Particle Accelerators, 1994, Vol. 48, pp. 19–41
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POTENTIAL WELL DISTORTION EFFECTS FOR A PARTIALLY FILLED RING

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(Received 6 December 1993; in final form 20 June 1994)

A novel analytic treatment of the multi bunch potential well distortion for a partially filled ring is presented. We start with a standard set of wake field-coupled equations of motion describing a train of M consecutive bunches in a storage ring of harmonic number N, where the wake field effects are separated into: the mode loss, the synchrotron tune shift (both due to potential well distortion) and the coherent multi-bunch coupling. Unlike in the case of completely filled ring, now, the first two quantities vary from bunch to bunch. Here, we evaluate both quantities analytically (using contour integration technique) for a general situation of a partially filled ring ($M<N$), where individual bunches are mutually interacting via wake fields generated by resonant structures. Resulting simple analytic formulas describe the mode loss and the synchrotron tune shift experienced by a given bunch within the train, as a function of the resonance frequency, ω_r, and the quality factor of the coupling impedance, Q. The first formula reveals resonance frequency regions in the vicinity of the integer multiples of the r.f. frequency, $N\omega_o$, where the mode loss response is still equal for all bunches (its absolute value scales as M). It also identifies the second (denser) set of characteristic resonant frequencies, spaced by the multiples of $N\omega_o/M$, at which the mode loss is not only bunch independent, but also considerably smaller (it scales as MQ^{-2}). Conversely, our analytic formula identifies frequency regions, where bunch-to-bunch variation of the mode loss is the strongest (ω_r at odd multiples of $N\omega_o/2M$). Similarly, an analogous analytic formula describing the amount of synchrotron tune shift suffered by different bunches was derived. Both analytic expressions can give one an insight as to optimizing various schemes; e.g. how to modify the existing configuration of parasitic cavity resonances (via frequency tuning), so that the resulting bunch-to-bunch spread of the mode loss and/or the synchrotron tune spread could be instrumental in stabilizing (via Landau damping) some unstable modes of the coupled bunch instability. A number of other possible applications of the presented formalism emerge from the fact that for a given configuration of cavity resonances one can get immediately a simple quantitative answer in terms of the mode loss and the synchrotron tune shift experienced by each bunch along the train.

KEY WORDS: Collective effects, storage rings

1 INTRODUCTION

While coupled multi-bunch motion for a symmetric configuration of populated buckets in a storage ring has been extensively studied and the stability problem has a closed analytic solution[1] for most standard wake fields a fully populated ring is rarely the case for any

* Operated by the Universities Research Association, Inc., under a contract with the U.S. Department of Energy.

operational mode of a realistic synchrotron. As a beam is injected and extracted to and from a storage ring there is always a gap of missing bunches to accommodate injection/extraction kickers. Even a small gap breaks down the symmetry of a coupled multi-bunch motion. Quite often, an extremely non symmetric situation is present in high energy colliders, where a relatively short train of bunches is being accelerated in a long storage ring; e.g. during bunch coalescing.

Here, we present a rigorous treatment of the potential well distortion effects for a non symmetric configuration of populated buckets. We formulate the problem in the framework of a system of differential equations of motion for individual bunches coupled via wake fields. First, we extract the non symmetric mode loss and the synchrotron tune shift terms from the rest of the multi-bunch coupling. The core of this paper is an analytic method, involving contour integration in complex frequency domain, which yields a pair of closed expressions describing the mode loss and the synchrotron tune shift, driven by a general resonant impedance, for the case of partially filled ring.

Both quantities, the mode loss and the synchrotron tune shift, are expressed in terms of explicit functions of: the bunch index, the resonance frequency and the quality factor of the impedance peak. Superimposing many parasitic cavity modes one can use the above formulas to choose appropriate tuning of existing configuration of parasitic modes to minimize mode loss effects.

2 COUPLED MULTI-BUNCH MOTION

We assume a storage ring of a harmonic number N populated by a train of M consecutive bunches, where $M \leq N$. We will confine our consideration the dipole mode of the longitudinal motion only. Therefore, it will be sufficient to model each bunch as a macro particle combining intensity of the entire bunch. To describe a coupled motion of a system of M bunches one can represent a state of the system at a given time by the following vector in the M-dimensional configuration space $\mathbf{R}^M$

$$|y(t)\rangle = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_M(t) \end{pmatrix}, \tag{2.1}$$

where the n-th component of the above vector describes the longitudinal coordinate of the n-th bunch (y_n is the phase of the bunch with respect to the center of an unperturbed bucket). The steady state value of y_n is in general non-zero — except for the case of a machine operating with harmonic number one.

Collective synchrotron motion of the system on M bunches coupled via wake fields can be described by the following set of equations of motion[1,2]

$$\frac{\partial^2}{\partial t^2} y_n(t) + \omega^2 y_n(t) = A \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W' \left(- \left(k + \frac{m-n}{N} \right) T_0 + \frac{1}{c} \left[y_n(t) - y_m \left(t - \left(k + \frac{m-n}{N} \right) T_0 \right) \right] \right) , \tag{2.2}$$

where

$$A = \frac{N_0 r_0 \eta \omega_0}{2\pi\gamma} .$$

Here W' is the time derivative of the wake function, ω is the unperturbed synchrotron frequency, η is the phase slip factor, c is the velocity of light, r_0 is the classical proton radius, ω_0 is the revolution frequency and T_0 is the revolution period. The index k gives the sum of the wake fields from all previous turns.

The argument of the wake function in the right hand side of Equation (2.2) can be separated into a large part given by the bunch separation and a small correction of the order of the relative bunch displacement due to the synchrotron motion. Taylor expansion of the wake function with respect to the difference of bunch displacements (up to the linear term) allows one to rewrite the set of equations of motion as follows

$$\frac{\partial^2}{\partial t^2} y_n(t) + \omega^2 y_n(t) = A \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W' \left(- \left(k + \frac{m-n}{N} \right) T_0 \right) + \tag{2.3}$$

$$+ A \frac{1}{c} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W'' \left(- \left(k + \frac{m-n}{N} \right) T_0 \right) \left[y_n(t) - y_m \left(t - (k + \frac{m-n}{N} \right) T_0 \right] .$$

We can identify the first term in the right hand side of Equation (2.3) with the mode loss[3,4] suffered by the n-th bunch. It explicitly depends on the bunch index n. One can notice in passing, that the mode loss can be viewed as a shift of the minimum of the potential well (a trivial distortion). It is simply related to a physical observable that is readily measurable; that is the synchronous phase shift. We will denote the mode loss by f_n.

$$f_n = A \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W' \left(- \left(k + \frac{m-n}{N} \right) T_0 \right) . \tag{2.4}$$

Two ingredients of last term in Equation (2.3) have the following physical interpretation. The term in $W''y_n$ indicates how the wake field changes with respect to variations of the observation point y_n on the present turn, while the wake field sources $y_m = 0$ are held constant. Similarly, the term in $W''y_m$ indicates how the wake field at location $y_n = 0$ (on the present turn) changes with respect to variations of the sources y_m on the present and all previous turns. The difference in sign between the two terms arises because the wake field depends on the separation of its sources and the observation points, that is the difference of their locations. Furthermore, the term proportional to y_n in the right hand side of Equation (2.3) can be absorbed by the synchrotron frequency. This is known as the synchrotron frequency shift due to the potential well distortion[4,5] (change of its curvature); it will be denoted by $\Delta\omega_n^2$. This term can be absorbed to redefine the perturbed synchrotron frequency corrected according to the following expression

$$\omega_n^2 = \omega^2 - \Delta\omega_n^2 \tag{2.5}$$

where

$$\Delta\omega_n^2 = A\frac{1}{c}\sum_{k=-\infty}^{\infty}\sum_{m=0}^{M-1} W''\left(-\left(k+\frac{m-n}{N}\right)T_0\right)$$

Here, ω_n – the corrected synchrotron frequency of the n-th bunch explicitly depends on the bunch index n. Both f_n and $\Delta\omega_n^2$ will be evaluated explicitly in the next section.

Now the set of equations of motion, Equation (2.2), can be rewritten as follows

$$\frac{\partial^2}{\partial t^2} y_n(t) + \left(\omega^2 - \Delta\omega_n^2\right) y_n(t) = f_n + \tag{2.6}$$

$$-A\frac{1}{c}\sum_{k=-\infty}^{\infty}\sum_{m=0}^{M-1} W''\left(-\left(k+\frac{m-n}{N}\right)T_0\right) \quad y_m\left(t-\left(k+\frac{m-n}{N}\right)T_0\right),$$

where the last term in the right hand side of Equation (2.6) represents the coherent multi bunch wake field coupling term, which may result in a collective synchrotron motion of bunches — a multi bunch instability.

The resulting set of equations of motion, Equation (2.6), along with a convenient representation of the wake field coupling in terms of the longitudinal impedance, will be analyzed in the complex frequency domain later in the paper.

3 MODE LOSS AND SYNCHROTRON TUNE SHIFT – FULL VS PARTIALLY FILLED RING

To evaluate both the mode loss term and the synchrotron frequency shift due to the potential well distortion it is convenient to express them in terms of the longitudinal coupling impedance. The time derivative of the wake function is related to the longitudinal impedance via the inverse Fourier transform as follows

$$W'(t) = \frac{c}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} Z_{\parallel}(\omega)\ . \tag{3.1}$$

Similar relationship holds for the second time derivative of the wake function

$$W''(t) = -\frac{c}{2\pi i} \int_{-\infty}^{\infty} d\omega \; \omega \; e^{i\omega t} Z_{\|}(\omega) \; . \tag{3.2}$$

One may substitute Equation (3.1) and (3.2) into Equation (2.4) and (2.5) respectively. The resulting expressions are written as follows

$$f_n = \frac{Ac}{2\pi} A \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} \int_{-\infty}^{\infty} d\omega \; e^{-i\omega(k+\frac{m-n}{N} T_o} Z_{\|}(\omega) \; , \tag{3.3}$$

and

$$\Delta\omega_n^2 = \frac{A}{2\pi i} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} \int_{-\infty}^{\infty} d\omega \; \omega \; e^{-i\omega\left(k+\frac{m-n}{N}\right)T_o} Z_{\|}(\omega) \; , \tag{3.4}$$

Infinite summation over k can be carried out explicitly using a trivial version the Poisson sum identity:

$$\sum_{k=-\infty}^{\infty} e^{-i\omega k T_o} = \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - p\omega_0) \tag{3.5}$$

Substituting Equation (3.5) in Eqs. (3.3) and (3.2) one can also carry out integration over ω. The resulting expressions are given by

$$f_n = \omega_0 \frac{Ac}{2\pi} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} Z_{\|}(p\omega_0) \; e^{2\pi i p \frac{n-m}{N}} , \tag{3.6}$$

and

$$\Delta\omega_n^2 = \omega_0 \frac{A}{2\pi i} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} (p\omega_0) \; Z_{\|}(p\omega_0) \; e^{2\pi i p \frac{n-m}{N}} \; . \tag{3.7}$$

Applying a simple sum identity, Equation (A1), to Eqs. (3.6) and (3.7) one can rewrite them in the following form

$$f_n = \omega_0 \frac{Ac}{2\pi} \sum_{q=-\infty}^{\infty} \sum_{l=0}^{M-1} e^{2\pi i l \frac{n}{N}} \; Z_{\|}\left((Nq+l)\omega_0\right) \sum_{m=0}^{M-1} e^{-2\pi i l \frac{m}{N}} , \tag{3.8}$$

and

$$\Delta\omega_n^2 = \omega_0 \frac{A}{2\pi i} \sum_{q=-\infty}^{\infty} \sum_{l=0}^{M-1} e^{2\pi i l \frac{n}{N}} \left[(Nq+l)\omega_0\right] \; Z_{\|}\left(Nq+l\right) \omega_0) \sum_{m=0}^{M-1} e^{-2\pi i l \frac{m}{N}} , \tag{3.9}$$

The last summation (over m) in Equation (3.8) and (3.9) can be carried out explicitly, Equation (A3) and (A5), (see Appendix A). The resulting formula is written as follows

$$B_l = \sum_{m=0}^{M-1} e^{-2\pi i l \frac{m}{N}} = e^{\pi i \frac{l}{N}(M-1)} \frac{\sin \pi \frac{lM}{N}}{\sin \pi \frac{l}{N}}, \qquad l = 0 \ldots N-1. \tag{3.10}$$

One can immediately see that in case of the full ring ($M = N$) the above expression, Equation (3.10), reduces to the following simple form

$$B_l = N \delta_{l,0}, \qquad l = 0 \ldots N-1 \, . \tag{3.11}$$

Substitution Equation (3.11) into Equation (3.8) and (3.9) and carrying summation over l yields the following set of expressions for the full ring

$$(f)_{\text{full}} = \omega_0 \frac{Ac}{2\pi} N \sum_{q=-\infty}^{\infty} Z_{\|}(Nq\omega_0) \, , \tag{3.12}$$

and

$$\left(\Delta\omega^2\right)_{\text{full}} = \omega_0 \frac{A}{2\pi i} N \sum_{q=-\infty}^{\infty} (Nq\omega_0) \;\; Z_{\|}(Nq\omega_0) \, . \tag{3.13}$$

We notice in passing that in the above expressions, Equation (3.12) and (3.13), the bunch index, n, is no longer present. This simplicity, granted by the symmetry of the bunch configuration is inherent to the full ring case only. For the general partially filled ring case ($M < N$) both quantities: f_n and $\Delta\omega_n^2$, given by Equation (3.8) and (3.9) vary with the bunch location within the sequence. In the next two sections we will derive analytically a simple set of closed formulas describing f_n and $\Delta\omega_n^2$ in the case of partially filled ring ($M < N$) driven by a general resonant impedance.

4 FUNDAMENTAL AND PARASITIC MODE LOSS

We wish to evaluate the general mode loss function, f_n, given by Equation (3.6), where $n = 0, \ldots, M-1$ for the case of partially filled ring ($M < N$). Summing explicitly over m, one can rewrite Equation (3.6) into the following convenient form

$$f_n = \omega_0 \frac{Ac}{2\pi} \sum_{p=-\infty}^{\infty} e^{2\pi i (p\omega_o)\frac{n}{N\omega_o}} \;\; Z_{\|}(p\omega_0) \;\; \frac{1 - e^{-2\pi i (p\omega_o)\frac{M}{N\omega_o}}}{1 - e^{-2\pi i (p\omega_o)\frac{1}{N\omega_o}}} = \sum_{p=-\infty}^{\infty} F(p\omega_0) \tag{4.1}$$

The last part of Equation (4.1) highlights generic sampling structure of the above expression. Applying the following form of the Poisson sum identity

$$\sum_{p=-\infty}^{\infty} F(p\omega_0) = \sum_{q=-\infty}^{\infty} \frac{1}{\omega_o} \int_{-\infty}^{\infty} d\omega \;\; e^{2\pi i q \frac{w}{w_o}} \;\; F(\omega) \, , \tag{4.2}$$

to Equation (4.1) one can express it in the following form

$$f_n = iAc \sum_{q=-\infty}^{\infty} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \, Z_{\|}(\omega) \frac{e^{2\pi i\left(q+\frac{n}{N}\right)\frac{\omega}{\omega_0}} - e^{2\pi i\left(q-\frac{M-n}{N}\right)\frac{\omega}{\omega_0}}}{1-e^{-2\pi i\frac{\omega}{N\omega_0}}} . \tag{4.3}$$

Introducing two kinds of generic integrals, namely:

$$I^{+}(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \;\; Z_{\|}(\omega) \frac{e^{2\pi ik\frac{\omega}{N\omega_0}}}{1-e^{-2\pi i\frac{\omega}{N\omega_0}}} , \qquad k \geq 0 \tag{4.4}$$

and

$$I^{-}(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \;\; Z_{\|}(\omega) \frac{e^{-2\pi ik\frac{\omega}{N\omega_0}}}{1-e^{-2\pi i\frac{\omega}{N\omega_0}}} , \qquad k \geq 0 \tag{4.5}$$

one can express the mode loss term, Equation (4.3), in the following compact form

$$\begin{aligned} f_n =& iAc \left\{ I^{+}(n) - I^{-}(n) + \right) \sum_{q=1}^{\infty} \left[I^{+}(Nq+n) + I^{-}(Nq-n) + \right. \\ &\left. \left. - \; I^{+}(Nq-(M-n)) - I^{-}(Nq+(M-n)) \right] \right\} , \end{aligned} \tag{4.6}$$

Assuming general form of the longitudinal impedance of a resonant structure, given by the following standard expression:

$$Z_{\|}(\omega) = \frac{R}{1+iQ\left(\frac{\omega}{\omega_r}-\frac{\omega_r}{\omega}\right)} , \qquad Q \gg 1 \tag{4.7}$$

where R is the shunt resistance, Q is the quality factor of the resonator and ω_r is its resonance frequency, one can evaluate integrals $I^{+}(k)$ and $I^{-}(k)$ explicitly via contour integration (see Appendix B). The resulting expressions given by Equation (B5) and (B6) are summarized as below

$$I^{+}(k) = \frac{1}{2}\omega_0 \frac{1}{2\pi i} N \sum_{p=-\infty}^{\infty} Z_{\|}(Np\omega_0) - i\frac{R}{2Q} \left[\frac{\omega_{+} e^{2\pi ik\frac{\omega_{+}}{N\omega_0}}}{1-e^{-2\pi i\frac{\omega_{+}}{N\omega_0}}} - \frac{\omega_{-} e^{2\pi ik\frac{\omega_{-}}{N\omega_0}}}{1-e^{-2\pi i\frac{\omega_{-}}{N\omega_0}}} \right] , \tag{4.8}$$

and

$$I^{-}(k) = -\frac{1}{2}\omega_0 \frac{1}{2\pi i} N \sum_{p=-\infty}^{\infty} Z_{\|}(Np\omega_0) , \tag{4.9}$$

where the singularities of $Z_{\|}(\omega)$ are defined by the following pair of complex poles $\omega_{\pm}$ located in the upper half plane

$$\omega_{\pm} = \omega_r(\pm 1 + i\delta), \qquad \delta = \frac{1}{2Q} \ll 1 \,. \tag{4.10}$$

Substituting the above integrals, Equation (4.9)–(4.10) into Equation (4.6) one can carry out remaining infinite summation over q, which converges to the following simple expression

$$\sum_{q=1}^{\infty} e^{2\pi i q \frac{\omega_{\pm}}{\omega_o}} = \frac{e^{2\pi i k \frac{\omega_{\pm}}{\omega_o}}}{1 - e^{2\pi i \frac{\omega_{\pm}}{\omega_o}}} \,, \qquad \text{since Im } \omega_{\pm} > 0. \tag{4.11}$$

Finally, one can rewrite Equation (4.6) in the following form

$$f_n = \omega_0 \frac{Ac}{2\pi} N \sum_{p=-\infty}^{\infty} Z_{\|}(Np\omega_0) + \tag{4.12}$$

$$+Ac\frac{R}{2Q}\left[\frac{\omega_+ e^{2\pi\, in\frac{\omega_+}{N\omega_o}}\left(1 - e^{2\pi\, i(N-M)\frac{\omega_+}{N\omega_o}}\right)}{\left(1 - e^{2\pi\, i\frac{\omega_+}{N\omega_o}}\right)\left(1 - e^{2\pi\, i\frac{\omega_+}{\omega_o}}\right)} - \frac{\omega_- e^{2\pi\, in\frac{\omega_-}{N\omega_o}}\left(1 - e^{2\pi(N-M)\frac{\omega_-}{N\omega_o}}\right)}{\left(1 - e^{2\pi i\frac{\omega_-}{N\omega_o}}\right)\left(1 - e^{2\pi i\frac{\omega_-}{\omega_o}}\right)}\right] .$$

The first term in Equation (4.12) can be immediately identified with the mode loss term for a fully populated ring, $(f)_{\text{full}}$, given in the previous section by Equation (3.12). Furthermore, $(f)_{\text{full}}$ was evaluated explicitly for our model resonant impedance, Equation (4.7), at the and of the Appendix B (see Equation (B.14)). The result can be summarized as follows

$$(f)_{\text{full}} = \frac{AcR\omega_o}{\pi}\left(\frac{\pi\omega_r}{N\omega_o}\right) N\delta^2 \frac{\left(\frac{\pi\omega_r}{N\omega_o}\right) - \frac{1}{2}\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2 \delta^2} \,. \tag{4.13}$$

The above quantity is plotted for our numerical example in Figure 1. One can see, that the peak value scales with the total number of bunches, N, and the width (FWHH) of the peak in Figure 1 is determined by the following identity:

$$\Delta\omega_r = \pm N\omega_o \frac{k}{2Q} \,. \tag{4.14}$$

Introducing a new function $\Gamma(\omega)$ defined by:

$$\Gamma(\omega) = \frac{e^{2\pi\, in\frac{\omega}{N\omega_o}}\left(1 - e^{2\pi\, i(N-M)\frac{\omega}{N\omega_o}}\right)}{\left(1 - e^{2\pi\, i\frac{\omega}{N\omega_o}}\right)\left(1 - e^{2\pi\, i\frac{\omega}{\omega_o}}\right)} \,, \tag{4.15}$$

one can express f_n in the following compact form

$$f_n = (f)_{\text{full}} + Ac\frac{R}{2Q}\left[\omega_+\Gamma(\omega_+) - \omega_-\Gamma(\omega_-)\right] \,. \tag{4.16}$$

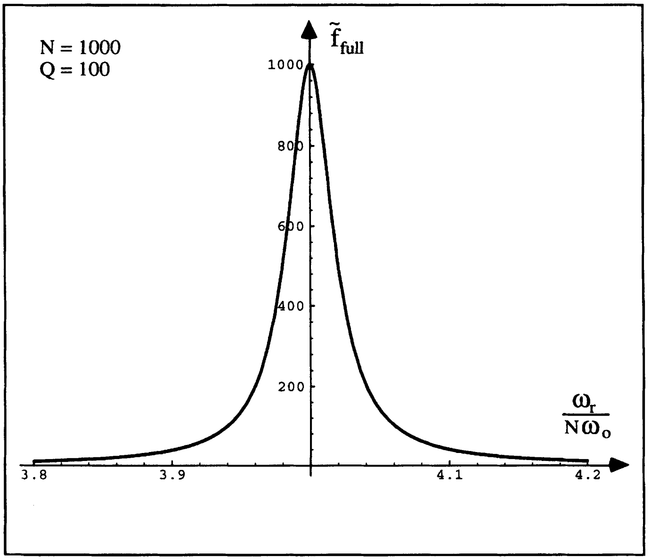

FIGURE 1: Completely filled ring – dimensionless mode loss, $\tilde{f}$, as a function of the resonance frequency, ω_r, of the coupling impedance.

Employing symmetry of $\Gamma(\omega)$, namely

$$\Gamma(-\omega^*) = \Gamma^*(\omega), \tag{4.17}$$

and the fact that $\omega_- = -\omega_+^*$, one can rewrite Equation (4.16) as follows

$$f_n = (f)_{\text{full}} + Ac\frac{R}{2Q}\, 2\text{Re}\left[\omega_+ \Gamma(\omega_+)\right] . \tag{4.18}$$

Cutting through some tedious algebra explicit expressions for $\text{Re}[\Gamma(\omega_+)]$ and $\text{Im}[\Gamma(\omega_+)]$ were worked out as an expansion in δ (keeping up to quadratic terms in δ). Substituting them along with Equation (4.13) into Equation (4.18) one obtains the final formula for f_n

$$f_n = \frac{AcR\omega_o}{\pi}\left\{\left(\frac{\pi\omega_r}{N\omega_o}\right)\delta^2 \frac{M\left(\frac{\pi\omega_r}{N\omega_o}\right) - \frac{1}{2}N\,\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2} + \right. \tag{4.19}$$

$$\left. - \frac{\sin\left(\frac{\pi\omega_r}{N\omega_o}\right)\,\sin\left(\frac{\pi\omega_r}{N\omega_o}M\right)\,\cos\left(\frac{\pi\omega_r}{N\omega_o}(2n-M-1)\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2}\right\}, \quad \delta = \frac{1}{2Q} \ll 1 .$$

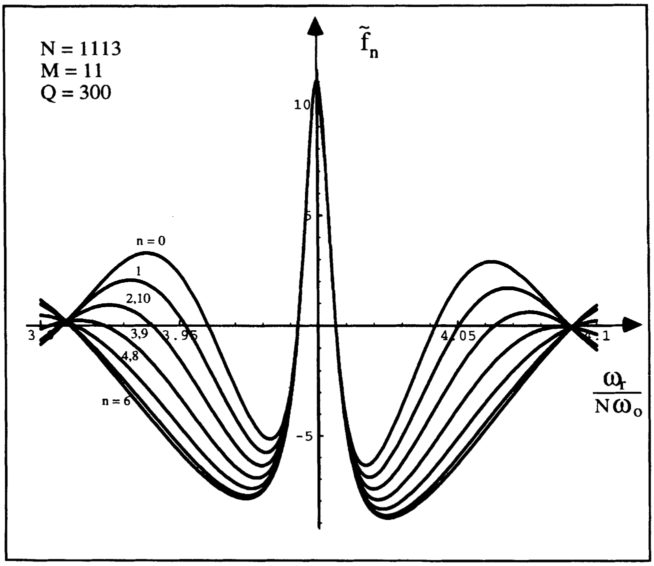

FIGURE 2: Partially filled ring – dimensionless mode loss, $\tilde{f}$, acting on the n-th bunch as a function of the resonance frequency, ω_r, of the coupling impedance.

Denoting the expression in curly bracket by $\tilde{f}_n$, one can introduce a dimensionless mode loss. Figure 2 illustrates a family of curves for different values of n, calculated according to Equation (4.19). As one can see Equation (4.19) has a simple asymptotics for the resonace frequencies, ω_r, in the vicinity of the integer multiples of the r.f. frequency, $kN\omega_o$, and away from them. These two asymptotic regions are determined by the relative strength of the expressions appearing in the denominator of Equation (4.19), namely: $\sin^2(\pi x)$ and $(\pi x)^2\,\delta^2$. It is convenient to introduce a dimensionless resonance frequency, x, (in units of the r.f. frequency) namely,

$$x = \frac{\omega_r}{N\omega_o} \tag{4.20}$$

Now, 'the immediate vicinity of the integer multiple of the r.f. frequency' is defined by the following inequality

$$\sin^2(\pi x) \ll (\pi x)^2\,\delta^2\ , \tag{4.21}$$

which can be rewritten into the following simple form

$$|x - k| \ll k\delta\ . \tag{4.22}$$

The remainder of the frequency domain, namely resonance frequencies given by

$$|x - k| \gg k\delta\ , \tag{4.23}$$

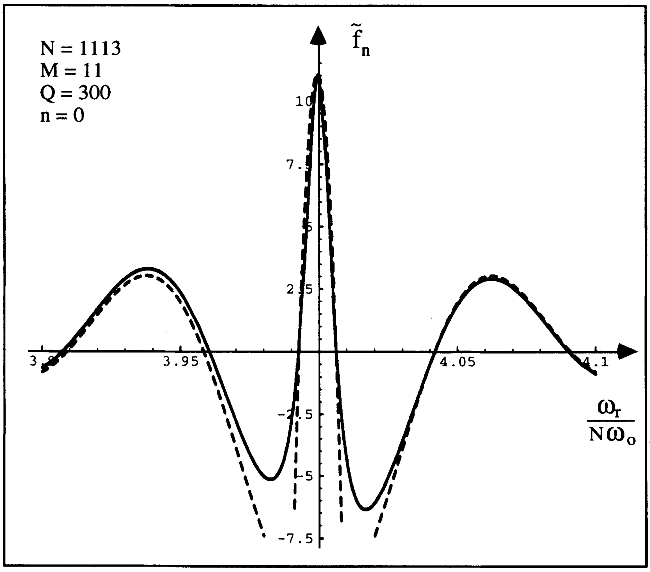

FIGURE 3: Asymptotics of the dimensionless mode loss, $\tilde{f}_n$, acting on the 0-th bunch for resonance frequencies, ω_r, at the '*immediate vicinity*' and '*away*' from the multiples of the r.f. frequency.

are considered to be 'away from the multiples of the r.f. frequency' – the inequality given by Equation (4.21) reversed.

Applying the above asymptotics, Equation (4.21)–(4.23), to Equation (4.19) (neglecting either $\sin^2(\pi x)$ or $(\pi x)^2\,\delta^2$, term in the denominator) reduces Equation (4.19) to the following simple expression

$$\left(\tilde{f}_n\right)^{\text{asym}} = \begin{cases} M - N\frac{\sin(2\pi x)}{2\pi x} - 4Q^2(-1)^{k(M+1)}\ \frac{\sin(\pi x)}{\pi x}\,\frac{\sin(\pi x M)}{\pi x} & \text{for } |x-k| \ll k\delta \\ -\frac{\sin(\pi x M)}{\sin(\pi x)}\ \cos(\pi x(2n-M-1)) & \text{for } |x-k| \gg k\delta \end{cases} \tag{4.24}$$

Figure 3 illustrates a comparison between the exact formula, $\tilde{f}_n(x)$, Equation (4.19), and its asymptotic version, given by Equation (4.24). One can notice, that for the resonance frequencies in 'the immediate vicinity of the integer multiple of the r.f. frequency' (the first asymptotic region in Equation (4.24)) the resulting *mode loss does not depend on the bunch index*, n, and it is governed by the quality factor, Q. Conversely, for the resonance frequencies 'away from the immediate vicinity of the integer multiple of the r.f. frequency' (the second asymptotic region in Equation (4.24)) the resulting *mode loss does not depend on the quality factor*, Q, and it is governed strictly by the bunch index, n. Therefore, for parasitic modes at resonance frequencies 'away from the immediate vicinity of the integer multiple of the r.f. frequency', which is usually the case, the so called 'de-Q-ing' of the modes does not have any effect on the beam loading forces experienced by individual bunches; see Equation (4.24).

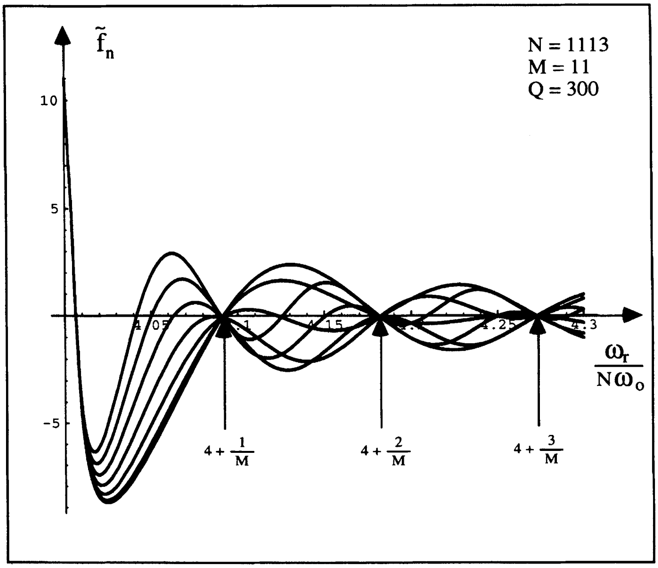

FIGURE 4: Dimensionless mode loss, $\tilde{f}_n$, acting on the n-th bunch for resonance frequencies, ω_r, '*away from the immediate vicinity*' of the multiple of the r.f. frequency. The mode loss vanishes for a discrete set of resonant frequencies defined by the fractional, $\frac{\ell}{M}$, multiples of $N\omega_o$.

Furthermore, the structure of Equation (4.24) (zeros of $\sin(\pi x M)$) reveals another finer level of symmetry governed by the fractional, $\frac{\ell}{M}$, multiples of $N\omega_o$. Indeed, as seen in Figure 4, the mode loss vanishes up to terms of $O(\delta^2)$, for a discrete set of resonance frequencies defined by

$$\omega_r = \left(k + \frac{\ell}{M}\right) N\omega_o\,, \quad \ell = 1, 2, \ldots, M-1\,. \tag{4.25}$$

These resonance frequencies are clearly marked in Figure 4 (arrows). Similarly, one can find frequency regions where bunch-to-bunch variation of the mode loss is the strongest — they are defined by the extremes of $\sin(\pi x M)$, which is also illustrated in Figure 4.

5 SYNCHROTRON TUNE SHIFT

We wish to evaluate general form of the synchrotron tune shift, $\Delta\omega_n^2$, given by Equation (3.6), where $n = 0, \ldots, M-1$ for the case of partially filled ring ($M < N$). Summing explicitly over m, one can rewrite Equation (3.6) into the following convenient form

$$\Delta\omega_n^2 = \omega_0 \frac{Ac}{2\pi i} \sum_{p=-\infty}^{\infty} e^{2\pi i (p\omega_o)\frac{n}{N\omega_o}} \, (p\omega_0)\, Z_{\|}(p\omega_0) \frac{1 - e^{-2\pi i (p\omega_o)\frac{M}{N\omega_o}}}{1 - e^{-2\pi i (p\omega_o)\frac{1}{N\omega_o}}} = \sum_{p=-\infty}^{\infty} G(p\omega_0) \tag{5.1}$$

The last part of Equation (5.1) highlights generic sampling structure of the above formula.

The above expression, Equation (5.1), resembles Equation (4.1) – the starting point of the previous section; in fact one can apply exactly the same method, as used in Sec. 4, to evaluate the synchrotron tune shift. Repeating a whole sequence of steps from the previous section, described by Equation (4.2) – (4.11), one obtains an analog of Equation (4.12), which can be written as follows

$$\Delta\omega_n^2 = \omega_0 \frac{Ac}{2\pi i} N \sum_{p=-\infty}^{\infty} (Np\omega_0)\ Z_{\|}(Np\omega_0) + \tag{5.2}$$

$$-Ai\frac{R}{2Q}\left[\frac{\omega_+ 2e^{2\pi\, in\frac{\omega_+}{N\omega_o}}\left(1-e^{2\pi\, i(N-M)\frac{\omega_+}{N\omega_o}}\right)}{\left(1-e^{2\pi\, i\frac{\omega_+}{N\omega_o}}\right)\left(1-e^{2\pi\, i\frac{\omega_+}{\omega_o}}\right)} - \frac{\omega_- 2e^{2\pi\, in\frac{\omega_-}{N\omega_o}}\left(1-e^{2\pi i(N-M)\frac{\omega_-}{N\omega_o}}\right)}{\left(1-e^{2\pi i\frac{\omega_-}{N\omega_o}}\right)\left(1-e^{2\pi i\frac{\omega_-}{\omega_o}}\right)}\right].$$

Similarly, the first term in Equation (5.2) can be immediately identified with the synchrotron tune shift for a fully populated ring, $(\Delta\omega^2)_{\text{full}}$, given before by Equation (3.13). Furthermore, it was evaluated explicitly for our model resonant impedance, Equation (4.7), in the Appendix C (see Equation (C.12)). The result can be summarized as follows

$$\left(\Delta\omega^2\right)_{\text{full}} = \frac{1}{2}AR\omega_r^2\delta\left\{\frac{\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right)+2\delta^2\left(\frac{\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right)+\left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2} - \frac{2\cos\left(\frac{1}{2}\tau\omega_r\right)}{\frac{1}{2}\tau\omega_r}\right\}. \tag{5.3}$$

Here, τ is a characteristic bunch length in units of time. Using previously defined function $\Gamma(\omega)$, Equation (4.14), one can express $\Delta\omega_n^2$ in the following compact form

$$\Delta\omega_n^2 = (\Delta\omega^2)_{\text{full}} - Ai\frac{R}{2Q}\left[\omega_+^2\Gamma(\omega_+) - \omega_-^2\Gamma(\omega_-)\right]. \tag{5.4}$$

Employing symmetry of $\Gamma(\omega)$, Equation (4.16), and the fact that $\omega_- = -\omega_+^*$, one can rewrite Equation (5.4) as follows

$$\Delta\omega_n^2 = \left(\Delta\omega_n^2\right)_{\text{full}} + A\frac{R}{2Q}\,2\text{Im}\left[\omega_+^2\,\Gamma(\omega_+)\right]. \tag{5.5}$$

Substituting the explicit expressions for $\text{Re}[\Gamma(\omega_+)]$ and $\text{Im}[\Gamma(\omega_+)]$ along with Equation (5.3) into Equation (5.5) one obtains the final formula for $\Delta\omega_n^2$

$$\Delta\omega_n^2 = \frac{A\omega_o\omega_r}{\pi}\frac{R}{Q}\left\{-\frac{T_o}{2\tau}\cos\left(\pi N\frac{\tau}{T_o}\right)+\left(\frac{\pi\omega_r}{N\omega_o}\right)\frac{M\delta^2\left(\frac{\pi\omega_r}{N\omega_o}\right)+\frac{1}{4}N\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right)+\left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2}+\right.$$

$$\left.-\frac{1}{2\delta}\frac{\sin\left(\frac{\pi\omega_r}{N\omega_o}\right)\sin\left(\frac{\pi\omega_r}{N\omega_o}M\right)\sin\left(\frac{\pi\omega_r}{N\omega_o}(2n-M-1)\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right)+\left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2}\right\}. \tag{5.6}$$

The above expression is still divergent with respect to $\tau \to 0$. It is worth noting, that part of the problem derives from using a resonator model over the infinite frequency domain, whereas this model is only an accurate model of a fundamental or parasitic cavity mode close to resonance. Denoting the expression in curly bracket by $\Delta\tilde{\omega}_n^2$, one can introduce a dimensionless square of the synchrotron tune shift. Its asymptotic behavior for the resonant frequencies, ω_r, in the vicinity of the integer multiples of the r.f. frequency, $N\omega_o$, namely:

$$\omega_r = kN\omega_o + \Delta\omega_r \,, \tag{5.7}$$

is determined by the relative strength of the expressions appearing in the denominator of Equation (5.6). As before, we define the immediate vicinity of the integer multiple by the inequalities given by Equation (4.21) or Equation (4.22). If these conditions are satisfied, Equation (5.6), reduces to the following simple expression:

$$\Delta\tilde{\omega}_n^2 = -\frac{T_o}{2\tau}\cos\left(\pi N\frac{\tau}{T_o}\right) + M \,, \tag{5.8}$$

where a bunch independent square of the synchrotron frequency shift scales as the total number of bunches, M. As an example, we assign a realistic value of a bunch length, in terms of the bunching factor, $N\frac{\tau}{T_o}$, equal to 0.1, the first term in Equation (5.8) can be evaluated approximately as follows

$$\frac{T_o}{2t}\cos\left(\pi N\frac{\tau}{T_o}\right) \approx 5N \,. \tag{5.9}$$

Outside the immediate vicinity of the integer multiple (inequality, Equation (4.21), reversed) our expression, Equation (5.6), assumes the following asymptotic form:

$$\Delta\tilde{\omega}_n^2 = -5N + \frac{1}{2}N\left(\frac{\pi\omega_r}{N\omega_o}\right)\cot\left(\frac{\pi\omega_r}{N\omega_o}\right) - \frac{Q\,\sin\left(\frac{\pi\omega_r}{N\omega_o}M\right)\,\sin\left(\frac{\pi\omega_r}{N\omega_o}(2n-M-1)\right)}{\sin\left(\frac{\pi\omega_r}{N\omega_o}\right)} \,, \tag{5.10}$$

which does not depend explicitly on the resonance width, δ. Apart from the integer multiples of the r.f. frequency, $N\omega_o$, the structure of Equation (5.6) (zeros of $\sin\left(\frac{\pi\omega_r}{N\omega_o}M\right)$) reveals another finer level of symmetry governed by the fractional, $\frac{l}{M}$, multiples of $N\omega_o$. Indeed, the third (bunch index dependent) term in Equation (5.6) vanishes for a discrete set of resonant frequencies defined by

$$\omega_r = \left(k + \frac{l}{M}\right)N\omega_o \,, \qquad l = 1, 2, \ldots, M-1 \tag{5.11}$$

The amount of the synchrotron tune shift for these resonant frequencies does not depend on the bunch index. Conversely, one can find frequency regions where bunch-to-bunch variation of the synchrotron tune is the strongest. From Equation (5.10) one can easily identify them with the extremes of $\sin\left(\frac{\pi\omega_r}{N\omega_o}\right)$, namely

$$\omega_r = \left(k + \frac{2l+1}{2M}\right) N\omega_o, \qquad l = 1, 2, \dots, M-1 \tag{5.13}$$

One can see from Equation (5.6) and (5.10) that the synchrotron frequency shift in case of a partially and completely filled ring are almost the same, since the third term in both equations is very small compare to the first two. Figure 5 illustrates the case of a completely filled ring (for $N = 1000$ and $Q = 100$), where the synchrotron frequency shift denoted by $(\Delta\tilde{\omega}^2)_{\text{full}}$, extracted from Equation (5.6) is plotted as a function of the resonance frequency, ω_r.

To illustrate a small bunch-to-bunch variation effect, a family of curves $\Delta\tilde{\omega}^2 - (\Delta\tilde{\omega}^2)_{\text{full}}$ is plotted as a function of ω_r in Figure 6 ($N = 1000$, $M = 10$ and $Q = 100$). All the asymptotic features of the synchrotron tune shift, we discussed above, are visible in our example.

6 SUMMARY

Both potential well distortion characteristics: the mode loss and the synchrotron tune shift experienced by a given bunch within the train, were calculated analytically (using contour integration technique) for a partially filled ring ($M < N$). Here individual bunches are mutually interacting via wake fields generated by resonant structures. Resulting simple analytic formulas express both quantities, as a function of the resonance frequency, ω_r, and the quality factor of the coupling impedance, Q.

Both formulae reveal, that for resonator natural frequencies in the vicinities of the integer multiples of the r.f. frequency, $N\omega_o$, the beam loading response is equal for all bunches (its absolute value scales as M). It also identifies the second set of characteristic resonant frequencies, spaced by the multiples of $N\omega_o/M$, at which the potential well distortion characteristics are not only bunch independent, but also considerably smaller (it scales as MQ^{-2}). Further we identified another interesting set of resonance frequencies (ω_r at odd multiples of $N\omega_o/2M$), where bunch-to-bunch variation of the mode loss and the synchrotron tune shift is the strongest.

Finally, for a given configuration of cavity resonances one can get immediately a simple quantitative answer in terms of the mode loss and the synchrotron tune shift experienced by each bunch along the train[6]. These analytic expressions give one an insight into various optimizing schemes; e.g. to modify the existing configuration of parasitic cavity resonances (via frequency tuning), so that the resulting potential well distortion effects are minimized.

ACKNOWLEDGEMENTS

The author wishes to thank Bill Ng, Francois Ostiguy and Pat Colestock for stimulating discussions and comments. Many useful comments from Shane Koscielniak, regarding terminology and jargon used in this paper, improved its clarity; they are gratefully acknowledged.

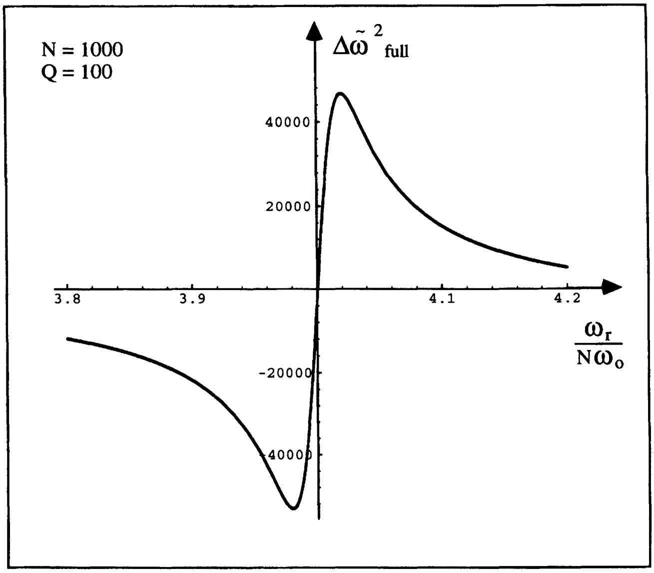

FIGURE 5: Completely filled ring – dimensionless square of the synchrotron tune shift, $\Delta\tilde{\omega}^2$, as a function of the resonance frequency, ω_r, of the coupling impedance.

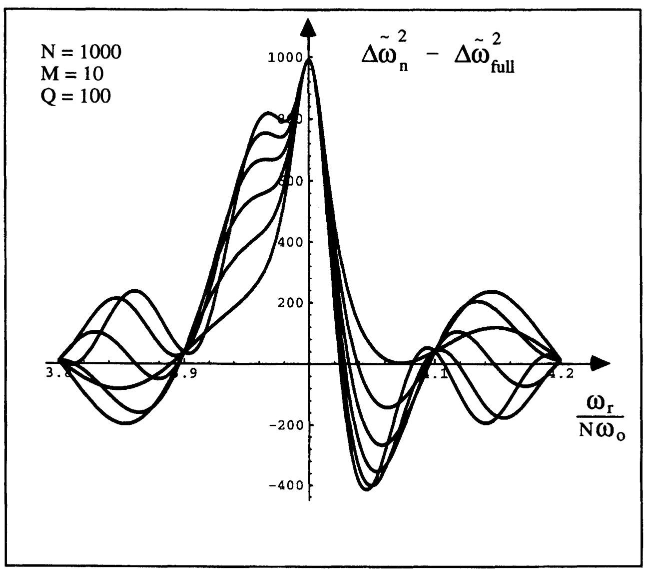

FIGURE 6: Partially filled ring – dimensionless square of the synchrotron tune shift, $\Delta\tilde{\omega}_n^2$, experienced by the n-th bunch as a function of the resonance frequency, ω_r, of the coupling impedance.

APPENDIX A

The following useful summation identity can be proven by inspection

$$\sum_{p=-\infty}^{\infty} F(p\omega_0) = \sum_{q=-\infty}^{\infty} \sum_{l=0}^{N-1} F\left((Nq+l)\omega_0\right) \ , \qquad \text{(A.1)}$$

where F is an arbitrary continuous function and N is a positive integer. Indeed, any integer p can be written as $p = Nq + l$, where the numbers q and l are unique (there is one-to-one correspondence between p and a pair (q, l)). Therefore summation over p is equivalent to a double summation over q and l.

Let us evaluate the following sum of the first M N-th roots of unity, $M \leq N$

$$A_{\mu\nu} = \sum_{m=0}^{M-1} e^{2\pi i m \frac{\mu-\nu}{N}}, \qquad \begin{array}{l} \mu = 0 \dots N-1 \\ \nu = 0 \dots N-1 \end{array} \qquad \text{(A.2)}$$

The above sum is in fact a sum of a geometric series, which can be easily evaluated as

$$A_{\mu\nu} = e^{\pi \frac{\mu-\nu}{N}(M-1)} \frac{\sin \pi(\mu-\nu)\frac{M}{N}}{\sin \pi \frac{\mu-\nu}{N}} \ . \qquad \text{(A.3)}$$

One can see that for $M = N$, the above equation simplifies as follows

$$A_{\mu\nu} = \sum_{m=0}^{N-1} e^{2\pi\, i m \frac{\mu-\nu}{N}} = N\delta_{\mu\nu} \ . \qquad \text{(A.4)}$$

We notice in passing that the general form of $A_{\mu\nu}$, given by Equation (A.3), depends on the difference of μ and ν. Therefore, one can identify expression B_l, (see Section 2), with A_{l0}.

$$B_l = A_{l0}, \qquad \text{(A.5)}$$

APPENDIX B

We wish to calculate the following pair of integrals:

$$I^{+}(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \ Z_{\parallel}(\omega) \frac{e^{2\pi\, ik \frac{\omega}{N\omega_o}}}{1 - e^{-2\pi\, i \frac{\omega}{N\omega_o}}} \ , \qquad k \geq 0 \qquad \text{(B.1)}$$

and

$$I^{-}(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \ Z_{\parallel}(\omega) \frac{e^{-2\pi\, ik \frac{\omega}{N\omega_o}}}{1 - e^{-2\pi\, i \frac{\omega}{N\omega_o}}} \ , \qquad k \geq 0 \qquad \text{(B.2)}$$

Assuming general form of the longitudinal impedance of a resonant structure, given Equation (4.7), one can rewrite it in the following form

$$Z_{\|}(\omega) = -iR\frac{\omega_r}{Q}\frac{\omega}{(\omega-\omega_+)(\omega-\omega_-)} , \tag{B.3}$$

where the singularities of $Z_{\|}(\omega)$ are defined by the following pair of complex poles $\omega_\pm$ (located in the upper half plane)

$$\omega_\pm = \omega_r(\pm 1 + i\delta), \quad \delta = \frac{1}{2Q} \ll 1 . \tag{B.4}$$

Here R is the shunt resistance, Q is the quality factor of the resonator and ω_r is its resonance frequency.

Both integrants, written explicitly in Equation (B1) and (B2), have the same configuration of singularities in the complex ω-plane. It includes a pair of poles, $\omega_\pm$, in the upper half-plane, introduced by $Z_{\|}(\omega)$ and an infinite array of poles located on the real axis at integer multiples of $N\omega_o$. This last set of poles is introduced by the zeros of the following denominator: $1 - e^{-2\pi i \frac{\omega}{N\omega_o}}$, which appears in both Equation (B1) and (B2). Furthermore, the integrant of $I^+(k)$ restricted to a semi-circle of radius R, closed in the upper half-plane, vanishes exponentially with $R \to \infty$ (faster than $\frac{1}{R}$). Similar property holds for the integrant of $I^-(k)$ in the lower half plane.

Now we are ready to employ Cauchy's integral theorem to evaluate principle value integrals $I^+(k)$ and $I^-(k)$ explicitly. Figure 7 illustrates a complete set of singularities along with the appropriate choice of integration contours for both $I^+(k)$ and $I^-(k)$; C^+ and C^- respectively. Carrying out integration along these contours via Cauchy's integral theorem reduces integrals $I^+(k)$ and $I^-(k)$ to the following sum of residuum

$$I^+(k) = \frac{1}{2}\omega_0\frac{1}{2\pi i}N\sum_{p=-\infty}^{\infty} Z_{\|}(Np\omega_0) - i\frac{R}{2Q}\left[\frac{\omega_+ e^{2\pi ik\frac{\omega_+}{N\omega_o}}}{1-e^{-2\pi i\frac{\omega_+}{N\omega_o}}} - \frac{\omega_- e^{2\pi ik\frac{\omega_-}{N\omega_o}}}{1-e^{-2\pi i\frac{\omega_-}{N\omega_o}}}\right] , \tag{B.5}$$

and

$$I^-(k) = -\frac{1}{2}\omega_0\frac{1}{2\pi i}N\sum_{p=-\infty}^{\infty} Z_{\|}(Np\omega_0) , \tag{B.6}$$

The infinite sum over p appearing in both Equation (B.5) and (B.6) can be rewritten via the Poisson sum identity, Equation (4.2) as follows

$$N\omega_0\frac{1}{2\pi i}\sum_{p=-\infty}^{\infty} Z_{\|}(Np\omega_0) = \sum_{q=-\infty}^{\infty}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \; e^{2\pi iq\frac{\omega}{N\omega_o}} \; Z_{\|}(\omega) , \tag{B.7}$$

Since both singularities of $Z_{\|}$, $\omega_\pm$, are located in the upper half plane the following integral

$$I_q = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \; e^{2\pi iq\frac{\omega}{N\omega_o}} \; Z_{\|}(\omega) , \tag{B.8}$$

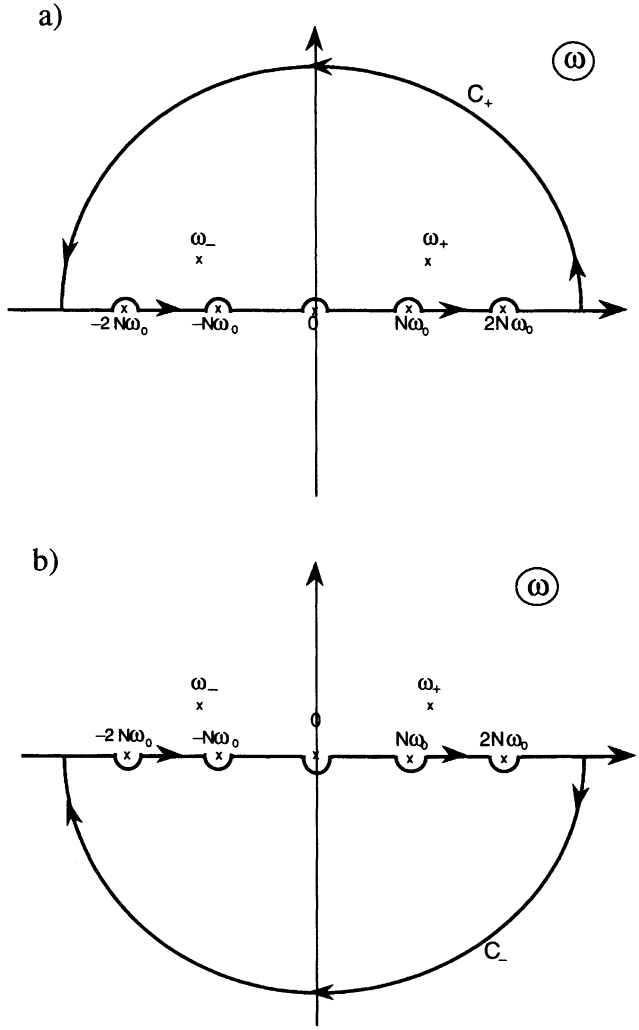

FIGURE 7: Complete set of singularities along with the appropriate choice of integration contours for both $I^+(k)$ and $I^-(k)$; C^+ and C^- respectively. A pair of poles, $\omega_\pm$, (in the upper half-plane) corresponds to the singularities of the coupling impedance, $Z_\parallel$, while an infinite array of equally spaced poles, $N\omega_o$, (on the real axis) corresponds to the zeros of $1-e^{-2\pi i \frac{\omega}{N\omega_o}}$.

vanishes identically for $q < 0$. Simple application of Cauchy's integral theorem along the contours $C_\pm$, illustrated in Figure 7, yields the following expression for I_q, if $q > 0$

$$I_q = -i\frac{R}{2Q}\left[\omega_+ e^{2\pi i k \frac{\omega_+}{N\omega_o}} - \omega_- e^{2\pi i k \frac{\omega_-}{N\omega_o}}\right], \quad \text{for } q > 0. \tag{B.9}$$

The remaining nontrivial case of $I_q (q = 0)$ can be evaluated via Cauchy's integral theorem as follows

$$I_0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \, \mathrm{Re}\left[Z_{\parallel}(\omega)\right] = -i\omega_r \frac{R}{2Q} \, . \tag{B.10}$$

Substituting Equation (B.9) and (B.10) into Equation (B.7) one can carry out the remaining summation over q employing the following convergence formula

$$\sum_{q=1}^{\infty} e^{2\pi i q \frac{\omega_\pm}{N\omega_o}} = \frac{e^{2\pi i \frac{\omega_\pm}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_\pm}{N\omega_o}}} \, , \quad \text{since } \mathrm{Im}\, \omega_\pm > 0 \, . \tag{B.11}$$

The resulting expression can be summarized as follows

$$N\omega_0 \frac{1}{2\pi i} \sum_{p=-\infty}^{\infty} Z_{\parallel}(Np\omega_0) = -i\frac{R}{2Q} \left[\omega_r + \omega_+ \frac{e^{2\pi i \frac{\omega_+}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_+}{N\omega_o}}} - \omega_- \frac{e^{2\pi i \frac{\omega_-}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_-}{N\omega_o}}} \right] , \tag{B.12}$$

where

$$\omega_\pm = \omega_r(\pm 1 + i\delta), \qquad \delta = \frac{1}{2Q} \, . \tag{B.13}$$

After some algebra one can rewrite Equation (B.12) into the following convenient form

$$N\omega_0 \frac{1}{2\pi i} \sum_{q=-\infty}^{\infty} Z_{\parallel}(Nq\omega_0) = -i\omega_r \delta^2 \frac{\left(\frac{\pi\omega_r}{N\omega_o}\right) - \sin\left(\frac{\pi\omega_r}{N\omega_o}\right) \cos\left(\frac{\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2 \delta^2} \, . \tag{B.14}$$

APPENDIX C

We wish to evaluate an analog of Equation (B.7), given by the following expression

$$J = N\omega_0 \frac{1}{2\pi i} \sum_{p=-\infty}^{\infty} (Np\omega_0) \;\, Z_{\parallel}(Np\omega_0) \, , \tag{C.1}$$

The infinite sum over p appearing in Equation (C.1) can be rewritten via the Poisson sum identity, Equation (4.2) as follows

$$J = \sum_{q=-\infty}^{\infty} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \;\, e^{2\pi i q \frac{\omega}{N\omega_o}} \, \omega \, Z_{\parallel}(\omega) \, , \tag{C.2}$$

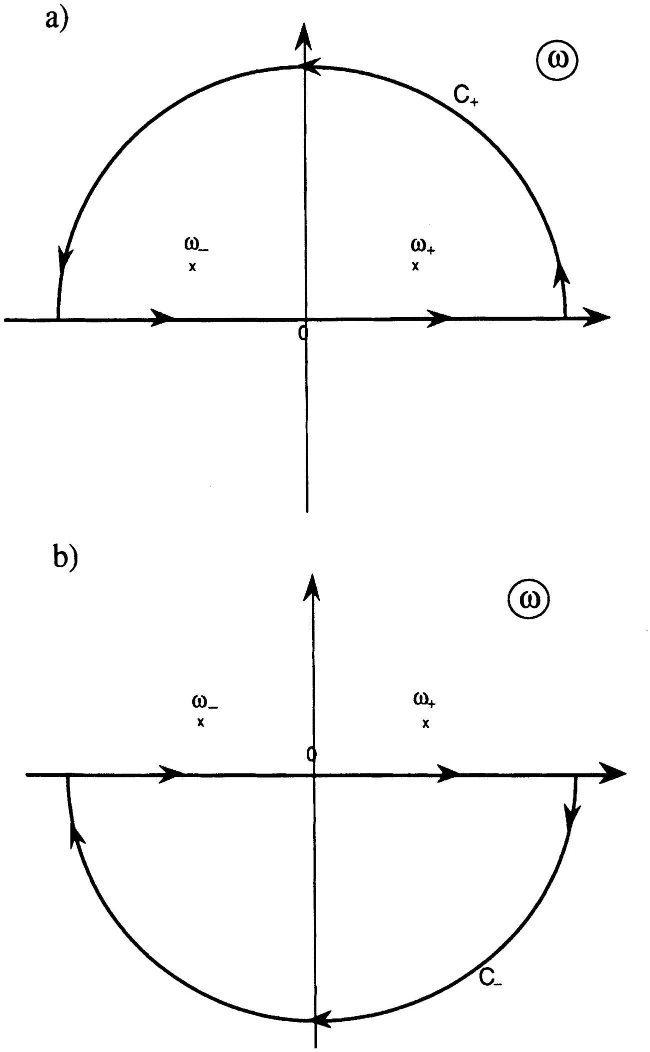

FIGURE 8: Complete set of singularities along with the appropriate choice of integration contours for J_q's; (C^+ for q>0 andC^- for q<0). A pair of poles, $\omega_\pm$, (in the upper half-plane) corresponds to the singularities of the coupling impedance, $Z_\parallel$.

Since both singularities of $Z_\parallel$, $\omega_\pm$, are located in the upper half plane the following integral

$$J_q = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \; e^{2\pi i q \frac{\omega}{N\omega_0}} \; \omega \; Z_\parallel(\omega) \; , \tag{C.3}$$

vanishes identically for $q < 0$ (see contour C_- illustrated in Figure 8b). Simple application of Cauchy's integral theorem along the contour C_+, illustrated in Figure 8a, yields the following expression for J_q, if $q > 0$

$$J_q = -i\frac{R}{2Q}\left[\omega_+^2 e^{2\pi i k \frac{\omega_+}{N\omega_o}} - \omega_-^2 e^{2\pi i k \frac{\omega_-}{N\omega_o}}\right], \quad \text{for } q > 0. \tag{C.4}$$

The remaining nontrivial case of $J_q (q = 0)$ can be written as follows

$$J_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega\, \mathrm{Im}\left[Z_{\parallel}(\omega)\right]. \tag{C.5}$$

One can notice that the above integral diverges for our model impedance, given by Equation (B.3). Indeed, for large values of ω, $\mathrm{Im}[Z_{\parallel}(\omega)] \sim \frac{1}{\omega}$, therefore, $J_0 \sim \int_{-\infty}^{\infty} d\omega\, \omega\, \frac{1}{\omega} \to \infty$. This unphysical divergence can be removed assuming finite bunch length rather than a point like bunch structure. Assuming a simple rectangular bunch of length τ (in time units) one should redefine J_0 as follows

$$\tilde{J}_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \rho(\omega)\, \omega\, \mathrm{Im}\left[Z_{\parallel}(\omega)\right]. \tag{C.6}$$

where the bunch spectrum is given by

$$\rho(\omega) = \frac{\sin\left(\frac{1}{2}\omega\tau\right)}{\frac{1}{2}\omega\tau}. \tag{C.7}$$

Simple application of Cauchy's integral theorem along the contours $C_\pm$, illustrated in Figure 8, yields the following expression for $\tilde{J}_0$

$$\tilde{J}_0 = -i\omega_r \frac{R}{2Q}\frac{\cos\left(\frac{1}{2}\omega_r\tau\right)}{\frac{1}{2}\omega_r\tau}. \tag{C.8}$$

Substituting Equation (C.4) and (C.8) into Equation (C.2) one can carry out the remaining summation over q employing the following convergence formula

$$\sum_{q=1}^{\infty} e^{2\pi i q \frac{\omega_\pm}{N\omega_o}} = \frac{e^{2\pi i \frac{\omega_\pm}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_\pm}{N\omega_o}}}, \quad \text{since } \mathrm{Im}\,\omega_\pm > 0. \tag{C.9}$$

The resulting expression can be summarized as follows

$$J = i\frac{R}{2Q}\left[i\omega_r^2 \frac{\cos\left(\frac{1}{2}\omega_r\tau\right)}{\frac{1}{2}\omega_r\tau} - \omega_+^2 \frac{e^{2\pi i \frac{\omega_+}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_+}{N\omega_o}}} + \omega_-^2 \frac{e^{2\pi i \frac{\omega_-}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_-}{N\omega_o}}}\right], \tag{C.10}$$

where

$$\omega_{\pm} = \omega_r(\pm 1 + i\delta), \quad \delta = \frac{1}{2Q}. \tag{C.11}$$

After some algebra one can rewrite Equation (C.10) into the following convenient form

$$J = \frac{1}{2} R\omega_r^2 \delta \left\{ \frac{\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right) + 2\delta^2\left(\frac{\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2 \delta^2} - \frac{2\,\cos\left(\frac{1}{2}\tau\omega_r\right)}{\frac{1}{2}\tau\omega_r} \right\}. \tag{C.12}$$

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