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THE PANOFSKY-WENZEL THEOREM AND GENERAL RELATIONS FOR THE WAKE POTENTIAL

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Some general relations are derived for the wake potential, that is, the integrated Lorentz force acting on a unit charge that passes, with constant velocity, through a region of electromagnetic fields. The wake potential is irrotational and nearly "source-free." The longitudinal component satisfies the Laplace equation for the transverse coordinates in the case of highly relativistic motion.

KEY WORDS: Electromagnetic field calculations, electron beam devices, wake-field accelerators

In 1956 Panofsky and Wenzel[1] considered the transverse kick experienced by a relativistic charge crossing an rf cavity. In the meantime this work has been applied to wake potentials (see, for example Reference 2), that is, the integrated Lorentz force that acts on a unit charge. The relation between the longitudinal derivation of the transverse wake potential and the transverse gradient of the longitudinal potential is referred to as the Panofsky-Wenzel theorem. Here we will derive it in a different, more general way.

Let us assume a probing charge traveling with constant velocity $\mathbf{v}$. Then we obtain for

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{v}(\nabla \cdot \mathbf{B}) - (\mathbf{v} \cdot \nabla)\mathbf{B} = -(\mathbf{v} \cdot \nabla)\mathbf{B}\,, \tag{1}$$

which we introduce into Maxwell's equation,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - (\mathbf{v} \cdot \nabla)\mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B})$$

obtaining

$$\nabla \times (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \nabla \times \mathbf{F} = -\frac{d\mathbf{B}}{dt}\,, \tag{2}$$

where we have used

$$\frac{d\mathbf{B}}{dt} = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{B}\,. \tag{3}$$

If the fields are excited by a driving charge q traveling along the same trajectory as the probing charge, then the wake potential is defined essentially as the force integral along the trajectory, but at a distance s behind the driving charge. In order to get an expression for the wake potential we integrate Equation (2) along the trajectory, which we have chosen parallel to the z-axis for convenience:

$$\int_{z_1}^{z_2} (\nabla \times \mathbf{F})_{x,y,z-s,t=z/v} dz = -\int_{z_1}^{z_2} \frac{d\mathbf{B}}{dt} dz = -v \int_{z_1}^{z_2} d\mathbf{B} = v(\mathbf{B}_1 - \mathbf{B}_2) . \tag{4}$$

The right-hand side of the equation vanishes if $z_1 - s$ and $z_2 - s$ are in a field-free region or if $\mathbf{B}_1 = \mathbf{B}_2$. In order to transform the left-hand side we introduce an operator ∇_s, where the subscript s means differentiation with respect to s instead of z. Then we can invert the order of differentiation and integration and obtain

$$\frac{1}{q} \int_{z_1}^{z_2} (\nabla \times \mathbf{F})_{x,y,z-s,t=z/v} dz = \frac{1}{q} \left(\nabla_t, -\frac{\partial}{\partial s} \right) \times \int_{z_1}^{z_2} \mathbf{F}(x, y, z - s, z/\mathbf{v}) dz = \nabla_s \times \mathbf{W} = 0. \tag{5}$$

In Equation (5) we have normalized the force integral with the exciting charge and introduced the wake potential $\mathbf{W}$ (remember that W_z is defined with a negative sign, such that the exciting charge always loses energy).

The equation states that $\mathbf{W}$ is irrotational. We call this the generalized Panofsky-Wenzel theorem. We obtain the usual form of it from (5) by splitting $\mathbf{W}$ in its longitudinal and transverse part and crossing it with $\mathbf{e}_z$:

$$\begin{aligned} \mathbf{e}_z \times (\nabla_s \times \mathbf{W}) =& \nabla_s (\mathbf{e}_z \cdot \mathbf{W}) - (\mathbf{e}_z \cdot \nabla_s)\mathbf{W} \\ =& -\frac{\partial \mathbf{W}_t}{\partial s} + \nabla_t W_z = 0, \end{aligned} \tag{6}$$

i.e., the longitudinal derivation of the transverse wake potential equals the transverse gradient of the longitudinal potential.

In Equation (5) we have shown that $\mathbf{W}$ is irrotational. Another relation, nearly expressing the divergence of $\mathbf{W}$, can be derived from the two remaining Maxwell's equations. From the difference

$$\nabla \cdot \mathbf{E} - \mathbf{v} \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \nabla \cdot \mathbf{F} = -\frac{1}{c^2} \mathbf{v} \cdot \frac{\partial \mathbf{E}}{\partial t} \tag{7}$$

and the total time derivative of $\mathbf{E}$, as given in Equation (3), we get

$$\nabla \cdot \mathbf{F} - \frac{1}{c^2} \mathbf{v} \cdot (\mathbf{v} \cdot \nabla)\mathbf{E} = \nabla_t \cdot \mathbf{F}_t + \left[1 - \left(\frac{v}{c} \right)^2 \right] \frac{\partial E_z}{\partial z} = \nabla_t \cdot \mathbf{F}_t + \frac{1}{\gamma^2} \frac{\partial E_z}{\partial z} = \frac{v}{c^2} \frac{dE_z}{dt} , \tag{8}$$

where γ is the Lorentz factor and $\mathbf{v}$ was again taken to be parallel to the z axis. In the above equations we have not taken into account charge density and currents. We consider only fields behind a point-like driving charge q: If we considered also ρ and $\mathbf{J}$ we would get the space-charge force due to the fields in free space.

After integrating Equation (8) along z,

$$\int_{z_1}^{z_2} [\nabla_t \cdot \mathbf{F}]_{x,y,z-s,t=z/v}\, dz + \frac{1}{\gamma^2} \int_{z_1}^{z_2} \left. \frac{\partial E_z}{\partial z} \right|_{x,y,z-s,t=z/v} dz$$

$$= -\frac{\mathbf{v}}{c^2} \int_{z_1}^{z_2} \frac{dE_z}{dt} dz = \left(\frac{v}{c}\right)^2 (E_{z1} - E_{z2})$$

and changing the order of integration and differentiation on the left-hand side,

$$\nabla_t \cdot \int_{z_1}^{z_2} \mathbf{F}_t(x, y, z-s, z/v) dz - \frac{1}{\gamma^2} \frac{\partial}{\partial s} \int_{z_1}^{z_2} \mathbf{F}_z(x, y, z-s, z/v) dz = \left(\frac{v}{c}\right)^2 (E_{z1} - E_{z2}) \ , \quad (9)$$

we can introduce the wake potential and obtain

$$\nabla_t \cdot \mathbf{W}_t + \frac{1}{\gamma^2} \frac{\partial W_z}{\partial s} = 0 \ , \quad (10)$$

if either $E_{z1} = E_{z2}$ or $z_1 - s$ and $z_2 - s$ are in a field-free region. Apart from the factor γ^{-2}, Equation (10) indicates that $\mathbf{W}$ is also "source-free." For highly relativistic motion, it follows from (10) that

$$\frac{\partial W_x}{\partial x} = -\frac{\partial W_y}{\partial y} \ . \quad (11)$$

Finally, because of Equation (5), we can derive $\mathbf{W}$ from a scalar potential Ψ:

$$\mathbf{W} = \nabla_s \Psi, \qquad \Psi(\xi_1, \xi_2, s) = T(\xi_1, \xi_2) Z(s) \ , \quad (12)$$

where we have assumed separability in the longitudinal coordinate.

For highly relativistic particles, $\gamma \to \infty$, we neglect the second term in Equation (10) and together with Equation (12), obtain

$$\nabla_t^2 \psi = \nabla^2 T = 0 \ . \quad (13)$$

In other words, T satisfies the Laplace equation.

Writing (12) in components,

$$W_1 = \frac{Z}{h_1}\frac{\partial T}{\partial \xi_1}, \quad W_2 = \frac{Z}{h_2}\frac{\partial T}{\partial \xi_2}$$
$$W_z = T\frac{dZ}{ds}, \tag{14}$$

we conclude also that the longitudinal component of the wake potential satisfies the 2-dimensional Laplace equation in (ξ_1, ξ_2). The transverse dependences of the transverse components follow from (14).

As an example, consider circular-cylinder coordinates. Then, φ periodic solutions of (13) are

$$\Psi^{(m)} = \left(A_m \rho^m + B_m \rho^{-m}\right) \cos m\varphi Z_m(s) . \tag{15}$$

If the axis, $\rho = 0$, is a possible trajectory, B has to be zero, and we obtain from Equation (14) and (15) the well-known relations

$$\begin{aligned} W_z^{(m)} &= \rho^m \cos m\varphi A_m Z_m'(s) \\ W_\rho^{(m)} &= m\rho^{m-1} \cos m\varphi A_m Z_m(S) \\ W_\varphi^{(m)} &= -m\rho^{m-1} \sin m\varphi A_m Z_m(S). \end{aligned} \tag{16}$$

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