1)/6}\end{aligned}$$

and

$$\begin{aligned}\int_{-\varepsilon^{(2\beta-1)/3}}^{-\varepsilon^{(2\beta-1)/12}} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}} \operatorname{Ai}^2(s)\,\mathrm{d}s &\leqslant C \int_{-\varepsilon^{(2\beta-1)/3}}^{-\varepsilon^{(2\beta-1)/12}} |s|^{-1/2}\, \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}}\, \mathrm{d}s \\ &\leqslant C \int_{-\varepsilon^{(2\beta-1)/3}}^{-1} |s|^{-1/2}\,\mathrm{d}s \\ &\leqslant C\varepsilon^{(2\beta-1)/6}\,.\end{aligned}$$

The third integral on the right-hand side of (4.14) can easily be bounded by $C\,\mathrm{e}^{\varepsilon^{(1-2\beta)/3}}$ using the boundedness of Ai. The final integral on the right-hand side of (4.14) converges to $\int_1^\infty \operatorname{Ai}^2(s)\,\mathrm{d}s$, which is just a constant. Combining these estimates tells us that the left-hand side of (4.14) is bounded above by $C\varepsilon^{(2\beta-1)/6}$.

To obtain a lower bound when $\beta > 0$, we use Lemma 4.2.6 with $a = \varepsilon^{(2\beta-1)/3}$ to find

$$\begin{aligned}\int_{T_-(\varepsilon,\beta)}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}} \operatorname{Ai}^2(s)\,\mathrm{d}s &\geqslant \int_{-\varepsilon^{(2\beta-1)/3}}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}} \operatorname{Ai}^2(s)\,\mathrm{d}s \\ &\geqslant C \int_{-\varepsilon^{(2\beta-1)/3}}^{\infty} \operatorname{Ai}^2(s)\,\mathrm{d}s \\ &\geqslant C\varepsilon^{(2\beta-1)/6}\,.\end{aligned}$$

If $\beta = 0$, we keep the lower integral limit as $T_-(\varepsilon,\beta)$. This gives us

$$\frac{m^\varepsilon(\beta,T)}{\sqrt{v^\varepsilon(\alpha,\beta,T)}} \asymp \varepsilon^{-\alpha}\varepsilon^{(1+\beta)/6}\varepsilon^{(1-2\beta)/12} = \varepsilon^{-\alpha+1/4}\,.$$

Case Four: $-1 < \beta < 0$

The behaviour of $m^\varepsilon(\beta,T)$ is treated as in Case Three. To obtain an upper bound for the integral in (4.11), we decompose it as

$$\int_{T_-(\varepsilon,\beta)}^{\infty} = \int_{T_-(\varepsilon,\beta)}^{-\varepsilon^{-(1+\beta)/6}} + \int_{-\varepsilon^{-(1+\beta)/6}}^{\infty}\,.$$

The second term on the right-hand side can be bounded above by a constant as in Case Three. For the first term, the difference with Case Three is that now $-\varepsilon^{(2\beta-1)/3} < T_-(\varepsilon,\beta)$ and so the exponential term in the integrand remains of order one and we can treat it as a constant. Therefore,

$$\begin{aligned}\int_{T_-(\varepsilon,\beta)}^{-\varepsilon^{-(1+\beta)/6}} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}} \operatorname{Ai}^2(s)\,\mathrm{d}s &\leqslant C\int_{T_-(\varepsilon,\beta)}^{-\varepsilon^{-(1+\beta)/6}} \operatorname{Ai}^2(s)\,\mathrm{d}s \\ &\leqslant C\int_{T_-(\varepsilon,\beta)}^{-1} |s|^{-1/2}\,\mathrm{d}s \\ &\leqslant C\varepsilon^{-(1+\beta)/6}\ .\end{aligned}$$

To obtain a lower bound, we use Lemma 4.2.6 to find

$$\int_{T_-(\varepsilon,\beta)}^{\infty} \mathrm{e}^{s\varepsilon^{(1-2\beta)/3}} \operatorname{Ai}^2(s)\,\mathrm{d}s \ \geqslant\ C\int_{T_-(\varepsilon,\beta)}^{\infty} \operatorname{Ai}^2(s)\,\mathrm{d}s \ \geqslant\ C\varepsilon^{-(1+\beta)/6}\ .$$

In contrast to the first three cases, this now gives us

$$\frac{m^\varepsilon(\beta,T)}{\sqrt{v^\varepsilon(\alpha,\beta,T)}} \asymp \varepsilon^{-\alpha}\varepsilon^{(1+\beta)/6}\varepsilon^{(1+\beta)/12} = \varepsilon^{-\alpha+(1+\beta)/4}\ ,$$

which completes the proof of Theorem 4.1.2. □

4.3 The Overdamped Model

The proof of Theorem 4.1.3 consists of two parts. Firstly, we show that given an arbitrary compact set $\mathcal{X}$ containing a neighbourhood of the origin, e.g. $\mathcal{X} = [-1,1]$, and given an arbitrary negative time $-T$, the process q_t starting at 0 at time $-\infty$ will be in $\mathcal{X}$ with very high probability at time $-T$. Secondly, we use the Markov property to restart q_t at time $-T$ and to show that the conclusion of Theorem 4.1.3 holds uniformly over initial conditions belonging to $\mathcal{X}$ at time $-T$. These two steps are formulated as Proposition 4.3.1 and Proposition 4.3.2 below. Theorem 4.1.3 is then an immediate consequence of these two results.

Recalling that $\mathbb{P}^s$ denotes the law of q_t with initial condition $q_s = 0$, we have

Proposition 4.3.1. *Let $\mathcal{X} = [-1,1]$, let $T \geqslant 1$ and let q_t solve (4.4). Then we have*

$$\lim_{\sigma,\varepsilon\to 0}\liminf_{s\to-\infty} \mathbb{P}^s(q_{-T}\in\mathcal{X}) = 1\ .$$

Proof. Applying Itô's formula to the function $q \mapsto q^2$ gives

$$\mathrm{d}q_t^2 = \left(\frac{2t}{\varepsilon}q_t^2 - \frac{2}{\varepsilon}q_t^4 + 2q_t + \frac{\sigma^2}{\varepsilon}\right)\mathrm{d}t + \mathrm{d}M(t) ,$$

where M is some continuous martingale. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(q_t^2) = \mathbb{E}\left(\frac{2t}{\varepsilon}q_t^2 - \frac{2}{\varepsilon}q_t^4 + 2q_t + \frac{\sigma^2}{\varepsilon}\right) \leqslant -\frac{1}{\varepsilon}\mathbb{E}(q_t^2) + \varepsilon + \frac{\sigma^2}{\varepsilon} ,$$

where we used the inequality $2q - q^2/\varepsilon \leqslant \varepsilon$ and that $t \leqslant -1$. It follows that $\mathbb{E}(q_t^2) \leqslant \sigma^2 + \varepsilon^2$. Applying Chebyshev's inequality and letting $\varepsilon \to 0$ and $\sigma \to 0$ gives the result. □

The remainder of this section is devoted to the proof of the following statement, where we denote by $\mathbb{P}^{-T,x}$ the law of (4.4) with initial condition $q_{-T} = x \in \mathcal{X}$.

Proposition 4.3.2. *Let q_t solve (4.4). There exist constants $c_1, \gamma > 0$ such that if $t_1 = c_1\sqrt{\varepsilon|\ln\sigma|}$ then*

1. *(Fast Pulling) for any $\sigma^{4/3}|\ln\sigma|^{2/3} \ll \varepsilon(\sigma) \ll 1$,*

$$\lim_{\sigma\downarrow 0}\inf_{x\in\mathcal{X}} \mathbb{P}^{-T,x}\left(\inf_{t_1 \leqslant t}\frac{q_t}{\sqrt{t}} > \gamma\right) = 1 .$$

2. *(Slow Pulling) for any $\sigma^2|\ln\sigma|^3 \lesssim \varepsilon(\sigma) \ll \sigma^{4/3}|\ln\sigma|^{-13/6}$,*

$$\lim_{\sigma\downarrow 0}\sup_{x\in\mathcal{X}}\left|\mathbb{P}^{-T,x}\left(\inf_{t_1 \leqslant t}\frac{q_t}{\sqrt{t}} > \gamma\right) - 1/2\right| = 0$$

and

$$\lim_{\sigma\downarrow 0}\sup_{x\in\mathcal{X}}\left|\mathbb{P}^{-T,x}\left(\sup_{t_1 \leqslant t}\frac{q_t}{\sqrt{t}} < -\gamma\right) - 1/2\right| = 0 .$$

Our approach in this section is based on that developed by Berglund and Gentz in [BG02, BG06]. They consider similar equations to (4.4), but with drift terms $(1/\varepsilon)f(q,t)$ such that $f(q,t) = -f(-q,t)$ and $f(0,0) = \partial_q f(0,0) = 0$. A simple example of such an f is $f(q,t) = tq - q^3$. Our additional drift term arising from pulling means that we cannot directly apply their results, except in a few cases as will be made clear.

4.3.1 Fast Pulling

We begin by considering the fast pulling regime from Proposition 4.3.2. In this case, the noise in the system is not strong enough to overcome the asymmetry caused by pulling and the qualitative behaviour of q_t is the same as that of the deterministic solution q_t^{det} of the ODE

$$\dot{q}_t^{\text{det}} = \frac{1}{\varepsilon}(tq_t^{\text{det}} - (q_t^{\text{det}})^3 + \varepsilon) \ , \quad q_{-T}^{\text{det}} = x \ . \tag{4.15}$$

In particular, we will see that q_t^{det} falls into the right-hand well by a time of order $\sqrt{\varepsilon|\ln\sigma|}$ after the bifurcation and so too does q_t. The strategy is as follows:

1. Show that q_t is of order $\sqrt{\varepsilon}$ when $t = \sqrt{\varepsilon}$.
2. Show that (q_t, t) then leaves the space-time set $\mathcal{K}(\kappa)$ (see (4.23)), by a time of order $\sqrt{\varepsilon|\ln\sigma|}$.
3. Show that q_t approaches the right-hand well and stays in a small neighbourhood of it up until any time $t_2 > 0$.
4. Show that by taking t_2 large enough, q_t stays in a neighbourhood of the right-hand well of order $t^{1/2-\gamma}$ for any $0 < \gamma < 1/2$ and all $t \geqslant t_2$.

Step One:

We begin by describing how q_t^{det} behaves.

Lemma 4.3.3. *Let q_t^{det} be the solution of (4.15). Then we have, uniformly for all initial conditions $x \in \mathcal{X}$,*

$$q_t^{\text{det}} \asymp \begin{cases} \varepsilon/|t| \ \textit{for} \ -T + \varepsilon|\ln\varepsilon| \leqslant t \leqslant -\sqrt{\varepsilon} \\ \sqrt{\varepsilon} \ \textit{for} \ -\sqrt{\varepsilon} \leqslant t \leqslant \sqrt{\varepsilon} \end{cases}$$

and there is a constant $C > 0$ such that for all $x \in \mathcal{X}$, $|q_t^{\text{det}}| < C$ for all $-T \leqslant t \leqslant -T + \varepsilon|\ln\varepsilon|$.

Proof. Consider the equation

$$tq_+^*(t) - q_+^*(t)^3 + \varepsilon = 0 \ . \tag{4.16}$$

By fixing t and differentiating the left-hand side with respect to q_+^*, we see that for $t < 0$ it has no turning points and so admits a unique real-valued solution. Furthermore, we can check that $q_+^*(t) \asymp \varepsilon$ and $(q_+^*)'(t) = q_+^*(t)/(3q_+^*(t)^2 - t) \asymp \varepsilon$ for negative t bounded away from zero.

Suppose first that the initial condition satisfies $x \leqslant q_+^*(-T)$. Define $z_t = q_t^{\det} - q_+^*(t)$. As long as $z_t \geqslant 0$, we have $\dot{z}_t \leqslant tz_t/\varepsilon$ so that

$$0 \leqslant q_t^{\det} - q_+^*(t) \leqslant (x - q_+^*(-T))\,\mathrm{e}^{(t^2-T^2)/2\varepsilon} \ .$$

Let $t_0 = -T + \varepsilon|\ln\varepsilon|$. If $z_t < 0$ for some $-T < t < t_0$, which means that $q_t^{\det} \asymp \varepsilon$, then the analysis below for $t \geqslant t_0$ can be applied from that time. Otherwise, the above inequality shows that $q_{t_0}^{\det} \asymp \varepsilon$.

If $x \leqslant q_+^*(-T)$ then we define $z_t = q_+^*(t) - q_t^{\det}$. As $(q_+^*)'(t) > 0$ for negative t bounded away from zero, we have $z_t \geqslant 0$ for such t. In this case, there is $c_1 > 0$, independent of $x \in \mathcal{X}$, such that

$$\dot{z}_t \leqslant c_1\varepsilon + \frac{1}{\varepsilon}(tz_t + 3q_+^*(t)z_t^2)$$

for all $-T \leqslant t \leqslant t_0$. Furthermore, as long as $z_t \leqslant -t/6q_+^*(t)$ (which is satisfied by $z(-T)$ for all $x \in \mathcal{X}$ by taking ε sufficiently small) then $3q_+^*(t)z_t^2 \leqslant -tz_t/2$ and so

$$\dot{z}_t \leqslant c_1\varepsilon + \frac{1}{2\varepsilon}tz_t \ .$$

This tells us

$$0 \leqslant q_+^*(t) - q_t^{\det} \leqslant (q^*(-T) - x)\,\mathrm{e}^{(t^2-T^2)/4\varepsilon} + c_1\varepsilon\int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/4\varepsilon}\,\mathrm{d}s \ ,$$

which shows that $z_t \leqslant -t/6q_+^*(t)$ for all $-T \leqslant t \leqslant t_0$ and we can use the above inequality to again see that $q_{t_0}^{\det} \asymp \varepsilon$.

We now analyse the behaviour for $t \geqslant t_0$. As $q_{t_0}^{\det} > 0$ and $\dot{q}_t^{\det} = 1$ whenever $q_t^{\det} = 0$, it follows that $q_t^{\det} \geqslant 0$ for all $t \geqslant t_0$. Therefore, for $t \geqslant t_0$ we have

$$\dot{q}_t^{\det} \leqslant \frac{1}{\varepsilon}(tq_t^{\det} + \varepsilon)$$

and so

$$q_t^{\det} \leqslant q_{t_0}^{\det}\,\mathrm{e}^{(t^2-t_0^2)/2\varepsilon} + \int_{t_0}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,\mathrm{d}s \leqslant \begin{cases} c_2\varepsilon/|t| \text{ for } t_0 \leqslant t \leqslant -\sqrt{\varepsilon} \\ c_2\sqrt{\varepsilon} \text{ for } -\sqrt{\varepsilon} \leqslant t \leqslant \sqrt{\varepsilon} \end{cases} \tag{4.17}$$

for some constant $c_2 > 0$. To obtain the lower bound, we use that for $t \leqslant 0$, $tq - q^3 \geqslant 2tq$ as long as $q^2 \leqslant |t|$. By taking ε sufficiently small, we have $0 < q_{t_0} \leqslant \sqrt{|t_0|}$ for all initial conditions $x \in \mathcal{X}$. As long as $0 \leqslant q_t^{\det} \leqslant \sqrt{|t|}$, then

$$\dot{q}_t^{\rm det} \;\geqslant\; \frac{1}{\varepsilon}(2tq_t^{\rm det} + \varepsilon)$$

and

$$q_t^{\rm det} \;\geqslant\; q_{t_0}\,{\rm e}^{(t^2-t_0^2)/\varepsilon} + \int_{t_0}^t {\rm e}^{(t^2-s^2)/\varepsilon}\,{\rm d}s\;.$$

By (4.17), we certainly have $q_t^{\rm det} \;\leqslant\; \sqrt{|t|}$ for $t \;\leqslant\; -\sqrt{\varepsilon}$ and so the above inequality gives the corresponding lower bound for $q_t^{\rm det}$ up until this time. For $-\sqrt{\varepsilon} \;\leqslant\; t \;\leqslant\; \sqrt{\varepsilon}$, we then have $tq_t^{\rm det} - (q_t^{\rm det})^3 \;\geqslant\; -C\varepsilon$ for some constant $C > 0$, so that $q_t^{\rm det}$ remains of order $\sqrt{\varepsilon}$ in this interval. This completes the proof.

□

We now show that the deviation process $y_t \;:=\; q_t - q_t^{\rm det}$ satisfies $|y(\sqrt{\varepsilon})| < h\varepsilon^{-1/4}$ for some $h \ll \varepsilon^{3/4}$, which will complete Step One. The process y_t solves

$${\rm d}y_t = \frac{1}{\varepsilon}[a(t)y_t + b(y_t,t)]\,{\rm d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,{\rm d}W_t\;, \quad y(-T) = 0\;, \tag{4.18}$$

where $a(t) = t - 3(q_t^{\rm det})^2$ and $b(y_t,t) = -3q_t^{\rm det}y_t^2 - y_t^3$. For all pairs $(y,t) \in \mathcal{B}(h)$ for a choice of $h = \mathcal{O}(\varepsilon^{1/4})$ (see (4.21) and Lemma 4.3.4), we have $|b(y,t)| \;\leqslant\; My^2$. Solving (4.18) gives

$$y_t = \frac{\sigma}{\sqrt{\varepsilon}}\int_{-T}^t {\rm e}^{\alpha(t,s)/\varepsilon}\,{\rm d}W_s + \frac{1}{\varepsilon}\int_{-T}^t {\rm e}^{\alpha(t,s)/\varepsilon}\,b(y_s,s)\,{\rm d}s \;=:\; y_t^0 + y_t^1\;, \tag{4.19}$$

where $\alpha(t,s) = \int_s^t a(u)\,{\rm d}u$.

We now define the space-time set $\mathcal{B}(h)$ mentioned above. If we write ${\rm Var}(y_t^0) = \sigma^2 v(t)$, then we find that $v(t)$ solves the ODE

$$\varepsilon\dot{v} = 2a(t)v + 1\;, \quad v(-T) = 0\;.$$

Let $\xi(t)$ be the particular solution of this ODE, with nonzero initial condition, given by

$$\xi(t) = \xi(-T)\,{\rm e}^{2\alpha(t,-T)/\varepsilon} + \frac{1}{\varepsilon}\int_{-T}^t {\rm e}^{2\alpha(t,s)/\varepsilon}\,{\rm d}s\;, \qquad \xi(-T) = \frac{1}{2|a(-T)|}\;. \tag{4.20}$$

Then we define

$$\mathcal{B}(h) = \{(y,t) : -T \;\leqslant\; t \;\leqslant\; \sqrt{\varepsilon}, |y| < h\sqrt{\xi(t)}\} \tag{4.21}$$

and the stopping time $\tau_{\mathcal{B}(h)} \;=\; \inf\{t \;\geqslant\; -T : (y_t,t) \notin \mathcal{B}(h)\}$. Note that $\tau_{\mathcal{B}(h)}$

depends on the choice of initial condition x. Before estimating $\tau_{\mathcal{B}(h)}$, we must first understand how $a(t)$ and $\xi(t)$ behave:

Lemma 4.3.4. *Let $q_t^{\det}$ solve (4.15), define $a(t) = t - 3(q_t^{\det})^2$ and let $\xi(t)$ be given by (4.20). Then, uniformly for $x \in \mathcal{X}$, $a(t) \asymp t$ for $-T \leqslant t \leqslant -\sqrt{\varepsilon}$ and $|a(t)| = \mathcal{O}(\sqrt{\varepsilon})$ for $|t| \leqslant \sqrt{\varepsilon}$. We also have, uniformly for $x \in \mathcal{X}$,*

$$\xi(t) \asymp \frac{1}{|t| \vee \sqrt{\varepsilon}}$$

and $|\dot{\xi}(t)| = \mathcal{O}(1/\varepsilon)$ for all $-T \leqslant t \leqslant \sqrt{\varepsilon}$.

Proof. The assertions about $a(t)$ follow from Lemma 4.3.3. We can use this to tell us how $\xi(t)$ behaves, for which it is helpful to consider the case $a(t) = t$ as an example. Furthermore, since ξ solves $\varepsilon\dot{\xi} = 2a(t)\xi + 1$, this tells us that $|\dot{\xi}(t)| = \mathcal{O}(1/\varepsilon)$. □

Having established the behaviour of all relevant quantities, we can now prove the following proposition telling us that sample paths are likely to remain in $\mathcal{B}(h)$ for all times $t \leqslant \sqrt{\varepsilon}$.

Proposition 4.3.5. *There exists a constant $C > 0$ such that for all ε sufficiently small, all $\sigma < h \ll \varepsilon^{3/4}$, and all initial conditions $x \in \mathcal{X}$,*

$$\mathbb{P}\left(\tau_{\mathcal{B}(h)} < \sqrt{\varepsilon}\right) \leqslant \frac{C}{\varepsilon^2} \exp\left(-\frac{h^2}{2\sigma^2}\left(1 - r(h,\varepsilon)\right)\right),$$

where $r(h,\varepsilon) = \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(h\varepsilon^{-3/4})$ uniformly for $x \in \mathcal{X}$.

Remark 4.3.6. *Choosing $h = k\sigma\sqrt{|\ln\sigma|}$ with $k > 0$ large enough guarantees that the right-hand side tends to zero as $\sigma \downarrow 0$ and that $h\sqrt{\xi(\sqrt{\varepsilon})} \ll \sqrt{\varepsilon}$, in which case we may take $q(\sqrt{\varepsilon}) \asymp \sqrt{\varepsilon}$ uniformly for all $x \in \mathcal{X}$.*

Proof. Recall the decomposition $y_t = y_t^0 + y_t^1$ from (4.19). We have for all $t < \tau_{\mathcal{B}(h)} \wedge \sqrt{\varepsilon}$,

$$\begin{aligned}
\frac{|y_t^1|}{\sqrt{\xi(t)}} &\leqslant \frac{1}{\sqrt{\xi(t)}} \frac{1}{\varepsilon} \int_{-T}^{t} \mathrm{e}^{\alpha(t,u)/\varepsilon} |b(y_u, u)| \,\mathrm{d}u \\
&\leqslant \frac{Mh^2}{\sqrt{\xi(t)}} \left(\sup_{-T \leqslant u \leqslant t} \xi(u)\right) \frac{1}{\varepsilon} \int_{-T}^{t} \mathrm{e}^{\alpha(t,u)/\varepsilon} \,\mathrm{d}u \\
&\leqslant \frac{c_1 M h^2}{\varepsilon^{3/4}}
\end{aligned}$$

for some constant $c_1 > 0$, where we obtain the final inequality by bounding

$$\frac{1}{\varepsilon}\int_{-T}^{t} \mathrm{e}^{\alpha(t,u)/\varepsilon}\,\mathrm{d}u \leqslant \begin{cases} C/|t| \text{ for } t \leqslant -\sqrt{\varepsilon} \\ C/\sqrt{\varepsilon} \text{ for } |t| \leqslant \sqrt{\varepsilon} \end{cases}$$

and

$$\frac{1}{\sqrt{\xi(t)}}\left(\sup_{-T\leqslant u\leqslant t}\xi(u)\right) \leqslant \begin{cases} C/\sqrt{|t|} \text{ for } t \leqslant -\sqrt{\varepsilon} \\ C\varepsilon^{-1/4} \text{ for } |t| \leqslant \sqrt{\varepsilon} \end{cases}$$

Therefore, if $|y_t^0|/\sqrt{\xi(t)} < h(1 - c_1 M h\varepsilon^{-3/4})$ for all $-T \leqslant t \leqslant \sqrt{\varepsilon}$ then we must have $\tau_{\mathcal{B}(h)} > \sqrt{\varepsilon}$. Letting $H = h(1 - c_1 M h\varepsilon^{-3/4})$, we obtain exactly as in the proof of Proposition 4.3 from [BG02] that for sufficiently small ε,

$$\mathbb{P}\left(\sup_{-T\leqslant t\leqslant\sqrt{\varepsilon}} \frac{|y_s^0|}{\sqrt{\xi(t)}} > H\right) \leqslant \frac{C}{\varepsilon^2}\exp\left(-\frac{H^2}{2\sigma^2}(1-\mathcal{O}(\sqrt{\varepsilon}))\right) \tag{4.22}$$

for some $C > 0$. Note we cannot apply that proposition directly because our function $a(t)$ behaves differently for $|t| \leqslant \sqrt{\varepsilon}$ than the corresponding function there. In particular, here $a(t) < 0$ for $t \ll \varepsilon$, whereas in [BG02], $a(t) = t + \mathcal{O}(t^2)$. We now outline the proof and show that it extends to our case.

For a partition $-T = t_0 < t_1 < \ldots < t_J = \sqrt{\varepsilon}$ of the interval $[-T, \sqrt{\varepsilon}]$, which is chosen below, we have on each subinterval

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t_j\leqslant t\leqslant t_{j+1}} \frac{|y_t^0|}{\sqrt{\xi(t)}} > H\right) \\
&\leqslant \mathbb{P}\left(\sup_{t_j\leqslant t\leqslant t_{j+1}} \left|\int_{-T}^{t} \mathrm{e}^{-\alpha(u)/\varepsilon}\,\mathrm{d}W_u\right| \geqslant \frac{\sqrt{\varepsilon}}{\sigma}\inf_{t_j\leqslant t\leqslant t_{j+1}} H\sqrt{\xi(t)}\,\mathrm{e}^{-\alpha(t)/\varepsilon}\right) \\
&\leqslant 2\exp\left(-\frac{H^2}{2\sigma^2}\mathrm{e}^{2\alpha(t_{j+1},t_j)/\varepsilon}\inf_{t_j\leqslant t\leqslant t_{j+1}}\frac{\xi(t)}{\xi(t_{j+1})}\right),
\end{aligned}$$

where the final line follows by the inequality (3.14). Now we describe our choice of partition. Let $J = J_0 + J_1$ for

$$J_0 = \left\lceil \frac{-\alpha(-\sqrt{\varepsilon}, -T)}{\varepsilon^2} \right\rceil, \quad J_1 = \left\lceil \frac{2}{\sqrt{\varepsilon}} \right\rceil,$$

and define the partition by

$$\begin{aligned}
-\alpha(t_{j+1}, t_j) &= \varepsilon^2, \qquad 0 \leqslant j \leqslant J_0 - 2\,, \\
t_j &= -\sqrt{\varepsilon} + (j - J_0)\varepsilon\,, \qquad J_0 \leqslant j \leqslant J_0 + J_1 - 1\,.
\end{aligned}$$

Using Lemma 4.3.4, we can Taylor expand $\xi(t)$ around $\xi(t_{j+1})$ to find constants C_1, $C_2 > 0$ such that for $t_j \leqslant t < t_{j+1}$ and any $0 \leqslant j \leqslant J_0 - 1$,

$$\frac{\xi(t)}{\xi(t_{j+1})} \geqslant 1 - \frac{C_1}{\varepsilon}\frac{t_{j+1}-t_j}{\xi(t_{j+1})} \geqslant 1 - \frac{C_2}{\varepsilon}(t_{j+1}-t_j)|t_{j+1}| \geqslant 1 + \frac{C_2}{\varepsilon}(t_{j+1}^2 - t_j^2) \,.$$

Since $\alpha(t,s) \asymp t^2 - s^2$ for $s \leqslant t \leqslant -\sqrt{\varepsilon}$, we have $|t_{j+1}^2 - t_j^2| = \mathcal{O}(\varepsilon^2)$ uniformly for all $0 \leqslant j \leqslant J_0 - 1$. This tells us there is a constant $C > 0$ such that for all $0 \leqslant j \leqslant J_0 - 1$,

$$\mathbb{P}\left(\sup_{t_j \leqslant t \leqslant t_{j+1}} \frac{|y_t^0|}{\sqrt{\xi(t)}} > H\right) \leqslant 2\exp\left(-\frac{H^2}{2\sigma^2}\,\mathrm{e}^{-2\varepsilon}\,(1 - C\varepsilon)\right)$$

and we can further bound the right-hand side using the simple inequality $\mathrm{e}^{-2\varepsilon} \geqslant 1 - 2\varepsilon$. We can also find constants C_1, $C_2 > 0$ such that for $t_j \leqslant t < t_{j+1}$ and any $J_0 \leqslant j \leqslant J_0 + J_1 - 1$,

$$\frac{\xi(t)}{\xi(t_{j+1})} \geqslant 1 - \frac{C_1}{\varepsilon}\frac{t_{j+1}-t_j}{\xi(t_{j+1})} \geqslant 1 - \frac{C_2}{\sqrt{\varepsilon}}(t_{j+1}-t_j) \geqslant 1 - C_2\sqrt{\varepsilon} \,.$$

Note too that since $|a(t)| = \mathcal{O}(\sqrt{\varepsilon})$ for $|t| \leqslant \sqrt{\varepsilon}$, it follows that for all $J_0 \leqslant j \leqslant J_0 + J_1 - 1$, $|\alpha(t_{j+1}, t_j)| = \mathcal{O}(\varepsilon^{3/2})$ uniformly. Therefore, for all such j we have

$$\mathbb{P}\left(\sup_{t_j \leqslant t \leqslant t_{j+1}} \frac{|y_t^0|}{\sqrt{\xi(t)}} > H\right) \leqslant 2\exp\left(-\frac{H^2}{2\sigma^2}\,\mathrm{e}^{-c\sqrt{\varepsilon}}\,(1 - C\sqrt{\varepsilon})\right)$$

for constants c, $C > 0$. The bound (4.22) then follows by summing over the partition, which completes the proof.

$\square$

Step Two:

We define for $\kappa > 0$ the space-time set

$$\mathcal{K}(\kappa) = \left\{(q,t) : t \geqslant \sqrt{\varepsilon}\,,\, q^2 \leqslant (1-\kappa)t\right\} \,. \tag{4.23}$$

The boundary of $\mathcal{K}(\kappa)$ consists of the curves $(\pm\sqrt{(1-\kappa)t}, t)$. For the present case, we only need to consider $q \geqslant 0$ (for the slow pulling regime in Section 4.3.2, we will also consider $q < 0$). Let $t_0 \geqslant \sqrt{\varepsilon}$ and suppose that $0 < q(t_0) < \sqrt{(1-\kappa)t_0}$.

Then for $t > t_0$ and as long as $0 \leqslant q_t \leqslant \sqrt{(1-\kappa)t}$, we have $q_t \geqslant q_t^\kappa$, where

$$\mathrm{d}q_t^\kappa = \frac{1}{\varepsilon}\kappa t q_t^\kappa \,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t\,, \quad q^\kappa(t_0) = q(t_0)/2\,. \tag{4.24}$$

Solving this SDE gives

$$q_t^\kappa = \frac{1}{2}q(t_0)\,\mathrm{e}^{(t^2-t_0^2)/2\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}}\int_{t_0}^t \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,\mathrm{d}W_s\,.$$

We now state two lemmas that are analogues of Lemmas 3.2.10 and 3.2.11 from [BG06]. The proofs given here are essentially the same. For the present section, we will only need to take $t_0 = \sqrt{\varepsilon}$ and, by Step One, $q_0 = q(\sqrt{\varepsilon}) \asymp \sqrt{\varepsilon}$. For the slow pulling regime, other initial conditions will be considered. Let

$$\tau_{\mathcal{K}(\kappa)} = \inf\{t \geqslant t_0 : (q_t, t) \notin \mathcal{K}(\kappa)\}$$

and

$$\tau_\kappa^0 = \inf\{t \geqslant t_0 : q_t^\kappa \leqslant 0\}\,.$$

Note that if $q_0 < 0$ then we define $\tau_\kappa^0 = \inf\{t \geqslant t_0 : q_t^\kappa \geqslant 0\}$.

Lemma 4.3.7. *Assume that q_t starts at time $t_0 \geqslant \sqrt{\varepsilon}$ in $q_0 > 0$, where $(q_0, t_0) \in \mathcal{K}(\kappa)$. Then there is $C > 0$, independent of t_0 and q_0, such that for all $t \geqslant t_0 + \varepsilon/t_0$,*

$$\mathbb{P}(\tau_{\mathcal{K}(\kappa)} \geqslant t,\, \tau_0^\kappa \geqslant t) \leqslant \frac{C}{\sigma}\sqrt{t_0}\sqrt{t}\,\mathrm{e}^{-\kappa(t^2-t_0^2)/2\varepsilon}\,.$$

Proof. We know $q_t \geqslant q_t^\kappa$ for all $t \leqslant \tau_{\mathcal{K}(\kappa)} \wedge \inf\{u > t_0 : q_u < 0\}$, so q_t cannot reach 0 before q_t^κ does. Therefore,

$$\begin{aligned}
\mathbb{P}(\tau_{\mathcal{K}(\kappa)} \geqslant t,\, \tau_0^\kappa \geqslant t) &\leqslant \mathbb{P}(0 < q_u^\kappa < \sqrt{(1-\kappa)u} \text{ for all } u \in [t_0, t]) \\
&\leqslant \mathbb{P}(0 < q_t^\kappa < \sqrt{(1-\kappa)t}) \\
&\leqslant \frac{\sqrt{(1-\kappa)t}}{\sqrt{2\pi \mathrm{Var}(q_t^\kappa)}}\,,
\end{aligned}$$

where the final inequality follows since q_t^κ has a Gaussian distribution. If $t - t_0 \geqslant \varepsilon/t_0$, then we have

$$\mathrm{Var}(q_t^\kappa) = \frac{\sigma^2}{\varepsilon}\int_{t_0}^t \mathrm{e}^{\kappa(t^2-s^2)/\varepsilon}\,\mathrm{d}s \geqslant \mathrm{e}^{-1-2\kappa}\frac{\sigma^2}{t_0}\mathrm{e}^{\kappa(t^2-t_0^2)/\varepsilon}\,.$$

Therefore, the result holds with $C = (1-\kappa)^{1/2}(2\pi)^{-1/2}\,\mathrm{e}^{\kappa+1/2}$. □

Lemma 4.3.8. *Let q_t^κ start at time $t_0 \geqslant \sqrt{\varepsilon}$ in $q_0/2 > 0$. Then there exist constants C_1, $C_2 > 0$, independent of t_0 and q_0, such that for all $t \geqslant t_0$, the probability of reaching zero before time t satisfies the bound*

$$\mathbb{P}(\tau_0^\kappa < t) \leqslant \frac{C_1\sigma}{q_0\sqrt{t_0}} \exp\left(-\frac{C_2 q_0^2 t_0}{\sigma^2}\right) .$$

Proof. We have by the reflection principle (see Lemma B.4.1 from [BG06]) that

$$\begin{aligned}\mathbb{P}(\tau_0^\kappa < t) = 2\mathbb{P}(q_t^\kappa < 0) = 2\Phi\left(-\frac{q_0 \, \mathrm{e}^{\kappa(t^2-t_0^2)/2\varepsilon}}{2\sqrt{\mathrm{Var}(q_t^\kappa)}}\right) \\ \leqslant 2\Phi\left(-\frac{q_0\sqrt{\kappa t_0}}{\sigma\sqrt{2}}\right) ,\end{aligned}$$

where the final line follows by the inequality

$$\mathrm{Var}(q_t^\kappa) = \frac{\sigma^2}{\varepsilon}\int_{t_0}^t \mathrm{e}^{\kappa(t^2-s^2)/\varepsilon}\, \mathrm{d}s \leqslant \frac{\sigma^2}{2\kappa t_0}\, \mathrm{e}^{\kappa(t^2-t_0^2)/\varepsilon} .$$

Using the bound $\Phi(-x) \leqslant (2\pi)^{-1/2}x^{-1}\,\mathrm{e}^{-x^2/2}$, valid for $x > 0$, the result then follows with $C_1 = 2/\sqrt{\kappa\pi}$ and $C_2 = \kappa/4$. $\square$

This second lemma shows that when $q_0\sqrt{t_0} \gg \sigma$, the linear process q_t^κ is unlikely to return to zero for any time $t \geqslant t_0$, while the first lemma shows that as t increases the probability of q_t remaining in $\mathcal{K}(\kappa)$ decreases. Therefore, we have $q_t \geqslant q_t^\kappa$ until time $\tau_{\mathcal{K}(\kappa)}$ and so q_t must exit $\mathcal{K}(\kappa)$ through the curve $\sqrt{(1-\kappa)t}$. Indeed, there are constants C_1, $C_2 > 0$ such that for initial time $t_0 = \sqrt{\varepsilon}$ and any initial position $q_0 \asymp \sqrt{\varepsilon}$, we have for any $t \geqslant 2\sqrt{\varepsilon}$ that

$$\mathbb{P}(\tau_{\mathcal{K}(\kappa)} < t,\, \tau_0^\kappa > t) \geqslant 1 - \frac{C_1\varepsilon^{1/4}}{\sigma}\sqrt{t}\,\mathrm{e}^{-\kappa t^2/2\varepsilon} - \frac{C_1\sigma}{\varepsilon^{3/4}}\exp\left(-\frac{C_2\,\varepsilon^{3/2}}{\sigma^2}\right) . \tag{4.25}$$

In the present fast pulling regime, $\varepsilon \gg \sigma^{4/3}$ and so the third term on the right-hand side tends to zero as $\sigma \downarrow 0$. Picking $t = \sqrt{2k\varepsilon|\ln\sigma|}$, we see the second term also tends to zero as long as $k > 1/\kappa$. By a time of order $\sqrt{\varepsilon|\ln\sigma|}$, all paths will have left $\mathcal{K}(\kappa)$ through its upper boundary.

Step Three:

Firstly, we will see how deterministic solutions behave when started from the boundary of $\mathcal{K}(\kappa)$. For this, we let $q_+^*(t)$ be the same solution of (4.16) that we considered in the proof of Lemma 4.3.3, i.e. the unique real-valued solution existing for all times $t \geqslant -T$. For $t > 0$ and ε sufficiently small, we have $\sqrt{t} \leqslant q_+^*(t) \leqslant \sqrt{t} + \varepsilon/t$, which can easily be seen by the intermediate value theorem. For the sake of brevity, we shall write τ to mean $\tau_{\mathcal{K}(\kappa)}$. The following proposition tells us how $q_t^{\det,\tau}$ behaves, where $q_t^{\det,\tau}$ solves

$$\dot{q}_t^{\det,\tau} = \frac{1}{\varepsilon}(tq_t^{\det,\tau} - (q_t^{\det,\tau})^3 + \varepsilon) , \qquad q_\tau^{\det,\tau} = \sqrt{(1-\kappa)\tau} . \tag{4.26}$$

Proposition 4.3.9. *Assume that $\kappa \in (1/2, 2/3)$ and let $\eta = 2 - 3\kappa > 0$. There is a constant $C > 0$ such that the solution $q_t^{\det,\tau}$ of (4.26) satisfies*

$$0 \leqslant q_+^*(t) - q_t^{\det,\tau} \leqslant C\left(\frac{\varepsilon}{t^{3/2}} + (q^*(\tau) - q^{\det,\tau}(\tau))\,\mathrm{e}^{-\eta(t^2-\tau^2)/2\varepsilon}\right)$$

for all $t \geqslant \tau$ and ε sufficiently small.

Remark 4.3.10. *The condition $\kappa > 1/2$ guarantees that paths do not re-enter $\mathcal{K}(\kappa)$ after leaving, while $\kappa < 2/3$ ensures that the potential is convex outside of $\mathcal{K}(\kappa)$.*

Proof. The inequality $q_t^{\det,\tau} \leqslant q_+^*(t)$ follows since $q^{\det,\tau}(\tau) < q_+^*(\tau)$ and

$$(q_+^*)'(t) = \frac{q_+^*(t)}{3q_+^*(t)^2 - t} > 0 . \tag{4.27}$$

The proof of the other inequality follows along the same lines as that given in [BG02, Proposition 4.11]. Note, however, that unlike there we only need to take ε sufficiently small and not t. This is because in our case the value of a_0^*, which is defined in equation (4.99) of [BG02], is given by $-2(1 + o_\varepsilon(1))$, rather than $-2(1 + o_t(1))$. Similarly, $M^* = 3(1 + o_\varepsilon(1))$. As $q_+^*(t) \leqslant \sqrt{t} + \varepsilon/t \leqslant (3/2)\sqrt{t}$ for $t \geqslant \sqrt{\varepsilon}$ and ε small, we can use (4.27) to show $(q_+^*)'(t) \leqslant (3/4)t^{-1/2}$, giving $K^* = 3/4$, where K^* is also defined in (4.99). In [BG02], $K^* = 1/2$, but the proof just requires that $K^* < 1$. □

Now that we understand how $q_t^{\det,\tau}$ behaves, the final step is to show that q_t, starting at the same point, stays close. Having shown in Proposition 4.3.9 above that the analogue of Proposition 4.11 from [BG02] holds, the proofs of the subsequent bounds there can easily be extended to our case and we now show what these are. Let

$$\xi^\tau(t) = \frac{1}{2|a^\tau(\tau)|}\,\mathrm{e}^{2\alpha^\tau(t,\tau)/\varepsilon} + \frac{1}{\varepsilon}\int_\tau^t \mathrm{e}^{2\alpha^\tau(t,s)/\varepsilon}\,\mathrm{d}s ,$$

where $a^\tau(\tau) = t - 3(q_t^{\det,\tau})^2$ is the linearisation of the drift term around $q_t^{\det,\tau}$ and

$\alpha^\tau(t,s) = \int_s^t a(u)\,\mathrm{d}u$. As is shown in Lemma 4.12 from [BG02], it follows from Proposition 4.3.9 above that $|a^\tau(\tau)| \asymp t$, so that $\xi^\tau(t) \asymp 1/t$.

Now we write

$$\mathcal{A}^\tau(h) = \{(q,t) : t \geqslant \tau, |q - q_t^{\mathrm{det},\tau}| \leqslant h\sqrt{\xi^\tau(t)}\} \tag{4.28}$$

and let $\tau_{\mathcal{A}^\tau(h)} = \inf\{t \geqslant \tau : (q_t,t) \notin \mathcal{A}^\tau(h)\}$. The following bound on $\tau_{\mathcal{A}^\tau(h)}$ follows by the analogue of Theorem 2.12 in [BG02] applied to our situation. It tells us that for $\kappa \in (1/2, 2/3)$ and any $t_2 > 0$, there exist constants C, $h_0 > 0$ such that for $h < h_0\tau$ and ε sufficiently small,

$$\mathbb{P}^{\tau,\sqrt{(1-\kappa)\tau}}\left(\tau_{\mathcal{A}^\tau(h)} < t_2\right) \leqslant \frac{C}{\varepsilon^2}\exp\left(-\frac{h^2}{2\sigma^2}\left[1 - \mathcal{O}(\varepsilon) - \mathcal{O}\left(\frac{h}{\tau}\right)\right]\right) , \tag{4.29}$$

where $\mathbb{P}^{\tau,\sqrt{(1-\kappa)\tau}}$ denotes the law of q_t when started from $\sqrt{(1-\kappa)\tau}$ at time τ. The right-hand side becomes small by choosing $h = k\sigma\sqrt{|\ln\sigma|}$ for k large enough, for which we note that $\tau \geqslant \sqrt{\varepsilon}$ by definition so that $h \ll \tau$.

Step Four:

Let us suppose that $t_2 \geqslant 1$. By Step Three, we may write $q(t_2) = \sqrt{t_2} + \mathcal{O}(\sigma\sqrt{|\ln\sigma|}) + \mathcal{O}(\varepsilon)$ independently of $\tau_{\mathcal{K}(\kappa)}$. For a given $q(t_2)$, let q_t^{det} be the corresponding deterministic solution starting at the same point. We can again obtain a similar bound as in Proposition 4.3.9, but for simplicity let us just say that $(9/10)\sqrt{t} \leqslant q_t^{\mathrm{det}} \leqslant (11/10)\sqrt{t}$ for all $t \geqslant 1$. Letting $y_t = q_t - q_t^{\mathrm{det}}$, we define $\tau(\gamma) = \inf\{t \geqslant t_2 : |y_t| > t^{1/2-\gamma}\}$ for $0 < \gamma < 1/2$. We again decompose y_t into a linear part, y_t^0, and nonlinear part, y_t^1, as in (4.19). Then $a(t) = t - 3(q_t^{\mathrm{det}})^2 \asymp -t$ uniformly for all $t \geqslant 1$ and the function $b(y,t)$ containing the nonlinear terms now satisfies $|b(y_t,t)| < M\sqrt{t}\,y_t^2$ for all $t < \tau(\gamma)$ and some constant $M > 0$ independent of t_2. We will show that $\mathbb{P}(\tau(\gamma) < \infty) \to 0$. For $t \leqslant \tau(\gamma)$, we have

$$\begin{aligned}
|y_t^1| &\leqslant \frac{1}{\varepsilon}\int_{t_2}^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\,|b(y_u,u)|\,\mathrm{d}u \\
&\leqslant \frac{Mt^{3/2-2\gamma}}{\varepsilon}\int_{t_2}^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\,\mathrm{d}u \\
&< Ct^{1/2-2\gamma} ,
\end{aligned}$$

where the final inequality holds uniformly in t and the constant $C > 0$ is independent of t_2. Therefore, if $|y_t^0| < H(t)$ for all $t \geqslant t_2$, where $H(t) = t^{1/2-\gamma}(1 - Ct^{-\gamma})$, then we must have $\tau(\gamma) = \infty$. Note that $1 - Ct^{-\gamma} > 0$ and $\dot{H}(t) > 0$ for all $t \geqslant t_2$ by taking t_2 large enough. We have

$$\mathbb{P}\left(\sup_{t \geqslant t_2} \frac{|y_t^0|}{H(t)} > 1\right) \leqslant \sum_{j=0}^{\infty} \mathbb{P}\left(\sup_{s_j \leqslant t \leqslant s_{j+1}} \frac{|y_t^0|}{H(t)} > 1\right) ,$$

where $t_2 = s_0 < s_1 < \dots$ is chosen by $-\alpha(s_{j+1}, s_j) = \varepsilon^2$. Note that, uniformly in j, we have $s_{j+1}^2 - s_0^2 \asymp -\alpha(s_{j+1}, s_0) = -\alpha(s_{j+1}, s_j) - \dots - \alpha(s_1, s_0) = (j+1)\varepsilon^2$, which shows that $s_j \to \infty$ as $j \to \infty$. Call the summand on the right-hand side above P_j. As $H(t)$ is increasing, we can further bound P_j by replacing $H(t)$ with $H(s_j)$. We can also use for $s_j \leqslant t \leqslant s_{j+1}$ the inequality

$$|y_t^0| = \left|\frac{\sigma}{\sqrt{\varepsilon}} \int_{s_0}^{t} \mathrm{e}^{\alpha(t,s)/\varepsilon} \, \mathrm{d}W_s\right| \leqslant \mathrm{e}^{\alpha(s_j)/\varepsilon} \left|\frac{\sigma}{\sqrt{\varepsilon}} \int_{s_0}^{t} \mathrm{e}^{-\alpha(s)/\varepsilon} \, \mathrm{d}W_s\right| .$$

This gives for all $j \geqslant 0$,

$$\begin{aligned}
P_j &\leqslant \mathbb{P}\left(\sup_{s_0 \leqslant t \leqslant s_{j+1}} \left|\int_{s_0}^{t} \mathrm{e}^{-\alpha(s)/\varepsilon} \, \mathrm{d}W_s\right| > \frac{\sqrt{\varepsilon}}{\sigma} \, \mathrm{e}^{-\alpha(s_j)/\varepsilon} H(s_j)\right) \\
&\leqslant 2\exp\left(-\frac{\varepsilon \, \mathrm{e}^{2\alpha(s_{j+1},s_j)/\varepsilon} H(s_j)^2}{2\sigma^2 \int_{s_0}^{s_{j+1}} \mathrm{e}^{2\alpha(s_{j+1},s)/\varepsilon} \, \mathrm{d}s}\right) \\
&\leqslant 2\exp\left(-\frac{c_1 s_j H(s_j)^2}{2\sigma^2}\right) ,
\end{aligned}$$

where the constant $c_1 > 0$ in the final inequality is independent of j. Note that the second inequality comes from (3.14) and the final inequality uses that $\alpha(s_{j+1}, s) \asymp -(s_{j+1}^2 - s^2)$ uniformly for all $s_0 \leqslant s \leqslant s_{j+1}$ and all j. Summing over $j \geqslant 1$ and using that $s_j - s_{j-1} \geqslant C\varepsilon^2/s_j$ uniformly in j, we have

$$\begin{aligned}
\sum_{j=1}^{\infty} P_j &= \sum_{j=1}^{\infty} P_j \frac{s_j - s_{j-1}}{s_j - s_{j-1}} \\
&\leqslant \frac{C}{\varepsilon^2} \sum_{j=1}^{\infty} P_j s_j (s_j - s_{j-1}) \\
&\leqslant C \int_{t_2}^{\infty} s P(s) \, \mathrm{d}s ,
\end{aligned}$$

where

$$P(s) = 2\exp\left(-\frac{c_1 s H(s)^2}{2\sigma^2}\right) \leqslant 2\exp\left(-\frac{c_2 s^{2(1-\gamma)}}{\sigma^2}\right)$$

and $c_2 > 0$ is a constant. We have used that $sP(s)$ is decreasing when bounding the series by the integral above. Then

$$\int_{t_2}^{\infty} sP(s)\,\mathrm{d}s \;\leqslant\; C\sigma^2 \exp\left(-\frac{c_2 t_2^{2(1-\gamma)}}{\sigma^2}\right)$$

for some constant $C > 0$ depending on t_2 and γ, so that

$$\mathbb{P}\left(\sup_{t \geqslant t_2} \frac{|y_t^0|}{H(t)} > 1\right) \;\leqslant\; P_0 + \frac{C\sigma^2}{\varepsilon^2} \exp\left(-\frac{c_2 t_2^{2(1-\gamma)}}{\sigma^2}\right)$$

and the right-hand side tends to zero as $\sigma \downarrow 0$.

4.3.2 Slow Pulling

We now consider the slow pulling regime from Proposition 4.3.2. In this case, the noise dominates the dynamics and cancels out the asymmetry caused by pulling. The process q_t should, therefore, behave similarly to $\tilde{q}_t$, where

$$\mathrm{d}\tilde{q}_t = \frac{1}{\varepsilon}\left(t\tilde{q}_t - \tilde{q}_t^3\right)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t\;, \qquad \tilde{q}(-T) = 0\;. \tag{4.30}$$

As we have chosen $\tilde{q}(-T) = 0$, the law of $\tilde{q}$ is entirely symmetric about zero. The strategy is as follows:

1. Recall from [BG02, BG06] that $\tilde{q}_t$ stays close to the origin with high probability. At time $t = \sqrt{\varepsilon}$, its typical spreading is of order $\sigma\varepsilon^{-1/4}\sqrt{|\ln\sigma|}$.
2. Show that paths of q_t stay close to those of $\tilde{q}_t$ until $\tilde{q}_t$ leaves the diffusion-dominated strip $\mathcal{S}(h)$, defined in (4.31) below.
3. Show that q_t then exits the slightly larger strip $\mathcal{K}(\kappa)$, defined previously in (4.23), without returning to the origin.
4. Show finally that q_t then falls into the potential well on the same side as it left $\mathcal{K}(\kappa)$ and remains there.

Step One:

This step is the same as Step One from the previous section, except now we are analysing the behaviour of $\tilde{q}_t$. Unlike in the previous section, we can use directly the results of [BG02, BG06], which we now summarise. We again define the function

$\xi(t)$ as in the last section and now $a(t) := t - 3(\tilde{q}_t^{\mathrm{det}})^2$, where

$$\dot{\tilde{q}}_t^{\mathrm{det}} = \frac{1}{\varepsilon}(t\tilde{q}_t^{\mathrm{det}} - (\tilde{q}_t^{\mathrm{det}})^3) , \qquad \tilde{q}_{-T}^{\mathrm{det}} = 0 .$$

Clearly, $\tilde{q}^{\mathrm{det}} \equiv 0$ and so now $a(t) \equiv t$. Again, $\xi(t) \asymp 1/(|t| \wedge \sqrt{\varepsilon})$ for $-T \leqslant t \leqslant \sqrt{\varepsilon}$. We define the space-time domain

$$\mathcal{B}(h) = \{(\tilde{q}, t) : -T \leqslant t \leqslant \sqrt{\varepsilon}, |\tilde{q}| < h\sqrt{\xi(t)}\}$$

and the stopping time $\tau_{\mathcal{B}(h)} = \inf\{t \geqslant -T : (\tilde{q}_t, t) \notin \mathcal{B}(h)\}$. Applying Theorem 2.10 from [BG02], we see that there exist constants C, $h_0 > 0$ such that for ε sufficiently small and $h \leqslant h_0\sqrt{\varepsilon}$,

$$\mathbb{P}(\tau_{\mathcal{B}(h)} < \sqrt{\varepsilon}) \leqslant \frac{C}{\varepsilon^2} \exp\left(-\frac{h^2}{2\sigma^2}\left[1 - \mathcal{O}(\sqrt{\varepsilon}) - \mathcal{O}\left(\frac{h^2}{\varepsilon}\right)\right]\right) .$$

Choosing $h = k\sigma\sqrt{|\ln \sigma|}$ for k large enough, the right-hand side tends to zero. At time $\sqrt{\varepsilon}$, we may take $|\tilde{q}(\sqrt{\varepsilon})| = \mathcal{O}(\sigma\varepsilon^{-1/4}\sqrt{|\ln \sigma|})$.

Step Two:

In the fast pulling section, we saw that at time $\sqrt{\varepsilon}$, $q(\sqrt{\varepsilon}) \asymp \sqrt{\varepsilon}$, from which we could then show its subsequent exit from $\mathcal{K}(\kappa)$. Now we are considering the exit of $\tilde{q}_t$ from $\mathcal{K}(\kappa)$, with values of $\tilde{q}(\sqrt{\varepsilon})$ as found in Step One. For this, we must first consider the exit of $\tilde{q}_t$ from a smaller strip. We define for $t \geqslant \sqrt{\varepsilon}$ the diffusion-dominated strip

$$\mathcal{S}(h) = \left\{(\tilde{q}, t) : t \geqslant \sqrt{\varepsilon}, |\tilde{q}| < \frac{h}{\sqrt{t}}\right\} \tag{4.31}$$

and the stopping time $\tau_{\mathcal{S}(h)} = \inf\{t \geqslant \sqrt{\varepsilon} : (\tilde{q}_t, t) \notin \mathcal{S}(h)\}$. See Figure 3 in [BG02] and Figure 3.12 in [BG06] for an illustration of $\mathcal{S}(h)$ and $\mathcal{K}(\kappa)$ (note that $\mathcal{K}(\kappa)$ is denoted $\mathcal{D}(\kappa)$ in [BG02]). Let $h^* := h_0\,\sigma\sqrt{|\ln \sigma|}$, where $h_0 > 0$ is a constant sufficiently large so that $(\tilde{q}(\sqrt{\varepsilon}), \sqrt{\varepsilon}) \in \mathcal{S}(h^*)$. Applying Proposition 4.7 from [BG02] with the choices $h = h^*$ and $\mu = 2$, we see that there exists $C > 0$ such that for all σ sufficiently small and all initial conditions $(q_0, \sqrt{\varepsilon}) \in \mathcal{S}(h^*)$,

$$\mathbb{P}\left(\tau_{\mathcal{S}(h^*)} \geqslant t\right) \leqslant \left(\frac{h^*}{\sigma}\right)^2 \exp\left(-\frac{(t^2 - \varepsilon)}{3\varepsilon}\left[1 - \mathcal{O}\left(\frac{1}{\ln(h^*/\sigma)}\right)\right]\right) ,$$

as long as $\sigma|\ln \sigma|^{3/2} = \mathcal{O}(\sqrt{\varepsilon})$, which we already assume in the present slow pulling regime. We can check that by taking $t = \sqrt{2k\varepsilon\ln(h^*/\sigma)}$ with $k > 0$ sufficiently large, the right-hand side tends to zero. For such a choice of k, we define $t^* = \sqrt{2k\varepsilon \ln(h^*/\sigma)}$, and henceforth only consider cases where $\tau_{\mathcal{S}(h^*)} \leqslant t^*$.

The important point here is that, by symmetry, $\tilde{q}_t$ exits $\mathcal{S}(h^*)$ through either boundary with equal probability. Now that we understand the behaviour of $\tilde{q}_t$ up until its exit from $\mathcal{S}(h^*)$, we turn our attention to q_t. The following lemma shows that q_t is close to $\tilde{q}_t$ at time $\tau_{\mathcal{S}(h^*)}$.

Lemma 4.3.11. *Let q_t solve (4.4) with any initial condition $q(-T) = x \in \mathcal{X}$, and suppose $\varepsilon(\sigma) \ll \sigma^{4/3}|\ln\sigma|^{-13/6}$. Then for almost all paths in the set*

$$\{(\tilde{q}(\sqrt{\varepsilon}), \sqrt{\varepsilon}) \in \mathcal{S}(h^*),\, \tau_{\mathcal{S}(h^*)} \leqslant t^*\}\ ,$$

we have

$$q_{\tau_{\mathcal{S}(h^*)}} = \tilde{q}_{\tau_{\mathcal{S}(h^*)}}\left[1 + \mathcal{O}\left(\frac{\varepsilon^{3/4}|\ln\sigma|^{13/8}}{\sigma}\right)\right]\ .$$

The proof of this lemma is based on the following simple comparison of q_t and $\tilde{q}_t$.

Lemma 4.3.12. *Let q_t solve (4.4) with initial condition $q(-T) = x \in \mathcal{X}$ and let $\tilde{q}_t$ solve (4.30). We have, almost surely, for all $t \geqslant -T$,*

$$\tilde{q}_t + x\,\mathrm{e}^{(t^2-T^2)/2\varepsilon} \leqslant q_t \leqslant \tilde{q}_t + \int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,\mathrm{d}s \tag{4.32}$$

if $x \leqslant 0$, and

$$\tilde{q}_t \leqslant q_t \leqslant \tilde{q}_t + \int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,\mathrm{d}s + x\,\mathrm{e}^{(t^2-T^2)/2\varepsilon} \tag{4.33}$$

if $x > 0$.

Proof. Solving (4.4) gives

$$\begin{aligned} q_t = x\,\mathrm{e}^{(t^2-T^2)/2\varepsilon} &+ \int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,\mathrm{d}s \\ &- \frac{1}{\varepsilon}\int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,q_s^3\,\mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}}\int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,\mathrm{d}W_s\ . \end{aligned} \tag{4.34}$$

By the comparison principle for SDEs, $q_t \geqslant \hat{q}_t$ almost surely, where $\hat{q}_t$ solves

$$\mathrm{d}\hat{q}_t = \frac{1}{\varepsilon}(t\hat{q}_t - \hat{q}_t^3)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t\ , \qquad \hat{q}(-T) = x\ .$$

Using this lower bound for q in the second integral on the right-hand side of (4.34) gives

$$q_t \leqslant \hat{q}_t + \int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon} \, \mathrm{d}s \, .$$

When $x \leqslant 0$, we have $\hat{q}_t \leqslant \tilde{q}_t$ almost surely, which gives the upper bound in (4.32). For the lower bound, we have

$$\begin{aligned}
\hat{q}_t &= x\,\mathrm{e}^{(t^2-T^2)/2\varepsilon} - \frac{1}{\varepsilon}\int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,\hat{q}_s^3 \,\mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}}\int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,\mathrm{d}W_s \\
&\geqslant x\,\mathrm{e}^{(t^2-T^2)/2\varepsilon} - \frac{1}{\varepsilon}\int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,\tilde{q}_s^3 \,\mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}}\int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon}\,\mathrm{d}W_s \\
&= x\,\mathrm{e}^{(t^2-T^2)/2\varepsilon} + \tilde{q}_t \, .
\end{aligned}$$

The case $x \geqslant 0$ is easier and does not involve $\hat{q}_t$. It follows along similar lines. □

Proof of Lemma 4.3.11. For all $\sqrt{\varepsilon} \leqslant t \leqslant t^*$,

$$\int_{-T}^{t} \mathrm{e}^{(t^2-s^2)/2\varepsilon} \,\mathrm{d}s \leqslant C_1\sqrt{\varepsilon}\left(\frac{h^*}{\sigma}\right)^4 \leqslant C_2\sqrt{\varepsilon}\,|\ln\sigma|^2 \, , \tag{4.35}$$

and for all $x \in \mathcal{X}$,

$$|x|\,\mathrm{e}^{(t^2-T^2)/2\varepsilon} \leqslant C_1\left(\frac{h^*}{\sigma}\right)^4 \mathrm{e}^{-T^2/2\varepsilon} \leqslant C_2\,|\ln\sigma|^2\,\mathrm{e}^{-T^2/2\varepsilon} \, .$$

Of these two estimates, (4.35) gives the larger upper bound. Next observe that

$$\tau_{\mathcal{S}(h^*)} \leqslant t^* \leqslant C\varepsilon^{1/2}|\ln\sigma|^{1/4} \, .$$

Therefore,

$$\frac{1}{|\tilde{q}_{\tau_{\mathcal{S}(h^*)}}|} = \frac{\sqrt{\tau_{\mathcal{S}(h^*)}}}{h^*} \leqslant C\frac{\varepsilon^{1/4}}{\sigma|\ln\sigma|^{3/8}} \, .$$

Using (4.35) and the above inequality together with Lemma 4.3.12 gives the result. □

Step Three:

Now we analyse the behaviour of q_t, rather than $\tilde{q}_t$, for $t > \tau_{\mathcal{S}(h^*)}$. As we saw in the fast pulling case, if q_t starts from $q_0 > 0$ at time $t_0 \geqslant \sqrt{\varepsilon}$, where $(q_0, t_0) \in \mathcal{K}(\kappa)$, then $q_t \geqslant q_t^\kappa$ as long as $0 < q_t < \sqrt{(1-\kappa)t}$, where q_t^κ was defined in (4.24). Unlike there, we now have to also consider negative initial conditions. We would like a bound of the form $q_t \leqslant q_t^\kappa$ in such cases, but now the bias of q_t in the positive direction makes this more difficult. In order to obtain a corresponding comparison with q_t^κ, we need an additional assumption on t_0 and q_0 as set out in the following lemma.

Lemma 4.3.13. *Suppose that at time $t_0 \geqslant \sqrt{\varepsilon}$, q_t starts at $q_0 < 0$, where $(q_0, t_0) \in \mathcal{K}(\kappa)$ and $|q_0| \gg \varepsilon/t_0$. Then we have $q_t \leqslant q_t^\kappa$ as long as $-\sqrt{(1-\kappa)t} \leqslant q_t \leqslant 0$ and ε is sufficiently small.*

Proof. For $-\sqrt{(1-\kappa)t_0} < q_0 < 0$, it is certainly true by comparison of the drift and initial conditions that q_t is bounded above by solutions of

$$\mathrm{d}z_t^\kappa = \frac{1}{\varepsilon}(\kappa t z_t^\kappa + \varepsilon)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t\,, \qquad z_{t_0}^\kappa = q_0\,,$$

as long as $-\sqrt{(1-\kappa)t} \leqslant q_t \leqslant 0$. The result follows since

$$\begin{aligned}
z_t^\kappa &= q_0\,\mathrm{e}^{\kappa(t^2-t_0^2)/2\varepsilon} + \int_{t_0}^t \mathrm{e}^{\kappa(t^2-s^2)/2\varepsilon}\,\mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}}\int_{t_0}^t \mathrm{e}^{\kappa(t^2-s^2)/2\varepsilon}\,\mathrm{d}W_s \\
&\leqslant (q_0 + C\varepsilon/t_0)\,\mathrm{e}^{\kappa(t^2-t_0^2)/2\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}}\int_{t_0}^t \mathrm{e}^{\kappa(t^2-s^2)/2\varepsilon}\,\mathrm{d}W_s \\
&\leqslant \frac{q_0}{2}\,\mathrm{e}^{\kappa(t^2-t_0^2)/2\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}}\int_{t_0}^t \mathrm{e}^{\kappa(t^2-s^2)/2\varepsilon}\,\mathrm{d}W_s \\
&= q_t^\kappa\,.
\end{aligned}$$

□

This lemma shows that, under suitable conditions, we may compare q_t and q_t^κ for both postive and negative initial conditions q_0. For $q_0 < 0$ we get analogous bounds to those in Lemmas 4.3.7 and 4.3.8. In the previous section, we applied those lemmas with $t_0 = \sqrt{\varepsilon}$ and $q_0 \asymp \sqrt{\varepsilon}$. We now apply these lemmas with $t_0 = \tau_{\mathcal{S}(h^*)}$ and $|q_0| \asymp h^*/\sqrt{\tau_{\mathcal{S}(h^*)}}$ (see Lemma 4.3.11). Note that if $\sqrt{\varepsilon} \leqslant \tau_{\mathcal{S}(h^*)} \leqslant t^*$ then $\varepsilon/t_0 \ll |q_0|$, and so the conditions of Lemma 4.3.13 are satisfied. We obtain a bound similar to (4.25) and again see that $\mathcal{K}(\kappa)$ is left by a time of order $\sqrt{\varepsilon|\ln\sigma|}$.

Step Four:

When q_t exits $\mathcal{K}(\kappa)$ on the positive side, this part is exactly the same as Step Three from the fast pulling section. The other case when q_t exits $\mathcal{K}(\kappa)$ on the negative side is similar. Firstly, we introduce $q_-^*(t)$, another real-valued solution of (4.16) existing for $t \geqslant \sqrt{\varepsilon}$ and satisfying the bounds $-\sqrt{t} \leqslant q_-^*(t) \leqslant -\sqrt{t} + \varepsilon/t$ and

$(q^*_-)'(t) < 0$ for all such t and ε small. Note there is also a third real-valued solution of (4.16) between q^*_- and q^*_+, which is an unstable equilibrium branch. By taking ε small, we have $q^*_-(t) < -\sqrt{(1-\kappa)t}$ for all $t \geqslant \sqrt{\varepsilon}$. Again writing $\tau = \tau_{\mathcal{K}(\kappa)}$, we need to check that the deterministic solution $q_t^{\det,\tau}$ of (4.26), with initial condition $q^{\det,\tau}(\tau) = -\sqrt{(1-\kappa)\tau}$, satisfies a bound as in Proposition 4.3.9 of the form

$$0 \leqslant q_t^{\det,\tau} - q^*_-(t) \leqslant C\left(\frac{\varepsilon}{t^{3/2}} + (q^{\det,\tau}(\tau) - q^*(\tau))\,\mathrm{e}^{-\eta(t^2-\tau^2)/2\varepsilon}\right)$$

for all $t \geqslant \tau$. The main thing to ensure is that $q_t^{\det,\tau}$, which has a bias in the positive direction, does not re-enter the set $\mathcal{K}(\kappa)$ after having left. To see that this is indeed the case, first note that the derivative with respect to t of the boundary curve $-\sqrt{(1-\kappa)t}$ is given by $-\frac{1}{2}t^{-1/2}\sqrt{1-\kappa}$. The derivative of $q_t^{\det,\tau}$ when on the boundary of $\mathcal{K}(\kappa)$ is given by $\frac{1}{\varepsilon}t^{3/2}(-\kappa+\varepsilon)\sqrt{1-\kappa}$. Using that $t \geqslant \sqrt{\varepsilon}$, we see that the inequality

$$\frac{1}{\varepsilon}t^{3/2}(-\kappa+\varepsilon)\sqrt{1-\kappa} < -\frac{1}{2}t^{-1/2}\sqrt{1-\kappa}$$

holds when $\kappa > 1/2 + \varepsilon$, which is true for all $\kappa > 1/2$ by taking ε sufficiently small. Having established that $q^*_-(t) \leqslant q_t^{\det,\tau} \leqslant -\sqrt{(1-\kappa)t}$ for all $t \geqslant \tau$ and ε sufficiently small, the rest of the proof follows like Proposition 4.11 from [BG02]. The subsequent estimate (4.29) above showing the concentration of q_t in the set $\mathcal{A}^\tau(h)$ then follows, where $\mathcal{A}^\tau(h)$ was defined in (4.28). Finally, we can show as in Step Four from the fast pulling section that q_t stays in a neighbourhood of $-\sqrt{t}$ for all $t \geqslant t_2$.

4.4 The Full Solution

We now consider the full equation (4.2) and show that for sufficiently small mass (large β), it behaves like the overdamped solution of the previous section. The general strategy is as in Section 4.3, namely we first show that if we start the system at the origin at time $s \ll -1$, then the solution at time -1 belongs to a suitable set. This is done in the following two propositions. We then provide a result that is uniform over all solutions starting from the set in question. The first step is achieved by the following statement.

Proposition 4.4.1. *Let* $\hat{\mathcal{X}} = [-\varepsilon^{1/2-\beta}, \varepsilon^{1/2-\beta}]$ *and* $\hat{\mathcal{V}} = [-\varepsilon^{-(1+3\beta)/2}, \varepsilon^{-(1+3\beta)/2}]$.

Let $T \geqslant 1$ be a constant and let q_t solve (4.2) with $\beta > 1/2$. Then we have

$$\lim_{\sigma,\varepsilon\to 0} \liminf_{s\to-\infty} \mathbb{P}^s\big(q_{-2T} \in \hat{\mathcal{X}},\, p_{-2T} \in \hat{\mathcal{V}}\big) = 1 \;.$$

Proof. We fix some arbitrary starting time $s < -2T$ and we consider the solution to (4.2) with initial condition $q_s = p_s = 0$. We define the function $\Psi(p, q, t)$ by

$$\Psi(p,q,t) = \frac{\varepsilon^\beta}{2}p^2 - \frac{t}{2\varepsilon}q^2 + \frac{1}{4\varepsilon}q^4 - q + \frac{1}{2}pq \;.$$

Applying Itô's formula to $\Psi(p_t, q_t, t)$, we obtain

$$\mathrm{d}\Psi(p_t,q_t,t) \;\leqslant\; \Big(-\varepsilon^{-\beta}\Psi(p_t,q_t,t) + \frac{1}{2}\sigma^2\varepsilon^{-1-\beta} - \frac{1}{2\varepsilon}q_t^2 - \frac{1}{2\varepsilon^\beta}q_t\Big)\,\mathrm{d}t + \mathrm{d}M(t) \;,$$

where M is some continuous martingale. Using the inequality $\varepsilon^{\beta-1}q^2 + q \geqslant -\varepsilon^{1-\beta}/4$, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\Psi(p_t,q_t,t) \;\leqslant\; -\varepsilon^{-\beta}\mathbb{E}\Psi(p_t,q_t,t) + \frac{1}{2}\sigma^2\varepsilon^{-1-\beta} + \frac{1}{8}\varepsilon^{1-2\beta} \;.$$

This tells us that $\mathbb{E}\Psi(p_t,q_t,t) \;\leqslant\; \sigma^2/(2\varepsilon) + \varepsilon^{1-\beta}/8$. For $t \;\leqslant\; -2T \;\leqslant\; -2$, we have

$$\Psi(p,q,t) \;\geqslant\; \frac{\varepsilon^\beta}{4}p^2 + \frac{1}{\varepsilon}q^2 + \frac{1}{4\varepsilon}q^4 - \frac{1}{4\varepsilon^\beta}q^2 - q \;\geqslant\; \frac{\varepsilon^\beta}{4}p^2 + \frac{1}{2\varepsilon}q^2 - \varepsilon - \frac{1}{16}\varepsilon^{1-2\beta} \;,$$

where we used the inequality $2pq \;\geqslant\; -\varepsilon^\beta p^2 - \varepsilon^{-\beta}q^2$, which tells us that

$$\begin{aligned}
\mathbb{E}(q_t^2) &\;\leqslant\; 2\varepsilon^2 + \varepsilon^{2-2\beta}/8 + \sigma^2 + \varepsilon^{2-\beta}/4 \;,\\
\mathbb{E}(p_t^2) &\;\leqslant\; 4\varepsilon^{1-\beta} + \varepsilon^{1-3\beta}/4 + 2\sigma^2\varepsilon^{-1-\beta} + \varepsilon^{1-2\beta}/2 \;.
\end{aligned}$$

The result then follows from Chebyshev's inequality, noting that $\beta > 1/2$. □

Now we use the previous proposition to restart the process at time $-2T$. We denote by $\mathbb{P}^{\hat{x},\hat{v}}$ the law of the solution of (4.2) starting at time $-2T$ with $q_{-2T} = \hat{x}$, $p_{-2T} = \hat{v}$.

Proposition 4.4.2. *Let $\mathcal{X} = [-1, 1]$ and $\mathcal{V} = [-\varepsilon^{-\beta}, \varepsilon^{-\beta}]$. Let $T \geqslant 1$ be a constant and let q_t solve (4.2) with $\beta > 2$. Then we have*

$$\lim_{\sigma,\varepsilon\to 0} \inf_{\hat{x}\in\hat{\mathcal{X}},\hat{v}\in\hat{\mathcal{V}}} \mathbb{P}^{\hat{x},\hat{v}}\big(q_{-T} \in \mathcal{X},\, p_{-T} \in \mathcal{V}\big) = 1 \;.$$

Proof. Let

$$\Psi(p,q) = \frac{1}{4}q^2 + \frac{\varepsilon^\beta}{2}pq + \frac{\varepsilon^{2\beta}}{2}p^2 + \frac{\varepsilon^{\beta-1}}{4}q^4 \; .$$

Applying Itô's formula to $\Psi(p_t, q_t)$, we obtain

$$\mathrm{d}\Psi(p_t,q_t) = \frac{1}{\varepsilon}\left(-\frac{\varepsilon^{\beta+1}}{2}p_t^2 + \frac{t}{2}q_t^2 - \frac{1}{2}q_t^4 + \frac{\varepsilon}{2}q_t + \varepsilon^\beta t p_t q_t + \varepsilon^{\beta+1}p_t + \frac{\sigma^2}{2}\right)\mathrm{d}t + \mathrm{d}M(t) \; ,$$

where M is some continuous martingale. Using the inequality $2pq \leqslant \varepsilon^2p^2 + \varepsilon^{-2}q^2$, and that $t \in [-2T, -T]$ and $\beta > 2$, we have for sufficiently small ε,

$$\mathrm{d}\Psi(p_t,q_t) \leqslant \frac{1}{\varepsilon}\left(-\frac{1}{2}\Psi(p_t,q_t) + 4\varepsilon^{\beta+1} + \frac{\varepsilon^2}{2} + \frac{\sigma^2}{2}\right)\mathrm{d}t + \mathrm{d}M(t) \; .$$

It follows that $\mathbb{E}\Psi(p_t,q_t) \leqslant \mathrm{e}^{-(t+2T)/2\varepsilon}\,\mathbb{E}\Psi(p_{-2T},q_{-2T}) + \sigma^2 + 8\varepsilon^{\beta+1} + \varepsilon^2$. We can then use the bounds $\Psi(p,q) \geqslant \varepsilon^{2\beta}p^2/4$ and $\Psi(p,q) \geqslant q^2/8$, along with Chebyshev's inequality, to obtain the result. □

We would like to use a singular perturbation approach to show that for $t \geqslant -T$ and suitably large β, sample paths of q_t can be approximated by those of an overdamped equation starting at $-T$. For this, we need to first consider a process that is similar to q_t but has better regularity. To this end, we introduce the processes Q and P that solve

$$\begin{aligned} \mathrm{d}Q_t &= P_t\,\mathrm{d}t \; , \qquad Q_{-T} = P_{-T} = 0 \; , \\ \varepsilon^\beta \mathrm{d}P_t &= -P_t\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \; . \end{aligned} \tag{4.36}$$

We find that

$$P_t = \sigma\varepsilon^{-1/2-\beta}\int_{-T}^t \mathrm{e}^{-(t-s)\varepsilon^{-\beta}}\,\mathrm{d}W_s$$

and

$$Q_t = \frac{\sigma}{\sqrt{\varepsilon}}W_t - \varepsilon^\beta P_t = \sigma\varepsilon^{-1/2-\beta}\int_{-T}^t \mathrm{e}^{-(t-s)\varepsilon^{-\beta}}\,W(s)\,\mathrm{d}s \; .$$

For δ, $t_2 > 0$, define the two events E_1 and E_2 by

$$\begin{aligned} E_1 &= \{|Q_t| > \sigma\varepsilon^{-1/2-\delta} \text{ for some } t \in [-T, t_2]\} \; , \\ E_2 &= \{|P_t| > \sigma\varepsilon^{-1/2-\beta/2-\delta} \text{ for some } t \in [-T, t_2]\} \; . \end{aligned}$$

The following lemma shows that these events are unlikely.

Lemma 4.4.3. *There exists $C > 0$ such that for all $\delta > 0$ and all $t_2 > 0$,*

$$\mathbb{P}(E_1 \cup E_2) \leqslant Ct_2(\mathrm{e}^{-\varepsilon^{-\delta/(t_2+T)}} + \mathrm{e}^{-\varepsilon^{-2\delta}/2})$$

holds for σ, $\varepsilon > 0$ sufficiently small.

Proof. Since $Q_t = \sigma\varepsilon^{-1/2-\beta} \int_{-T}^{t} \mathrm{e}^{-(t-s)\varepsilon^{-\beta}} W(s)\, \mathrm{d}s$, it follows that

$$\begin{aligned}\mathbb{P}(E_1) &\leqslant \mathbb{P}\left(|W_t| > \varepsilon^{-\delta} \text{ for some } t \in [-T, t_2]\right) \\ &\leqslant Ct_2\, \mathrm{e}^{-\varepsilon^{-\delta/(t_2+T)}} .\end{aligned}$$

where $C > 0$ is independent of δ and t_2. We also find as in the proof of Lemma 3.2.3 that there exists $C > 0$, independent of δ and t_2, such that for ε sufficiently small,

$$\mathbb{P}(E_2) \leqslant Ct_2\varepsilon^{-2\delta-1} \exp\left(-\frac{1}{2}\varepsilon^{-2\delta}\right) .$$

The result follows by combining these two estimates. □

Now let $y_t = q_t - Q_t$. It solves, almost surely, the second-order ODE

$$\varepsilon^{\beta}\ddot{y} = -\dot{y} + \frac{1}{\varepsilon}(t(y_t + Q_t) - (y_t + Q_t)^3 + \varepsilon) , \quad y(-T) = x , \dot{y}(-T) = v .$$

The following proposition shows that for almost all paths in $(E_1 \cup E_2)^c$, y may be approximated by the solution of a first-order ODE. Note that the condition on β is a little stronger than necessary, but is required later on in this section.

Proposition 4.4.4. *For all $t_2 > 1$, there exists $C = C(t_2) > 0$ such that for all $\beta > 2$, all $0 < \delta < \beta/2 - 1$, all $\sigma^{2/(1+2\delta)} \ll \varepsilon \ll 1$ and almost all paths in $(E_1 \cup E_2)^c$,*

$$\left|\dot{y} - \frac{1}{\varepsilon}(t(y_t + Q_t) - (y_t + Q_t)^3 + \varepsilon)\right| \leqslant C \max\{\varepsilon^{\beta-2-\delta}, \sigma\varepsilon^{-3/2+\beta/2-2\delta}\}$$

for all $t \in [-T + 2\varepsilon^{\beta-\delta}, t_2]$ and σ sufficiently small.

Proof. If we write $z = \dot{y}$, then almost surely the pair (y, z), which is differentiable, solves

$$\begin{aligned}\dot{y} &= z , \\ \varepsilon^{\beta}\dot{z} &= -z + \frac{1}{\varepsilon}g(t, y_t + Q_t) ,\end{aligned}$$

where $g(t, y_t + Q_t) = t(y_t + Q_t) - (y_t + Q_t)^3 + \varepsilon$. For $t > 0$, we have $g(t, \sqrt{t}) \approx 0$ so that we do not expect $y_t + Q_t$, or indeed y_t, to be much larger than $\sqrt{t}$. Therefore, we let $\tau = \inf\{t \geqslant -T : |y_t| > 2\sqrt{t_2}\}$. On $(E_1 \cup E_2)^c$, there is $C > 0$ depending on t_2 such that for all $-T \leqslant t \leqslant \tau \wedge t_2$, $|g(t, y_t + Q_t)| < C$. We solve the equation for z to give

$$z_t = v\,\mathrm{e}^{-(t+T)\varepsilon^{-\beta}} + \varepsilon^{-(1+\beta)} \int_{-T}^{t} \mathrm{e}^{-(t-s)\varepsilon^{-\beta}} g(s, y_s + Q_s)\,\mathrm{d}s \tag{4.37}$$

almost surely, from which we deduce that $|z_t| \leqslant \varepsilon^{-\beta}\,\mathrm{e}^{-(t+T)\varepsilon^{-\beta}} + C/\varepsilon$ for all $t \leqslant \tau \wedge t_2$. This immediately shows that for $-T \leqslant t \leqslant (-T + 2\varepsilon^{\beta-\delta}) \wedge \tau$ and sufficiently small ε,

$$|y_t - x| \leqslant 1 + C\varepsilon^{\beta-\delta-1}\,, \tag{4.38}$$

and so $\tau > -T + 2\varepsilon^{\beta-\delta}$.

For $-T + \varepsilon^{\beta-\delta} \leqslant t \leqslant \tau \wedge t_2$ (note we have written $\varepsilon^{\beta-\delta}$, not $2\varepsilon^{\beta-\delta}$), we have $|z_t| < C/\varepsilon$. Furthermore, for such t we find

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} g(t, y_t + Q_t)\right| \leqslant C \max\{\varepsilon^{-1}, \sigma\varepsilon^{-1/2-\beta/2-\delta}\}\,.$$

Now we apply the Laplace method to the integral in (4.37). For $t \geqslant -T + 2\varepsilon^{\beta-\delta}$, decompose the integral as

$$\int_{-T}^{t} = \int_{-T}^{t-\varepsilon^{\beta-\delta}} + \int_{t-\varepsilon^{\beta-\delta}}^{t}\,.$$

Then, by the boundedness of g, we have

$$\left|\int_{-T}^{t-\varepsilon^{\beta-\delta}} \mathrm{e}^{-(t-s)\varepsilon^{-\beta}} g(s, y_s + Q_s)\,\mathrm{d}s\right| < C\,\mathrm{e}^{-\varepsilon^{-\delta}}\,.$$

For the remaining integral, we use a Taylor expansion of g to give

$$g(s, y_s + Q_s) \leqslant g(t, y_t + Q_t) + C(t-s)\max\{\varepsilon^{-1}, \sigma\varepsilon^{-1/2-\beta/2-\delta}\}\,.$$

Then

$$\begin{aligned}\int_{t-\varepsilon^{\beta-\delta}}^{t} \mathrm{e}^{-(t-s)\varepsilon^{-\beta}} g(s, y_s + Q_s)\,\mathrm{d}s \leqslant\; & \varepsilon^{\beta} g(t, y_t + Q_t) + \\ & + C\varepsilon^{\beta}\,\mathrm{e}^{-\varepsilon^{-\delta}} + C\max\{\varepsilon^{2\beta-1-\delta}, \sigma\varepsilon^{-1/2+3\beta/2-2\delta}\}\,,\end{aligned}$$

which tells us that for $-T+2\varepsilon^{\beta-\delta} \leqslant t \leqslant \tau \wedge t_2$ and σ sufficiently small,

$$z_t \leqslant \frac{1}{\varepsilon} g(t, y_t + Q_t) + C \max\{\varepsilon^{\beta-2-\delta}, \sigma\varepsilon^{-3/2+\beta/2-2\delta}\} . \tag{4.39}$$

In a similar way, we can also show that

$$z_t \geqslant \frac{1}{\varepsilon} g(t, y_t + Q_t) - C \max\{\varepsilon^{\beta-2-\delta}, \sigma\varepsilon^{-3/2+\beta/2-2\delta}\} . \tag{4.40}$$

We will now show that the assumption $\tau \leqslant t_2$ leads to a contradiction. For this, we will show that if $y_\tau = +2\sqrt{t_2}$, then the right-hand side of (4.39) is strictly negative, whereas we should have $z_\tau \geqslant 0$ by continuity. The case $y_\tau = -2\sqrt{t_2}$ is similar. First, we note that if $y_\tau = +2\sqrt{t_2}$, then there are constants C_1, $C_2 > 0$, depending on t_2, such that

$$g(\tau, y_\tau + Q_\tau) \leqslant -C_1 + C_2\sigma\varepsilon^{-1/2-\delta} ,$$

and for σ sufficiently small the right-hand side is strictly negative and bounded away from zero. Then the conditions on ε, β and δ guarantee that the right-hand side of (4.39) is strictly negative. This means that we must have $\tau > t_2$ and so (4.39) and (4.40) hold for all $-T+2\varepsilon^{\beta-\delta} \leqslant t \leqslant t_2$, from which the result follows. □

Now we will use Proposition 4.4.4 to tell us something about the SDE (4.2).

Proposition 4.4.5. *For all $t_2 > 1$, there exists $C = C(t_2) > 0$ such that for all $\beta > 2$, all $0 < \delta < \beta/2 - 1$, all $\sigma^{2/(1+2\delta)} \ll \varepsilon \ll 1$ and almost all paths in $(E_1 \cup E_2)^c$,*

$$q_t^- \leqslant q_t + \varepsilon^\beta P_t \leqslant q_t^+$$

for all $t \in [-T+2\varepsilon^{\beta-\delta}, t_2]$ and σ sufficiently small, where

$$\mathrm{d}q_t^\pm = \frac{1}{\varepsilon}\left[t q_t^\pm - (q_t^\pm)^3 + \varepsilon(1 \pm r(\sigma))\right] \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \mathrm{d}W_t , \quad q^\pm(-T+2\varepsilon^{\beta-\delta}) = x \pm 3 , \tag{4.41}$$

with W_t the same Brownian motion appearing in (4.2) and

$$r(\sigma) = C \max\{\varepsilon^{\beta-2-\delta}, \sigma\varepsilon^{-3/2+\beta/2-2\delta}\} .$$

Proof of Theorem 4.4.5. We will show the upper bound. The lower bound is similar. Letting $t_0 = -T+2\varepsilon^{\beta-\delta}$, we know by Proposition 4.4.4 that there exists $C > 0$ and a process y_t^+ solving

$$\dot{y}_t^+ = \frac{1}{\varepsilon} g(t, y_t^+ + Q_t) + C \max\{\varepsilon^{\beta-2-\delta}, \sigma\varepsilon^{-3/2+\beta/2-2\delta}\} , \quad y_{t_0}^+ = x + 2 ,$$

such that $y_t \leqslant y_t^+$ for all $t \in [t_0, t_2]$, where $y_t = q_t - Q_t$. Note the initial condition for y_t^+ is chosen using (4.38). Therefore, $q_t \leqslant y_t^+ + Q_t$. In the same way as in Proposition 4.4.4, we can show $|y_t^+| < 4\sqrt{t_2}$ for all $t_0 \leqslant t \leqslant t_2$ by considering the sign of $\dot{y}_t^+$.

Now define $\eta_t := y_t^+ + Q_t + \varepsilon^\beta P_t$ and note that for all $t_0 \leqslant t \leqslant t_2$, $|\eta_t| < C$ for some constant $C > 0$ depending on t_2. It solves

$$\mathrm{d}\eta_t = \frac{1}{\varepsilon}[t(\eta_t - \varepsilon^\beta P_t) - (\eta_t - \varepsilon^\beta P_t)^3 + \varepsilon + C\max\{\varepsilon^{\beta-1-\delta}, \sigma\varepsilon^{-1/2+\beta/2-2\delta}\}]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t\ ,$$

with initial position $\eta_{t_0} \leqslant x + 3$.

Denote the drift term above by $f(t, \eta_t, P_t, \varepsilon)$. We will now show that f is bounded in such a way that allows us to use a comparison principle. As we are working on $(E_1 \cup E_2)^c$, we know that $|\varepsilon^\beta P_t| \leqslant \sigma\varepsilon^{-1/2+\beta/2-\delta}$ for all $t \in [t_0, t_2]$ by definition. Note also that by the conditions on δ and ε, $\max\{\varepsilon^{\beta-1-\delta}, \sigma\varepsilon^{-1/2+\beta/2-2\delta}\} \ll \varepsilon$. We then have

$$f(t, \eta_t, P_t, \varepsilon) \leqslant \frac{1}{\varepsilon}\left[t\eta_t - \eta_t^3 + \varepsilon(1 + r(\sigma))\right]\ ,$$

where $r(\sigma) = C\max\{\varepsilon^{\beta-2-\delta}, \sigma\varepsilon^{-3/2+\beta/2-2\delta}\}$ for some constant $C > 0$ depending on t_2. Let q_t^+ be the solution of

$$\mathrm{d}q_t^+ = \frac{1}{\varepsilon}\left[tq_t^+ - (q_t^+)^3 + \varepsilon(1 + r(\sigma))\right]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t\ , \quad q_{t_0}^+ = x + 3\ .$$

By Lemma C.0.4 below, $\eta_t \leqslant q_t^+$ for all $t \in [t_0, t_2]$ and almost all paths in $(E_1 \cup E_2)^c$. Therefore, $q_t + \varepsilon^\beta P_t \leqslant \eta_t \leqslant q_t^+$, which gives the upper bound. □

This proposition allows us to complete the proof of Theorem 4.1.4.

Proof of Theorem 4.1.4. Firstly, let us consider the fast pulling case. That is, $\sigma^{4/3}|\ln\sigma|^{2/3} \ll \varepsilon \ll 1$. In this case, when $\beta > 2$ the term $1 - r(\sigma)$ appearing in (4.41) is strictly positive and bounded away from zero for all σ sufficiently small. Then the analysis of Section 4.3.1 can be applied to q_t^- (the slightly different drift term does not matter). By Lemma 4.4.3, we can assume $(E_1 \cup E_2)^c$ to hold, in which case $|\varepsilon^\beta P_t| \leqslant \sigma\varepsilon^{-1/2+\beta/2-\delta}$ for all $t \in [-T + 2\varepsilon^{\beta-\delta}, t_2]$. Therefore, we find that $q_t^- - \varepsilon^\beta P_t > \gamma\sqrt{t}$ for all $c_1\sqrt{\varepsilon|\ln\sigma|} \leqslant t \leqslant t_2$, where c_1, $\gamma > 0$ are suitably chosen constants.

For the slow pulling case, let $0 < \delta < \beta/2 - 1$ and $\sigma^{2/(1+2\delta)} \ll \varepsilon \ll \sigma^{4/3}|\ln\sigma|^{-13/6}$. Again, $1 \pm r(\sigma)$ is positive and bounded away from zero for σ small and the analysis in Section 4.3.2 applies to q_t^+ and q_t^-. Note that, in the limit, q_t^- and q_t^+ must "go the same way" as t becomes positive, by comparison of their drift terms and initial positions, and we know by Proposition 4.3.2 that the probabilities are 1/2 in either direction. As above, the term $\varepsilon^\beta P_t$ does not change anything. Therefore, q_t behaves in the same way as q_t^- and q_t^+, which completes the proof. $\square$

4.5 Asymptotics of Airy Integrals

We give here the proofs of the four lemmas at the start of Section 4.2.2 that were used to prove Theorem 4.1.2.

Proof of Lemma 4.2.3. We trivially have

$$\int_a^\infty \mathrm{e}^{ps} \operatorname{Ai}(s)\,\mathrm{d}s \;\leqslant\; \int_0^\infty \mathrm{e}^{ps} \operatorname{Ai}(s)\,\mathrm{d}s\ .$$

As we saw previously in Section 4.2.1, $\int_{-\infty}^\infty \mathrm{e}^{ps} \operatorname{Ai}(s)\,\mathrm{d}s = \mathrm{e}^{p^3/3}$ for $p > 0$. In addition, from [dlm07] we see that $\int_{-\infty}^0 \mathrm{e}^{ps} \operatorname{Ai}(s)\,\mathrm{d}s \geqslant 0$. The upper bound then follows from these two inequalities.

For the lower bound, we have from [dlm07] that for $s \geqslant 1$,

$$\operatorname{Ai}(s) \;\geqslant\; \frac{1}{2\sqrt{\pi}} s^{-1/4}\,\mathrm{e}^{-2s^{3/2}/3}\left(1-\frac{5}{48}s^{-3/2}\right) \;\geqslant\; \frac{1}{4\sqrt{\pi}} s^{-1/4}\,\mathrm{e}^{-2s^{3/2}/3}\ .$$

Then

$$\begin{aligned}\int_a^\infty \mathrm{e}^{ps} \operatorname{Ai}(s)\,\mathrm{d}s \;&\geqslant\; \int_{p^2}^{p^2+p^{1/2}} \mathrm{e}^{ps} \operatorname{Ai}(s)\,\mathrm{d}s \\ &\geqslant\; \frac{1}{4\sqrt{\pi}} \int_{p^2}^{p^2+p^{1/2}} s^{-1/4}\,\mathrm{e}^{ps-2s^{3/2}/3}\,\mathrm{d}s \\ &\geqslant\; \frac{1}{8\sqrt{\pi}} p^{-1/2} \int_{p^2}^{p^2+p^{1/2}} \mathrm{e}^{ps-2s^{3/2}/3}\,\mathrm{d}s\ .\end{aligned}$$

Using the inequality $ps - 2s^{3/2}/3 \geqslant p^3/3 - 1/4$ for $s \in [p^2, p^2 + p^{1/2}]$ gives the result with $C = \pi^{-1/2}\,\mathrm{e}^{-1/2}/8$. $\square$

Proof of Lemma 4.2.4. For the lower bound, we have

$$\begin{aligned}\int_a^\infty \mathrm{e}^{2ps} \operatorname{Ai}^2(s)\,\mathrm{d}s \;&\geqslant\; \int_{p^2}^{p^2+p^{1/2}} \mathrm{e}^{2ps} \operatorname{Ai}^2(s)\,\mathrm{d}s \\ &\geqslant\; p^{-1/2}\left(\int_{p^2}^{p^2+p^{1/2}} \mathrm{e}^{ps} \operatorname{Ai}(s)\,\mathrm{d}s\right)^2\ ,\end{aligned}$$

where the second inequality follows by the Cauchy-Schwarz inequality. This final integral was bounded from below in the proof of Lemma 4.2.3 above, which gives the required lower bound with $C_1 = \pi^{-1} \mathrm{e}^{-1/2} /64$.

For the upper bound, we simply have

$$\int_a^\infty \mathrm{e}^{2ps} \operatorname{Ai}^2(s) \, \mathrm{d}s \ \leqslant \ \int_1^\infty \mathrm{e}^{2ps} \operatorname{Ai}^2(s) \, \mathrm{d}s \ .$$

and can then follow the proof of Lemma 4.2.2(i). □

Proof of Lemma 4.2.5. We have simply

$$\begin{aligned}\int_{-a}^\infty \mathrm{e}^{ps} \operatorname{Ai}(s) \, \mathrm{d}s &= \int_{-\infty}^\infty \mathrm{e}^{ps} \operatorname{Ai}(s) \, \mathrm{d}s - \int_{-\infty}^{-a} \mathrm{e}^{ps} \operatorname{Ai}(s) \, \mathrm{d}s \\ &= \mathrm{e}^{p^3/3} - \int_{-\infty}^{-a} \mathrm{e}^{ps} \operatorname{Ai}(s) \, \mathrm{d}s \ .\end{aligned}$$

Then by the boundedness of Ai,

$$\begin{aligned}\left| \int_{-\infty}^{-a} \mathrm{e}^{ps} \operatorname{Ai}(s) \, \mathrm{d}s \right| \ &\leqslant \ \int_{-\infty}^{-a} \mathrm{e}^{ps} \ |\operatorname{Ai}(s)| \ \mathrm{d}s \\ &\leqslant \ C \int_{-\infty}^{-a} \mathrm{e}^{ps} \ \mathrm{d}s \\ &= C p^{-1} \mathrm{e}^{-pa} \ .\end{aligned}$$

□

Proof of Lemma 4.2.6. We firstly bound the integral below by considering just $\int_{-a}^{-a/2}$. By (4.8), there exists a constant $C > 0$ such that for all $s < 0$ with $|s|$ sufficiently large,

$$\operatorname{Ai}^2(s) \ \geqslant \ \frac{1}{2\pi} |s|^{-1/2} \cos^2 \left(\frac{2}{3} |s|^{3/2} - \frac{\pi}{4} \right) - C|s|^{-2} \ .$$

We assume that a is large enough that this inequality applies for all $s \in [-a, -a/2]$. For each $n \in \mathbb{N}$, the cosine term has a zero j_n of the form

$$j_n = - \left(\frac{3n\pi}{2} + \frac{9\pi}{8} \right)^{2/3} \ .$$

Define the interval $J_n := [j_{n+1}, j_n]$ and let $\mathcal{N} := \{n \in \mathbb{N} : J_n \subset [-a, -a/2]\}$. By taking a large enough, we guarantee that $\mathcal{N} \neq \emptyset$ and that all $n \in \mathcal{N}$ satisfy $a^{3/2}/10 < n < 2a^{3/2}$. Then writing $D_\varepsilon := \bigcup_{n \in \mathcal{N}} J_n$, we futher bound the integral $\int_{-a}^{-a/2} \operatorname{Ai}^2(s) \, \mathrm{d}s \ \geqslant \ \int_{D_\varepsilon} \operatorname{Ai}^2(s) \, \mathrm{d}s$.

For all $n \geqslant 2$, the inequalities

$$j_n - j_{n+1} \leqslant \pi \left(\frac{3n\pi}{2} + \frac{9\pi}{8} \right)^{-1/3} \leqslant \pi n^{-1/3}$$

and

$$j_n - j_{n+1} \geqslant \pi \left(\frac{3(n+1)\pi}{2} + \frac{9\pi}{8} \right)^{-1/3} \geqslant \frac{1}{2}\pi n^{-1/3}$$

hold. Therefore, $a\pi^{-1}n^{1/3}/2 \leqslant |\mathcal{N}| \leqslant a\pi^{-1}n^{1/3}$ and so $|\mathcal{N}| \asymp a^{3/2}$ unifomly for all sufficiently large a. Then, uniformly in $n \in \mathcal{N}$ and a large, it follows that

$$\begin{aligned} \int_{J_n} |s|^{-1/2} \cos^2 \left(\frac{2}{3}|s|^{3/2} - \frac{\pi}{4} \right) \mathrm{d}s &\geqslant Cn^{-1/3} \int_{J_n} \cos^2 \left(\frac{2}{3}|s|^{3/2} - \frac{\pi}{4} \right) \mathrm{d}s \\ &\geqslant Cn^{-2/3} , \end{aligned}$$

so that

$$\int_{D_\varepsilon} |s|^{-1/2} \cos^2 \left(\frac{2}{3}|s|^{3/2} - \frac{\pi}{4} \right) \mathrm{d}s \geqslant Cn^{-2/3}|\mathcal{N}| \geqslant Ca^{1/2} . \tag{4.42}$$

Finally,

$$\int_{J_n} |s|^{-2} \, \mathrm{d}s \leqslant Cn^{-4/3}|J_n| \leqslant Cn^{-4/3}n^{-1/3}$$

so that

$$\int_{D_\varepsilon} |s|^{-2} \, \mathrm{d}s \leqslant Cn^{-4/3}n^{-1/3}|\mathcal{N}| \leqslant Ca^{-1} . \tag{4.43}$$

Combining (4.42) and (4.43) gives the result. □

Appendix A

First-Exit Problems

Here we present some results about first-exit problems, the theory on which the ideas of Chapter 2 were based. Throughout, we let $D \subset \mathbb{R}^n$ be an open, bounded domain with smooth boundary and consider on D the SDE

$$\mathrm{d}x_t^\sigma = -\nabla U(x_t^\sigma)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\ , \quad x_0^\sigma = x\ , \tag{A.1}$$

where $U : \mathbb{R}^n \to \mathbb{R}$ is some smooth potential function and $\sigma > 0$ is the noise intensity. In addition, we assume that U has a unique minimum at $0 \in D$. Many of the results below hold for more general drift terms not deriving from a potential and for more general diffusion coefficients depending on x_t^σ. Let

$$\tau = \tau^\sigma = \inf\{t \geqslant 0 : x_t^\sigma \notin D\}\ .$$

One of the main aims in the study of first-exit problems is to investigate the distribution of τ and the exit locations x_τ^σ. Below, we give an overview of the various methods that have been employed to tackle this and other questions related to stopping times like τ^σ.

A.1 Large Deviations

The SDE (A.1) can be viewed as a random perturbation of the deterministic ODE $\dot{x}_t^{\mathrm{det}} = -\nabla U(x_t^{\mathrm{det}})$, with the same initial condition. We assume that all deterministic trajectories, including those starting on ∂D, coverge to 0 as $t \to \infty$. For small noise we expect that x_t^σ will stay close to x_t^{det} and that x_t^σ will converge in some sense to x_t^{det} as $\sigma \downarrow 0$. The theory of large deviations tells us about the improbable event that they do not stay close and the rate at which this converges to zero. This can then be used to tell us something about τ.

The result stated in Theorem A.1.1 below was first obtained by Freidlin and Wentzell (see their monograph [FW98]) and later extended by Day [Day92] to deal with the characteristic boundary case, which means that the boundary consists of trajectories of the deterministic system. In the above setting, this would mean $\nabla U(z) = 0$ for some $z \in \partial D$, which may occur if ∂D is a separatrix between different domains of attraction of U.

An important role is played by a function $V(0, z)$, called the quasi-potential. This represents the "cost" of x_t^σ going from 0 to the point z and can be defined for diffusions with general drift terms. For gradient flows as above, it turns out that $V(0, z) = 2[U(z) - U(0)]$. As one would expect, the cost of reaching a point increases with the value of U at that point. Letting $\bar{V} = \inf_{z \in \partial D} V(0, z)$ and assuming $\bar{V} < \infty$, we have the following theorem, which is Theorem 5.7.11 from [DZ98].

Theorem A.1.1. *1. For all $x \in D$ and all $\delta > 0$,*

$$\lim_{\sigma \downarrow 0} \mathbb{P}^x \left(\mathrm{e}^{(\bar{V}-\delta)/\sigma^2} < \tau < \mathrm{e}^{(\bar{V}+\delta)/\sigma^2} \right) = 1 \ .$$

Moreover, for all $x \in D$,

$$\lim_{\sigma \downarrow 0} \sigma^2 \ln \mathbb{E}^x(\tau) = \bar{V} \ .$$

2. If $N \subset \partial D$ is a closed set and $\inf_{z \in N} V(0, z) > \bar{V}$, then for any $x \in D$,

$$\lim_{\sigma \downarrow 0} \mathbb{P}^x(x_\tau^\sigma \in N) = 0 \ .$$

In particular, if there exists $z^* \in \partial D$ such that $V(0, z^*) < V(0, z)$ for all $z \neq z^*$, $z \in \partial D$, then

$$\forall \delta > 0, \forall x \in D, \lim_{\sigma \downarrow 0} \mathbb{P}^x(|x_\tau^\sigma - z^*| < \delta) = 1 \ .$$

We immediately see the same type of exponential behaviour of τ that we saw in the Eyring-Kramers formula (2.2) in Chapter 2. We also see that the exit occurs at those points on the boundary with the least energy. The theorem does not, however, tell us which of these are more likely to be reached. This question is addressed below in Section A.4.

A.2 Infinitesimal Generator

Associated to the diffusion (A.1) is the infinitesimal generator $\mathcal{L}^\sigma$, defined by

$$\mathcal{L}^\sigma v(x,t) = \frac{\sigma^2}{2} \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2}(x,t) - \langle \nabla U(x), \nabla v(x,t) \rangle \ ,$$

where $\nabla = \nabla_x$. The following theorem, which is Corollary 5.7.4 from [DZ98], relates quantities of interest in the first-exit problem to solutions of partial differential equations involving $\mathcal{L}^\sigma$.

Theorem A.2.1. *Assume that for any $z \in \partial D$, there exists a ball $B(z)$ such that $\overline{D} \cap \overline{B(z)} = \{z\}$. Let $u_1(x) = \mathbb{E}^x(\tau^\sigma)$. Then u_1 is the unique solution of*

$$\mathcal{L}^\sigma u_1 = -1 \quad \text{in } D \ , \qquad u_1 = 0 \quad \text{on } \partial D \ . \tag{A.2}$$

Let $u_2(x) = \mathbb{E}^x(f(x_\tau^\sigma))$. Then for any f continuous, u_2 is the unique solution of

$$\mathcal{L}^\sigma u_2 = 0 \quad \text{in } D \ , \qquad u_2 = f \quad \text{on } \partial D \ . \tag{A.3}$$

If f is chosen to be 1 on a subset $A \subset \partial D$, where A is disconnected from the rest of the boundary, and 0 elsewhere, then the second part of the theorem tells us the probability of exiting through A. As noted in [DZ98], if we formally set $f(z) = \delta_{z^*}(z)$, then the fundamental solution of (A.3), considered as a function of z^*, gives the exit density on ∂D.

In general, (A.2) and (A.3) cannot be solved explicitly and even a numerical approach is not feasible due to the small values of σ that are of interest.

A.3 Potential-Theoretic Approach

We saw in Theorem A.1.1 the exponential behaviour of τ as seen in the classical Eyring-Kramers formula (2.2). However, it is only recently in [BEGK04] that this formula, including the exponential prefactor, has been proved in the multidimensional case. The approach is based on potential theory. We shall now assume that the domain D contains multiple minima and saddle points of U, all of which are non-degenerate, meaning that the Hessian of U at these points has only nonzero eigenvalues. The case of degenerate saddles has also been considered [BG10].

We shall now state Theorem 3.2 from [BEGK04]. Firstly, we need some definitions. For any two sets $A, B \subset D$, the saddle height between A and B is defined by

$$\hat{U}(A,B) \equiv \inf_{d:d(0)\in A, d(1)\in B} \sup_{t\in[0,1]} U(d(t)) \ ,$$

where the infimum is over all continuous paths $d \in D$. Let $\mathcal{P}(A,B)$ denote the set of minimal paths from A to B,

$$\mathcal{P}(A,B) \equiv \{d \in \mathcal{C}([0,1],D) : d(0) \in A,\ d(1) \in B,\ \sup_{t\in[0,1]} U(d(t)) = \hat{U}(A,B)\} \ .$$

A gate $G(A,B)$ is a minimal subset of $\mathcal{G}(A,B) \equiv \{z \in D : U(z) = \hat{U}(A,B)\}$ such that all minimal paths intersect $G(A,B)$. The set of saddle points $\mathcal{S}(A,B)$ is the union over all gates $G(A,B)$.

Theorem A.3.1. *Let x be a minimum of U and let E be any closed subset of D such that:*

(i) if $\mathcal{M} \equiv \{y_1,\ldots,y_k\}$ enumerates all those minima of U such that $U(y_j) \leqslant U(x)$, then $\cup_{j=1}^{k} B_{\sigma^2/2}(y_j) \subset E$;

(ii) dist$(\mathcal{S}(x,\mathcal{M}),E) \geqslant \delta > 0$, where $\mathcal{S}(x,\mathcal{M}) \equiv \{z_1^*,\ldots,z_l^*\}$.

Then

$$\mathbb{E}^x(\tau_E) = \frac{2\pi\, \mathrm{e}^{2[\hat{U}(x,\mathcal{M})-U(x)]/\sigma^2}}{\sqrt{\det(\nabla^2 U(x))}\sum_{j=1}^{l}\frac{|\lambda_1(z_j^*)|}{\sqrt{|\det(\nabla^2 U(z_j^*))|}}}(1+\mathcal{O}(\sigma|\ln\sigma|)) \ ,$$

where $\tau_E = \inf\{t \geqslant 0 : x_t \in E\}$ and $\lambda_1(z_j^*)$ denotes the unique negative eigenvalue of the Hessian of U at z_j^*.

A.4 Exit Measure

We saw in Theorem A.1.1 that paths typically exit those places on the boundary that have the least energy. Now we look more closely at the exit measure, which will enable us to identify what happens when several boundary points have the same energy. A thorough review of this topic is given by Day in [Day87].

A.4.1 Asymptotic Expansions

Matkowsky and Schuss [MS77] used formal asymptotic expansions to construct a solution of (A.3) and obtain an expression for the exit measure. We now outline this approach in the context of our SDE (A.1), again assuming that all deterministic trajectories starting on ∂D converge to 0 as t increases.

The aim is to determine the distribution of exit locations on ∂D, whose density is given by

$$p_\sigma(x,z)\,\mathrm{d}S_z = \mathbb{P}(x^\sigma_\tau \in \mathrm{d}S_z \,|\, x^\sigma_0 = x), \quad x \in D,\, z \in \partial D\ ,$$

where $\mathrm{d}S_z$ is a surface element on ∂D at z. In this case, the solution of (A.3) can be expressed as

$$u_\sigma(x) = \mathbb{E}^x(f(x^\sigma_\tau)) = \int_{\partial D} f(z) p_\sigma(x,z)\,\mathrm{d}S_z\ ,$$

which relates $u_\sigma(x)$ and $p_\sigma(x,z)$. Using an asymptotic expansion of $u_\sigma(x)$, they find that

$$\lim_{\sigma\downarrow 0} \int_{\partial D} f(z) p_\sigma(x,z)\,\mathrm{d}S_z = C_0\ , \tag{A.4}$$

where C_0 is some given constant, which allows us to compute $\lim_{\sigma\downarrow 0} p_\sigma(x,z)$. Furthermore, it implies the limit is independent of x so that $\lim_{\sigma\downarrow 0} p_\sigma(x,z) = p(z)$ for all $x \in D$.

It is again necessary to consider those points on the boundary where U is minimal. If this occurs on a set $\Gamma \subset \partial D$ such that $\Gamma^0 \neq \emptyset$ and $\Gamma \setminus \Gamma^0$ has zero measure in ∂D, then

$$p(z) = \frac{\|\nabla U(z)\| \chi_\Gamma(z)}{\int_\Gamma \|\nabla U(z)\|\,\mathrm{d}S_z}\ ,$$

where χ is the characteristic function. If the minimum is attained at distinct points $z_1, \dots, z_m \in \partial D$ and if

$$H_k = H(z_k) \equiv \det \left.\frac{\partial^2 U}{\partial z'_i \partial z'_j}\right|_{z_k} \neq 0\ , \quad i,j = 1,\dots,n-1\ ,$$

where $(z'_1, \dots, z'_{n-1})$ are coordinates on ∂D, then

$$p(z) = \frac{\sum_{k=1}^m H^{-1/2}(z_k) \|\nabla U(z_k)\| \delta(z - z_k)}{\sum_{k=1}^m H^{-1/2}(z_k) \|\nabla U(z_k)\|}\ . \tag{A.5}$$

These results obtained by Matkowsky and Schuss using formal methods were proved rigorously by Kamin [Kam79] in the case that $\nabla U(x) \cdot x \geqslant \lambda \|x\|^2$ in a neighbourhood of the boundary, for some constant $\lambda > 0$. This obviously includes the case that U is quadratic, as considered by Lee in [Lee09] (see also Section 2.2.1 and (2.7)).

A.4.2 Probabilistic Methods

Kifer [Kif74] obtained similar results using probabilistic considerations, in the case $-\nabla U(x) = Bx$ for B a real-valued matrix with eigenvalues having negative real parts. He was able to relate the exit measure on ∂D to the invariant measure of the diffusion x_t^σ. Day [Day84] demonstrates that these two approaches are related by showing that the constant C_0 in (A.4) can be expressed in terms of the density of this invariant measure.

Appendix B

Singular Perturbation Theory

We here state Definition 2.1.1 and Theorems 2.1.7 and 2.1.8 from [BG06] concerning singular pertubration theory.

Consider the slow-fast system

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y), \\ \dot{y} &= g(x,y), \end{aligned} \tag{B.1}$$

where $f \in \mathcal{C}^2(\mathcal{D},\mathbb{R}^n)$, $g \in \mathcal{C}^2(\mathcal{D},\mathbb{R}^m)$ and $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$ is a connected, open set.

Definition B.0.1. *Let $\mathcal{D}_0 \subset \mathbb{R}^m$ be a connected set of nonempty interior and assume that there exists a continuous function $x^* : \mathcal{D}_0 \to \mathbb{R}^n$ such that $(x^*(y), y) \in \mathcal{D}$ and $f(x^*(y), y) = 0$ for all $y \in \mathcal{D}_0$. Then the set $\mathcal{M} = \{(x,y) : x = x^*(y), y \in \mathcal{D}_0\}$ is called a slow manifold of the system (B.1).*

Let $A^*(y) = \partial_x f(x^*(y), y)$ denote the stability matrix of the associated system at $x^*(y)$. The slow manifold $\mathcal{M}$ is called uniformly asymptotically stable if all eigenvalues of $A^*(y)$ have negative real parts that are uniformly bounded away from zero for $y \in \mathcal{D}_0$.

In the case $m = 1$, the slow manifold is also called an equilibrium branch.

Theorem B.0.2. *Let $\mathcal{M} = \{(x,y) : x = x^*(y),\ y \in \mathcal{D}_0\}$ be a uniformly asymptotically stable slow manifold of the slow-fast system (B.1). Let f and g, as well as all their derivatives up to order two, be uniformly bounded in norm in a neighbourhood $\mathcal{N}$ of $\mathcal{M}$. Then there exist positive constants ε_0, c_0, c_1, $\kappa = \kappa(n)$, M such that, for $0 < \varepsilon \leqslant \varepsilon_0$ and any initial condition $(x_0, y_0) \in \mathcal{N}$ satisfying $\|x_0 - x^*(y_0)\| \leqslant c_0$, the bound*

$$\|x_t - x^*(y_t)\| \leqslant M\|x_0 - x^*(y_0)\|\, \mathrm{e}^{-\kappa t/\varepsilon} + c_1\varepsilon$$

holds as long as $y_t \in \mathcal{D}_0$.

Theorem B.0.3. *Let the slow manifold $\mathcal{M} = \{(x, y) : x = x^*(y),\, y \in \mathcal{D}_0\}$ of the slow-fast system (B.1) be uniformly asymptotically stable. Then there exists, for sufficiently small ε, a locally invariant manifold (or adiabatic manifold)*

$$\mathcal{M}_\varepsilon = \{(x, y) : x = \bar{x}(y, \varepsilon),\, y \in \mathcal{D}_0\},$$

where $\bar{x}(y, \varepsilon) = x^*(y) + \mathcal{O}(\varepsilon)$. In other words, if the initial condition is taken on $\mathcal{M}_\varepsilon$, that is, $x_0 = \bar{x}(y_0, \varepsilon)$, then $x_t = \bar{x}(y_t, \varepsilon)$ as long as $y_t \in \mathcal{D}_0$.

As explained in [BG06], this final theorem can be proved using the centre-manifold theorem. Furthermore, it is shown that

1. For each initial condition (x_0, y_0) sufficiently close to $\mathcal{M}_\varepsilon$, there exists a particular solution $\hat{y}_t$ of the equation $\dot{y} = g(\bar{x}(y, \varepsilon), y)$ (i.e. the equation satisfied by paths on $\mathcal{M}_\varepsilon$) such that

$$\|(x_t, y_t) - (\hat{x}_t, \hat{y}_t)\| \leqslant M\|(x_0, y_0) - (\hat{x}_0, \hat{y}_0)\| \,\mathrm{e}^{-\kappa t/\varepsilon} \tag{B.2}$$

   for some M, $\kappa > 0$, where $\hat{x}_t = \bar{x}(\hat{y}_t, \varepsilon)$.

2. $\bar{x}(y, \varepsilon)$ can be approximated by a power series in ε up to any order. In particular,

$$\bar{x}(y, \varepsilon) = x^*(y) + \varepsilon A^*(y)^{-1}\partial_y x^*(y) g(x^*(y), y) + \mathcal{O}(\varepsilon^2). \tag{B.3}$$

Note that Theorem B.0.3 also holds under slightly more general conditions on $A^*(y)$.

Appendix C

A Comparison Lemma

We present a lemma that is a slightly modified version of Theorems 5.1 and 5.2 appearing in [And72]. Our proof follows those given there.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a one-dimensional Brownian motion W adapted to a filtration $(\mathcal{F}_t)_{t \geqslant 0}$. For $t \geqslant 0$, let $X(t)$ and $Y(t)$ be two real-valued processes evolving according to

$$\begin{aligned} X(t) &= X(0) + \int_0^t a(s, X(s), Z(s))\,\mathrm{d}s + CW_t\,, \\ Y(t) &= Y(0) + \int_0^t c(s, Y(s))\,\mathrm{d}s + CW_t\,, \end{aligned} \tag{C.1}$$

where C is a constant, $a : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$, $c : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $Z(t)$ is an $\mathcal{F}_t$-adapted process with continuous sample paths almost surely and $Z(0) = z_0$, which is $\mathcal{F}_0$-adapted. We assume that $X(0) \leqslant Y(0)$ almost surely, where $X(0)$ and $Y(0)$ are $\mathcal{F}_0$-adapted.

Lemma C.0.4. *Suppose there are constants C_1, $C_2 > 0$ such that whenever $|x| < C_1$ and $|z| < C_2$, $a(t, x, z) < c(t, x)$ for all $t \geqslant 0$. If, almost surely, $|X(t)| < C_1$ and $|Z(t)| < C_2$ for all $t \geqslant 0$ then*

$$\mathbb{P}\left(Y(t) \geqslant X(t) \text{ for all } t \geqslant 0\right) = 1\,.$$

Proof. Define $\tau = \inf\{t > 0 : Y(t) - X(t) < 0\}$ and set $\tau = +\infty$ if $Y(t) \geqslant X(t)$ for all $t \geqslant 0$. This is a stopping time because if $t > 0$ then

$$\{\tau \geqslant t\} = \bigcap_{r \in [0,t] \cap \mathbb{Q}} \{Y(r) - X(r) \geqslant 0\} \in \mathcal{F}_t\,.$$

Put $D = \{\tau < +\infty\}$ and assume that $\mathbb{P}(D) > 0$. Then we can define a probability measure $\mathbb{Q}(\,\cdot\,) = \mathbb{P}(\,\cdot\,|\,D)$ on $\mathcal{F}$. By continuity, $\mathbb{Q}(X(\tau) = Y(\tau)) = 1$. For any $t \geqslant 0$, we can therefore write (almost surely with respect to $\mathbb{Q}$)

$$\begin{aligned} Y(t+\tau) - X(t+\tau) &= Y(\tau) - X(\tau) + \int_{\tau}^{t+\tau} c(s, Y(s)) - a(s, X(s), Z(s))\,\mathrm{d}s \\ &= \int_0^t c(s+\tau, Y(s+\tau)) - a(s+\tau, X(s+\tau), Z(s+\tau))\,\mathrm{d}s\ . \end{aligned}$$

The right-hand side is continuously differentiable in t and so

$$\mathbb{Q}\left(\lim_{t\to 0} \frac{Y(t+\tau) - X(t+\tau)}{t} = c(\tau, X(\tau)) - a(\tau, X(\tau), Z(\tau))\right) = 1\ .$$

Therefore, $\mathbb{Q}(Y(t+\tau) > X(t+\tau)$ for all sufficiently small $t > 0) = 1$. But due to continuity of X and Y and the definition of τ, this probability should be zero. This contradiction arises from the assumption that $\mathbb{P}(D) > 0$. Therefore, $\mathbb{P}(\tau = +\infty) = 1$.

$\square$

Appendix D

Symbols and Acronyms

Below is a partial list of symbols and acronyms used throughout this thesis. Other notation is explained within the thesis itself.

$f(x) = \mathcal{O}(g(x))$	$\lvert f(x)\rvert \leqslant M\lvert g(x)\rvert$ for all $0 < x < x_0$
$f(x) = o_x(1)$	$\lim_{x\to 0} f(x) = 0$
$f(x) \ll g(x)$	$\lim_{x\to 0} f(x)/g(x) = 0$
$f(x) \lesssim g(\sigma)$	$f(x) \ll g(x)$ or $f(x) = \mathcal{O}(g(x))$
$\mathrm{Var}(X)$, $\mathbb{E}(X)$	Variance, Expectation of a random variable X
$\lceil x \rceil$	The smallest integer bigger than or equal to x
$f(x) \asymp g(x)$	$c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)$ for all $0 < x < x_0$
$\Phi(x)$	Normal distribution function, $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \mathrm{e}^{-x^2/2}\,\mathrm{d}x$
ODE	Ordinary differential equation
SDE	Stochastic differential equation
DFS	Dynamic force spectroscopy

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