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THE APPLICATION OF LIE METHOD TO THE SPIN MOTION IN NONLINEAR COLLIDER FIELDS

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Lie operator method for handling the spin motion equation in collider nonlinear fields is used. The matrix presentation of spin Lie transformation in particle passage through collider elements is obtained. The formulas for combined several spin turn transformations are calculated in vector, matrix and operator forms for the zero, first and second powers component of dynamical variable vector. The expressions for frequency precession vector components in the zero, first and second powers on orbit motion and first powers on spin motion are obtained. The computer code algorithms for nonlinear spin motion calculation are discussed.

KEY WORD: Spin motion, Lie algebra, polarization

1. INTRODUCTION

Calculation of spin motion in accelerators and colliders is of interest in connection with different schemes of experiments with polarized beams (naturally, including the longitudinal polarization). Various devices are used in modern accelerators to improve beam parameters. These devices (sextupoles, octupoles etc.) strongly distort the linear motion, making the linear methods for calculation of orbital and spin motion insufficient. The Lie method, which was developed by A.Dragt, E.Forest and others,[1–3] allows to take into account nonlinear corrections for orbital motion. K. Yokoya[4] proposed using this technique in spin calculations. Unfortunately, no practical results have been obtained (the calculation formulas and algorithms, computer codes and etc.). This is due to two circumstances. First, the method of Lie operators allows to write a solution of the equation for spin motion in the form of a matrix operator, which acts on initial spin vector $\vec{S}(0)$. But this matrix operator depends on the synchrobetatron motion of the charged particle for each type of collider elements in rather complicated manner. The second obstacle consists in finding the rules of adding Lie operators to the two successive collider elements. These formulas are necessary for calculation of one-turn map of the entire ring and, hence, the spin transformation operator for practically any number of particle turns in the collider.

In this work an attempt has been made to solve these problems. Its plan is as follows: the first part contains a discussion of the method of Lie operators applied to the equation of spin motion. After that, a matrix form of the spin transformation operator is obtained. Further on, the rules of adding constant, linear and quadratic (in vector $\vec{Z} = (x, p_x, z, p_z, \sigma, p_\sigma)$) parts of the precession frequency vector $\vec{W}$ are calculated in operator forms. The next part contains the formulas for W components which are found by the use of the second (sextupole) order in orbital and the first order in spin component vector of dynamic variables $\vec{Y} = (\vec{Z}; \vec{S}) \equiv (\vec{Z}; S_x, S_z, S_\tau)$. In the last part, the algorithm for the application of the results obtained for the development of a computer code for nonlinear spin motion in colliders is discussed.

2. THE SPIN MOTION EQUATION

As is known,[5] the classical equation of spin motion in the collider is:

$$\frac{d\vec{S}}{ds} = [\vec{W}\vec{S}], \tag{1}$$

where $\vec{S}$ is a spin vector, s is an azimuth and the precession frequency vector $\vec{W}$ is defined by BMT's equation.[7] The Equation (7) is written in the frame $(\vec{e}_x, \vec{e}_z, \vec{\tau})$, fixed relative to the collider.

The usual "classical" approach to the solution of this equation is as follows: the precession frequency vector $\vec{W}$ is written for each type of collider elements, after that a system of differential equations for $\vec{S}$ is integrated by the method of sequential approximations. However, after taking into account synchrobetatron motion (SBM), the vector $\vec{W}$ depends on parameters of this motion in rather complicated manner. That is why, the analytical solution of the Equation (1) is possible in linear approximation (in SBM) only.

Another approach is based on the application of Lie operators technique. Vectors $\vec{S}$ and $\vec{W}$ in the equation (1) are considered as operators. Then, for a particle with the orbital Hamiltonian H_{orb} and spin Hamiltonian $H_{\text{sp}} = \vec{W}\vec{S}$, one can find the solution of this equation (the colons emphasize the operator nature of the expression[a]):

$$\vec{S}(s) = \exp\left[- : \int_0^s \left(H_{orb} + \vec{W}\vec{S}\right) ds' : \right] \vec{S}(0). \tag{2}$$

Here, as usual, the exponential operator is understood as a series:

$$\exp^{-:\mathcal{F}:} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} : \mathcal{F} :^n .$$

[a] The solutions have this form if the commutator of the total Hamiltonian is equal to zero at any time.

Each term of this series is the differential operator of n-th power, the action of which on an arbitrary function f is defined with the help of Poisson brackets[b]:

$$:\mathcal{F}: f \equiv \{\mathcal{F}, f\} = \frac{\partial \mathcal{F}}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial \mathcal{F}}{\partial p_i}\frac{\partial f}{\partial q_i}.$$

The operator, which is introduced in this manner, is referred to as a Lie operator and exponential series - Lie transformation.

For the total Hamiltonian $H_{\text{tot}} = H_{\text{orb}} + H_{\text{sp}}$, which does not depend on azimuth explicitly, instead of (2), one can find:

$$\vec{S}(s) = e^{-:s\cdot H_{\text{tot}}:}\vec{S}(0) \equiv \mathcal{M}\vec{S}(0), \tag{3}$$

where $\mathcal{M}$ is a total exponential operator. According to the Hamilton equations, this operator satisfies the equation $\frac{d\mathcal{M}}{ds} = \mathcal{M} : -H_{\text{tot}} :$. Let us present this operator as a product of three exponential operators[3]: $\mathcal{M} = \mathcal{M}_2 \cdot \mathcal{M}_1 \cdot \mathcal{M}_0$. To this let us expand the total Hamiltonian in a sum of homogeneous polynomials in powers of $\vec{Z}$:

$$H_{\text{tot}} = H^{(2)} + H^{(3)} + H^{(4)} + \vec{W}^{(0)}\vec{S} + \vec{W}^{(1)}\vec{S} + \vec{W}^{(2)}\vec{S},$$

where subscripts show powers of polynomials. It is important, that operators $: H^{(2)} :$ and $: \vec{W}^{(0)}\vec{S} :$ do not change the power of dependence on $\vec{Z}$ for any operands, but operators $: H^{(3)} :$ increase it by one. Let us consider the operator $: \vec{W}^{(1)}\vec{S} :$ acting on spin operand, which can be always presented as $\vec{S} \cdot f(\vec{Z})$. We have:

$$\begin{aligned} :\vec{W}^{(1)}\vec{S}: \left(\vec{S}\cdot f(\vec{Z})\right) &= \vec{W}^{(1)}\{\vec{S}, \vec{S}\cdot f(\vec{Z})\} + \vec{S}\{\vec{W}^{(1)}, \vec{S}\cdot f(\vec{Z})\} = \\ &= \vec{W}^{(1)} \cdot f(\vec{Z}) : \vec{S} : \vec{S} + \vec{S}^2 : \vec{W}^{1)} : f(\vec{Z}). \end{aligned}$$

Since the spin vector value is proportional to Planck constant h, one can omit the second term as compared to the first one, hence the operator $: \vec{W}^{(1)}\vec{S} :$ increases the power of operand dependence on $\vec{Z}$ by one as well. Similarly, the operators $: H^{(4)} :$ and $: \vec{W}^{(2)}\vec{S} :$ increase it by two. Therefore, it is convenient to introduce the following

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[b] As is known, Poisson brackets are: $\{q_i, q_k\} = \{p_i, p_k\} = 0$, $\{q_i, p_k\} = -\{p_i, q_k\} = \delta_{ik}$, where q, p - are conjugate dynamical variable pairs (x, p_x), (z, p_z), (σ, p_σ), and δ_{ik} – is the Kroneker symbol. Since the spin motion equation can't be linearized in spin canonical variables action-angle $(\mathcal{J}, \phi)$, it is useful to introduce the set of noncanonical variables[4]:

$$S_x = \sqrt{S^2 - \mathcal{J}^2}\cos\phi, S_z = \sqrt{S^2 - \mathcal{J}^2}\sin\phi, S_s = \mathcal{J},$$

here, $S^2 = S_x^2 + S_z^2 + S_s^2$. For this set, one can find: $\{S_i, S_j\} = e_{ijk}S_k$, where e_{ijk} – is the three-dimensional completely antisymmetric tensor. Besides that, for any i, j the following expression takes place: $\{Z_i, S_j\} = \{S_i, Z_j\} = 0$.

operators: $\mathcal{H}_1 \ =: \ H^{(3)} + \vec{W}^{(1)}\vec{S}\ :$ and $\mathcal{H}_2 \ =: \ H^{(4)} + \vec{W}^{(2)}\vec{S}\ :$. Let us divide the operators $: H_{\rm tot} :$ into operators $\mathcal{H}_0 =: H^{(2)} + \vec{W}^{(0)}\vec{S}\ :$ (which does not increase the operand power) and $\mathcal{H}_r = \mathcal{H}_1 + \mathcal{H}_2$, and $\mathcal{M}$ into $\mathcal{M}_0$ and $\mathcal{M}_r$. Then on the one hand,

$$\frac{d\mathcal{M}}{ds} = \mathcal{M} : -H_{\rm tot} := \mathcal{M}_r \cdot \mathcal{M}_0 \cdot (-\mathcal{H}_0 - \mathcal{H}_r) =$$

$$= \mathcal{M}_r \cdot \mathcal{M}_0 \cdot (-\mathcal{H}_0) + \mathcal{M}_r \cdot \mathcal{M}_0 \cdot (-\mathcal{H}_r) =$$

$$= \mathcal{M}_r \frac{d\mathcal{M}_0}{ds} + \mathcal{M}_r \cdot \mathcal{M}_0 \cdot (-\mathcal{H}_r),$$

and on the other hand,

$$\frac{d\mathcal{M}}{ds} = \mathcal{M}_r \frac{d\mathcal{M}_0}{ds} + \frac{d\mathcal{M}_r}{ds}\mathcal{M}_0,$$

so that,

$$\frac{d\mathcal{M}_r}{ds} \cdot \mathcal{M}_0 = \mathcal{M}_r \cdot \mathcal{M}_0(-\mathcal{H}_r).$$

Hence,

$$\frac{d\mathcal{M}_r}{ds} = \mathcal{M}_r \cdot \mathcal{M}_0(-\mathcal{H}_r) \cdot \mathcal{M}_0^{-1},$$

or, using the Lie transformations property,

$$\mathcal{M}_0 : g(\vec{Z}, \vec{S}) : \mathcal{M}_0^{-1} =: g(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) :, \tag{4}$$

one obtains

$$\frac{d\mathcal{M}_r}{ds} = \mathcal{M}_r\Big(- \mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_s \Big).$$

Integrating both parts of this expression, one can find ($\mathcal{J}$ is a unit operator) :

$$\mathcal{M}_r = \mathcal{J} + \int_0^s ds' \mathcal{M}_r(s')\Big(- \mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big).$$

This integral equation solution is easily obtained in series form if one substitutes in the integrand the right part step by step:

$$\mathcal{M}_r =$$

$$=\mathcal{J} + \int_0^s ds' \Big[\mathcal{J} + \int_0^{s'} ds'' \mathcal{M}_r(s'') \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s''} \Big)\Big] \cdot$$

$$\cdot \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) =$$

$$=\mathcal{J} + \int_0^s ds' \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) +$$

$$+ \int_0^s ds' \Big[\int_0^{s'} ds'' \mathcal{M}_r(s'') \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s''} \Big)\Big] \cdot$$

$$\cdot \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) =$$

$$=\mathcal{J} + \int_0^s ds' \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) +$$

$$+ \int_0^s ds' \Big\{ \int_0^{s'} ds'' \Big[\mathcal{J} + \int_0^{s''} ds''' \mathcal{M}_r(s''') \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'''} \Big)\Big] \cdot$$

$$\cdot \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s''} \Big)\Big\} \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) =$$

$$=\mathcal{J} + \int_0^s ds' \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) +$$

$$+ \int_0^s ds' \Big[\int_0^{s'} ds'' \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s''} \Big)\Big] \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) +$$

$$+ \int_0^s ds' \Big\{ \int_0^{s'} ds'' \Big[\int_0^{s''} ds''' \mathcal{M}_r(s''') \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'''} \Big)\Big] \cdot$$

$$\cdot \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s''} \Big)\Big\} \Big(-\mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) = \ldots$$

There ia a simple criterion for this series to break. It is associated with the order of its terms in powers of $\vec{Z}$. Indeed, the operator $\mathcal{H}_r \cdot \exp(- : \vec{W}^{(0)}\vec{S} :)$ increases the operand power by unit as a minimum. Restricting to the terms with a power not higher than two, one can omit the terms which contain $\mathcal{H}_r^3$ and break the series:

$$\mathcal{M}_r \simeq \mathcal{J} + \int_0^s ds' \Big(- \mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) + \tag{5}$$

$$+ \int_0^s ds' \int_0^{s'} ds'' \Big(- \mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s''} \Big) \Big(- \mathcal{H}_r(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big).$$

Since $\mathcal{M}_r$ is a product of two exponential operators $\mathcal{M}_2$ and $\mathcal{M}_1$, let us choose their powers as operators, which increase the operand power by one $(- : f_1 :)$ and two $(- : f_2 :)$. Then, expand the exponents into series and omit the third and higher power terms. In this way one obtains:

$$\mathcal{M}_r = \mathcal{M}_2 \cdot \mathcal{M}_1 = \exp(- : f_2 :) \cdot \exp(- : f_1 :) =$$

$$= \Big(\mathcal{J} - : f_2 : + \frac{: f_2 :^2}{2} - ... \Big) \cdot \Big(\mathcal{J} - : f_1 : + \frac{: f_1 :^2}{2} - ... \Big) \simeq$$

$$\simeq \mathcal{J} - : f_1 : - : f_2 : + \frac{: f_1 :^2}{2}.$$

Comparing this expression with (5), one can find:

$$: f_1 := - \int_0^s ds' \Big(- \mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) \tag{6}$$

and,

$$: f_2 := \frac{: f_1 :^2}{2} - \int_0^s ds' \Big(- \mathcal{H}_2(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) -$$

$$- \int_0^s ds' \int_0^{s'} ds'' \Big(- \mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s''} \Big) \cdot \Big(- \mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big).$$

But,

$$: f_1 :^2 = \int_0^s ds' \int_0^s ds'' \Big(- \mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s''} \Big) \cdot \Big(- \mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big),$$

hence separating the second integration over s'' in intervals from 0 to s' and from s' to s and changing the variable for the second interval this way, one can obtain

$$: f_1 :^2 =$$

$$= \int_0^s ds' \int_0^{s'} ds'' \Big(-\mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s''} \Big) \cdot \Big(-\mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) +$$

$$+ \int_0^s ds' \int_0^{s'} ds'' \Big(-\mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) \cdot \Big(-\mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s''} \Big).$$

Thus, for $: f_2 :$ one finally finds:

$$: f_2 := - \int_0^s ds' \Big(-\mathcal{H}_2(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}) \mid_{s'} \Big) - \tag{7}$$

$$- \frac{1}{2} \int_0^s ds' \int_0^{s'} ds'' \Big[-\mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}; \mid_{s''}), -\mathcal{H}_1(\mathcal{M}_0\vec{Z}, \mathcal{M}_0\vec{S}; \mid_{s'}) \Big],$$

where $\Big[, \Big]$ is an operator commutator.

So, the operator $\mathcal{M}$, which determines the solution (3) of the spin motion Equation (1), is equal to:

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_2 \cdot \mathcal{M}_1 \cdot \mathcal{M}_0 = \exp(- : f_2 :) \cdot \exp(- : f_1 :) \cdot \exp(- : f_0 :) = \\ &= \Big(\mathcal{J} - : f_1 : - : f_2 : + \frac{: f_1 :^2}{2} \Big) \cdot \mathcal{M}_0. \end{aligned} \tag{8}$$

In some cases it is convenient to separate the expression (8) in successive pure spin and orbital operators. For this, f_1 and f_2 are substituted in (8) as $h_3 + \vec{w}^{(1)}\vec{S}$ and $h_4 + \vec{w}^{(2)}\vec{S}$ respectively. The contribution of operators $: h_4 :, : h_3 :^2$ and $: \vec{w}^{(1)}\vec{S} : : h_3 :$ to (8) can be neglected: they act on an operand, which contains the first power of $\vec{Z}$ as a minimum, and increase this power by two. But operators $: \vec{w}^{(2)}\vec{S} :, : \vec{w}^{(1)}\vec{S} :^2$ and $: h_3 : : \vec{w}^{(1)}\vec{S} :$, can act on the operand with a zero power in $\vec{Z}$ and they must be taken into account. Besides, let us take into account that one can omit the operator $: \vec{w}^{(1)}\vec{S} : : h_3 :$ for the same reasons and hence, $: h_3 : : \vec{w}^{(1)}\vec{S} :=: (: h_3 : \vec{w}^{(1)}\vec{S}) :$. Finally, one obtains:

$$\mathcal{M} = \Big(\mathcal{J} - : h_3 : - : \vec{w}^{(1)}\vec{S} : - : \vec{w}^{(2)}\vec{S} : + \frac{: \vec{w}^{(1)}\vec{S} :^2}{2} + \frac{: h_3 : : \vec{w}^{(1)}\vec{S} :}{2} \Big) \cdot \mathcal{M}_0 =$$

$$= \exp \Big[- : \Big(\vec{w}^{(2)} - \frac{: h_3 : \vec{w}^{(1)}}{2} \Big) \vec{S} : \Big] \cdot \exp(- : \vec{w}^{(1)}\vec{S} :) \cdot \exp(- : h_3 :) \cdot \mathcal{M}_0.$$

Let us note that $:\vec{w}^{(0)}\vec{S}:$ (which is a part of the total operator $\mathcal{M}_0$) commutes with $:h_2:$ and $:h_3:$, hence, one can finally write:

$$\mathcal{M} = \exp\left[-:\left(\vec{w}^{(2)} - \frac{:h_3:\vec{w}^{(1)}}{2}\right)\vec{S}:\right]\cdot\exp\left(-:\vec{w}^{(1)}\vec{S}:\right)\cdot$$
$$\cdot\exp\left(-:\vec{w}^{(0)}\vec{S}:\right)\cdot\exp\left(-:h_3:\right)\cdot\exp\left(-:h_2:\right). \tag{9}$$

Sometimes it is necessary to represent the spin part of the expression (9) as one exponent with the precession frequency $\vec{W}^*$ which determines spin rotation for each element:

$$\mathcal{M} = \exp\left(-:\vec{W}^*\vec{S}:\right)\cdot\exp\left(-:h_3:\right)\cdot\exp\left(-:h_2:\right). \tag{10}$$

Let us obtain the "convolution" of three exponents' operators:

$$e^{-:sf:} = \exp\left(-:sf_2:\right)\exp\left(-:sf_1:\right)\exp\left(-:sf_0:\right). \tag{11}$$

Differentiating both parts with respect to s, one obtains:

$$\begin{aligned} e^{-:sf:}\cdot:-f: &= \exp\left(-:sf_2:\right)\cdot:-f_2:\cdot\exp\left(-:sf_1:\right)\cdot\exp\left(-:sf_0:\right)+ \\ &+\exp\left(-:sf_2:\right)\cdot\exp\left(-:sf_1:\right)\cdot:-f_1:\cdot\exp\left(-:sf_0:\right)+ \\ &+\exp\left(-:sf_2:\right)\cdot\exp\left(-:sf_1:\right)\cdot\exp\left(-:sf_0:\right)\cdot:-f_0:= \\ &=\exp\left(-:sf_2:\right)\cdot\exp\left(-:sf_1:\right)\cdot \\ &\cdot\Big(\exp\left(:sf_1:\right)\cdot:-f_2:\exp\left(-:sf_1:\right)\Big)\cdot\exp\left(-:sf_0:\right)+ \\ &+\exp\left(-:sf_2:\right)\cdot\exp\left(-:sf_1:\right)\cdot\exp\left(-:sf_0:\right)\cdot \\ &\cdot\Big(\exp\left(:sf_0:\right)\cdot:-f_1:\exp\left(-:sf_0:\right)\Big)+ \\ &+\exp\left(-:sf_2:\right)\cdot\exp\left(-:sf_1:\right)\cdot\exp\left(-:sf_0:\right)\cdot:-f_0:\,. \end{aligned}$$

The factor $e^{-:sf:}$ evidently arises in the second and third terms. Besides that, the expressions in brackets can be grouped with the help of the rule (4). Then:

$$\begin{aligned} e^{-:sf:}\cdot:-f: &= \exp\left(-:sf_2:\right)\cdot\exp\left(-:sf_1:\right)\cdot:-\exp\left(:sf_1:\right)f_2:\cdot \\ &\cdot\exp\left(-:sf_0:\right)+\exp\left(-:sf:\right)\cdot:-\exp\left(:sf_0:\right)f_1:+ \\ &+\exp\left(-:sf:\right)\cdot:-f_0:\,. \end{aligned}$$

Repeating similar action in the first terms with operator $\exp(-:sf_0:)$, one can obtain:

$$\begin{aligned}
e^{-:sf:}\cdot :-f:= & \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot\exp(-:sf_0:)\cdot \\
& \cdot :-\exp(:sf_0:)\cdot\exp(:sf_1:)f_2:+ \\
& +\exp(-:sf:)\cdot :-\exp(:sf_0:)f_1:+ \\
& +\exp(-:sf:)\cdot :-f_0:= \\
= & \exp(-:sf:)\cdot :-\exp(:sf_0:)\cdot\exp(:sf_1:)f_2:+ \\
& +\exp(-:sf:)\cdot :-\exp(:sf_0:)f_1:+ \\
& +\exp(-:sf:)\cdot :-f_0: .
\end{aligned}$$

The required equation for f can be found after that:

$$f = f_0 + \exp(:sf_0:)f_1 + \exp(:sf_0:)\cdot\exp(:sf_1:)f_2. \tag{12}$$

The contribution of the operator $\exp(:sf_1:)$ is not lower than the third power. Then, for $\vec{W}^*$, one finally obtains:

$$\vec{W}^* = \vec{w}^{(0)} + \exp(:\vec{w}^{(0)}\vec{S}:)\vec{w}^{(1)} + \exp(:\vec{w}^{(0)}\vec{S}:)\Big(\vec{w}^{(2)} - \frac{:h_3:\vec{w}^{(1)}}{2}\Big). \tag{12}$$

The operator $\exp(-:\vec{w}^{(0)}\vec{S}:)$, as shown below, can be represented in a matrix form.

Thus using the technique of Lie operators, it is necessary to substitute the expression for $H_{\rm orb}$ and $\vec{W}$ from BMT-formula (for each type of collider elements) in (6) and (7) and to find the transformed precession frequency $\vec{w}$ components and Hamiltonian h_{orb}. The obtained results should be substituted into (8) or (9). The operators $\exp(-:h_2:)$ and $\exp(-:\vec{w}^{(0)}:)\vec{S}$ are calculated by expanding to series. The equation (1) is indeed solved[c].

3. MATRIX FORM FOR OPERATOR OF SPIN TRANSFORMATION

The spin exponential operator, which was obtained in (9) and (13), has the form $e^{-:\vec{W}\vec{S}:}$, where $\vec{W}$ is some precession frequency. If the expression for $\vec{W}$ is known,

[c] It is necessary to note that an important problem exists anyway. It is as follows: when the orbit imperfections are taken into account, then the orbital Hamiltonian starts with the first order, but not the second order polynomial if canonical variables are defined so as to measure the distance from the design orbit. This problem can be avoided by measuring the transverse coordinates from the distorted orbit. But in this case the Hamiltonian becomes explicitly time-dependent. This problem is solved in.[6]

one can calculate the value of $e^{-:\vec{W}\vec{S}:}$ successively finding the low-order terms of this series. For the zero order, by definition, it is:

$$(-:\vec{W}\vec{S}:)^0 S_k = S_k, \tag{14}$$

The first order (taking into account the independence of spin vector components S_k on orbital variables $\vec{Z}$):

$$(-:\vec{W}\vec{S}:)^1 S_k = -W_i\{S_i, S_k\} = -e_{ikj}S_jW_i = [\vec{W},\vec{S}]_k,$$

where $[,]$ is the vector product. Thus,

$$(-:\vec{W}\vec{S}:)^1 S_k = [\vec{W},\vec{S}]_k. \tag{15}$$

Then, for the second order, one has:

$$\begin{aligned}(-:\vec{W}\vec{S}:)^2 S_k =&(-:\vec{W}\vec{S}:)^1(-:\vec{W}\vec{S}:)^1 S_k = (-:\vec{W}\vec{S}:)^1[\vec{W},\vec{S}]_k = \\ =&-:W_nS_n: e_{ijk}W_iS_j = -e_{ijk}W_iW_n\{S_n,S_j\}- \\ &- e_{ijk}S_jS_n\{W_n,W_i\} - e_{ijk}S_jW_n\{S_n,W_i\}- \\ &- e_{ijk}W_iS_n\{W_n,S_j\} = [\vec{W},[\vec{W},\vec{S}]]_k - [S_n\{W_n,\vec{W}\},\vec{S}]_k\,.\end{aligned}$$

The second term, which is proportional to the h^2 (h is the Planck constant), can be omitted (it is rather small compared to the first one). Hence, one finally has:

$$(-:\vec{W}\vec{S}:)^2 S_k = [\vec{W},[\vec{W},\vec{S}]]_k. \tag{16}$$

Further, for the third order (omitting the terms which are again proportional to h^2):

$$\begin{aligned}(-:\vec{W}\vec{S}:)^3 S_k =&: -W_nS_n : [\vec{W},[\vec{W},\vec{S}]]_k = \\ =&: -W_nS_n : \Big(W_kW_iS_i - S_kW_iW_i - \Big) = \\ =& -W_iW_kW_n\{S_n,S_i\} + W_iW_iW_n\{S_n,S_k\} = \\ =& -W_iW_kW_ne_{nij}S_j + W_iW_iW_ne_{nkj}S_j,\end{aligned}$$

that is, (W is an absolute value of the vector $\vec{W}$):

$$(-:\vec{W}\vec{S}:)^3 S_k = -W^2[\vec{W},\vec{S}]_k. \tag{17}$$

Thus, we have the possibility to deduce any term of this series:

$$(- : \vec{W}\vec{S} :)^{2n+1} S_k = (-1)^n W^{2n} [\vec{W}, \vec{S}]_k,$$

$$n = 0, 1, 2, ...;$$

$$(- : \vec{W}\vec{S} :)^{2n+2} S_k = (-1)^n W^{2n} [\vec{W}, [\vec{W}, \vec{S}]]_k .$$

Dividing the exponential series into two - even and odd degrees - we can sum it completely:

$$e^{-:\vec{W}\vec{S}:} S_k = \sum_{n=0}^{\infty} \frac{(- : \vec{W}\vec{S} :)^n}{n!} S_k = (: -\vec{W}\vec{S} :)^0 S_k +$$

$$+ \sum_{n=0}^{\infty} \left(\frac{(- : \vec{W}\vec{S} :)^{2n+1}}{(2n+1)!} + \frac{(- : \vec{W}\vec{S} :)^{2n+2}}{(2n+2)!} \right) S_k =$$

$$= S_k + \frac{\sin W}{W} \left[\vec{W}, \vec{S}\right]_k + \frac{1 - \cos W}{W^2} \left[\vec{W}, \left[\vec{W}\vec{S}\right]\right]_k .$$

Hence, the exponential operator $e^{-:\vec{W}\vec{S}:}$ has a matrix form:

$$S_i = \left(e^{-:\vec{W}\vec{S}:}\right)_{ij} S_j = \mathcal{T}_{ij} S_j,$$

where,

$$\mathcal{T}_{ij} = \cos W \delta_{ij} + \frac{\sin W}{W} e_{ijk} W_k + \frac{1 - \cos W}{W^2} W_i W_j. \tag{18}$$

This representation for the solution of the spin motion equation is exact, but the accuracy is determined by that of $\vec{W}$ calculation. Let us confine to the second order over synchro-betatron motion in this work. Then,

$$W_i = W_i^{(0)} + W_{im}^{(1)} Z_m + W_{imn}^{(2)} Z_m Z_n, \tag{19}$$

so, for the absolute value of the spin angle turn W we have ($W_0 = \sqrt{W_i^{(0)} W_i^{(0)}}$):

$$W = \sqrt{W_i^2} = W_0 \Big[1 + \frac{W_i^{(0)}}{W_0^2} W_{im}^{(1)} Z_m + \tag{20}$$

$$+ \left(\frac{W_i^{(0)} W_{imn}^{(2)}}{W_0^2} + \frac{W_{im}^{(1)} W_{in}^{(1)}}{2W_0^2} - \frac{W_i^{(0)} W_j^{(0)} W_{im}^{(1)} W_{jn}^{(1)}}{2W_0^4} \right) Z_m Z_n \Big].$$

Substituting (19) and (20) into (18) (one must decompose the sin and cos functions in relation to W_0), for the zero order of $\vec{Z}$ after some simple transformations one receives:

$$\mathcal{T}_{ij}^{(0)} = \cos W_0 \delta_{ij} + \frac{\sin W_0}{W_0} e_{ijk} W_k^{(0)} + \frac{1-\cos W_0}{W_0^2} W_i^{(0)} W_j^{(0)}; \tag{21}$$

for the first order:

$$\mathcal{T}_{ijm}^{(1)} = \tag{22}$$

$$= -\frac{\sin W_0}{W_0}\left(\delta_{ij} W_m^{(0)} - e_{ijk} W_{km}^{(1)}\right) + \frac{1-\cos W_0}{W_0^2}\left(W_i^{(0)} W_{jm}^{(1)} + W_j^{(0)} W_{im}^{(1)}\right) +$$

$$+ \frac{\cos W_0 - \frac{\sin W_0}{W_0}}{W_0^2} e_{ijk} W_k^{(0)} W_n^{(0)} W_{nm}^{(1)} + \frac{\frac{\sin W_0}{W_0} - 2\frac{1-\cos W_0}{W_0^2}}{W_0^2} W_i^{(0)} W_j^{(0)} W_k^{(0)} W_{km}^{(1)}$$

and at last, for the second order:

$$\mathcal{T}_{ijmn}^{(2)} = \frac{\sin W^{(0)}}{W_0}\left[-\delta_{ij}\left(W_k^{(0)} W_{kmn}^{(2)} + \frac{1}{2} W_{km}^{(1)} W_{kn}^{(1)}\right) + e_{ijk} W_{kmn}^{(2)}\right] +$$

$$+ \frac{1-\cos W_0}{W_0^2}\left(W_i^{(0)} W_{jmn}^{(2)} + W_j^{(0)} W_{imn}^{(2)} + W_{im}^{(1)} W_{jn}^{(1)}\right) +$$

$$+ \frac{\cos W_0 - \frac{\sin W_0}{W_0}}{W_0^2} \cdot \left[e_{ijk}\left(W_k^{(0)} W_p^{(0)} W_{pmn}^{(2)} + \frac{1}{2} W_k^{(0)} W_{pm}^{(1)} W_{pn}^{(1)} + \right.\right.$$

$$\left.\left. + W_p^{(0)} W_{km}^{(1)} W_{pn}^{(1)}\right) - \delta_{ij} W_k^{(0)} W_p^{(0)} W_{km}^{(1)} W_{pn}^{(1)}\right] + \tag{23}$$

$$+ \frac{\frac{\sin W_0}{W_0} - 2\frac{1-\cos W_0}{W_0^2}}{W_0^2} \cdot \left[W_i^{(0)} W_j^{(0)}\left(W_k^{(0)} W_{kmn}^{(2)} + \frac{1}{2} W_{pm}^{(1)} W_{pn}^{(1)}\right) + \right.$$

$$\left. + \left(W_i^{(0)} W_{jm}^{(1)} + W_j^{(0)} W_{im}^{(1)}\right) W_k^{(0)} W_{kn}^{(1)}\right] -$$

$$- \frac{\frac{\sin W_0}{W_0} + 3\frac{\cos W_0 - \sin(W_0)/W_0^2}{W_0^2}}{2W_0^2} \cdot e_{ijk} W_k^{(0)} W_p^{(0)} W_q^{(0)} W_{pm}^{(1)} W_{qn}^{(1)} +$$

$$+ \frac{8\frac{1-\cos W_0}{W_0^2} + \cos W_0 - 5\frac{\sin W_0}{W_0}}{2W_0^4} \cdot W_i^{(0)} W_j^{(0)} W_p^{(0)} W_q^{(0)} W_{pm}^{(1)} W_{qn}^{(1)} .$$

4. FORMULAS OF "ADDITION"

To combine the successive spin transformations during the beam passage through the magnetic system, let us deduce the formulas for addition of transformations in the form (10), i.e., let us define operators which obey the equation:

$$\begin{aligned} &\exp(-:\vec{W}\vec{S}:)\cdot\exp(-:h_3^w:)\cdot\exp(-:h_2^w:)=\\ &=\exp(-:\vec{U}\vec{S}:)\cdot\exp(-:h_3^u:)\cdot\exp(-:h_2^u:)\cdot\\ &\cdot\exp(-:\vec{V}\vec{S}:)\cdot\exp(-:h_3^v:)\cdot\exp(-:h_2^v:). \end{aligned} \tag{24}$$

In other words, let us find Lie operators, which transform spin vector from azimuth s_0 to azimuth s'' equivalent to sequential Lie operators, transforming the spin from s_0 to s' (the operators $\exp(-:\vec{V}\vec{S}:)\cdot\exp(-:h_3^v:)\cdot\exp(-:h_2^v:)$) and then from s' to s'' (the operators $\exp(-:\vec{U}\vec{S}:)\cdot\exp(-:h_3^u:)\cdot\exp(-:h_2^u:)$). Acting in a way similar to the calculation of formula (12) for "merging" of exponential operators product, one can easily obtain the following result:

$$\begin{aligned} e^{-:\vec{W}\vec{S}:}&=e^{-:\vec{U}\vec{S}:}\cdot e^{-:\vec{V}^*\vec{S}:},\\ \vec{V}^*&=\exp(-:h_3^u:)\cdot\exp(-:h_2^u:)\cdot\vec{V},\\ h_3^w&=h_3^u+\exp(-:h_2^u:)\cdot h_3^v,\\ \exp(-:h_2^w:)&=\exp(-:h_2^u:)\cdot\exp(-:h_2^v:). \end{aligned} \tag{25}$$

One has now the equation $\exp\left(-:\vec{W}\vec{S}:\right)=\exp\left(-:\vec{U}\vec{S}:\right)\cdot\exp\left(-:\vec{V}\vec{S}:\right)$ (the superscript "*" near $\vec{V}$ is omitted for simplicity). If one replaces the operators by matrices of the form (18) this equation takes the form:

$$T_{ij}^{(w)}=T_{ik}^{(u)}\cdot T_{kj}^{(v)}. \tag{26}$$

Since one is interested in a result in the series form in $\vec{Z}$ powers:

$$T_{ij}^{(w)}=T_{ij}^{(w0)}+T_{ijm}^{(w1)}\cdot Z_m+T_{ijmn}^{(w2)}\cdot Z_m\cdot Z_n, \tag{27}$$

Substitutions to (26) of the series for $T_{ik}^{(u)}$ and $T_{kj}^{(v)}$ are done in the same manner as in (27). After rewriting the result in powers of $\vec{Z}$, one obtains:

$$T_{ij}^{(w0)} = T_{ik}^{(u0)} \cdot T_{kj}^{(v0)},$$
$$T_{ijm}^{(w1)} = T_{ik}^{(u0)} \cdot T_{kjm}^{(v1)} + T_{ikm}^{(u1)} \cdot T_{kj}^{(v0)}, \tag{28}$$
$$T_{ijmn}^{(w2)} = T_{ik}^{(u0)} \cdot T_{kjmn}^{(v2)} + T_{ikm}^{(u1)} \cdot T_{kjn}^{(v1)} + T_{ikmn}^{(u2)} \cdot T_{kj}^{(v0)}.$$

5. SPIN PRECESSION FREQUENCY

As is known, the vector $\vec{W}_0$ of the spin precession frequency of a charged particle (e and m are its charge and mass and $\vec{\beta}$, γ are velocity vector and relativistic factor) in electromagnetic fields $\vec{H}$ and $\vec{E}$ is determined by the BMT expression[7]:

$$\vec{W}_0 = -\frac{e}{\gamma mc}\left((1+a\gamma)\vec{H} - \frac{a\gamma^2}{1+\gamma}\left(\vec{\beta}\vec{H}\right)\vec{\beta} - \left(a\gamma + \frac{\gamma}{1+\gamma}\right)\left[\vec{\beta}\vec{E}\right]\right), \tag{29}$$

where $a = 1.159... * 10^{-3}$ is the dimensionless part of the electron anomalous magnetic momentum. This precession frequency determines the known equation for the particle spin $\vec{S}$ precession:

$$\frac{d\vec{S}}{dt} = \left[\vec{W}_0\vec{S}\right]. \tag{30}$$

Let us pass from time t to other independent variable - the azimuth s, which is calculated along the equilibrium orbit. Then, in the frame $(\vec{e}_x, \vec{e}_z, \vec{\tau})$, which is connected with this orbit, the radius-vector of particle position is equal to:

$$\vec{r}(x,z,s) = \vec{r}_0 + x\vec{e}_x + z\vec{e}_z$$

and,

$$\frac{d}{dt} = \frac{dl}{dt}\frac{d}{dl} = c\beta\frac{ds}{dl}\frac{d}{ds} = \frac{c\beta}{l'}\frac{d}{ds},$$

where l is the arc length along the orbit and prime means of differentiation over s. Further on, one has:

$$\frac{d\vec{r}}{ds} = \vec{r}\,'_0 + x'\vec{e}_x + z'\vec{e}_z + x\vec{e}\,'_x + z\vec{e}\,'_z$$

and using the Frene formulae for plane orbit (!), one can obtain:

$$\vec{\tau} = \frac{d\vec{r}_0}{ds}, \frac{d\vec{\tau}}{ds} = -K\vec{n}, \frac{d\vec{n}}{ds} = K\vec{\tau}, \frac{d\vec{b}}{ds} = 0,$$

and hence,

$$\frac{d\vec{r}}{ds} = (1 + K_x + K_z)\vec{\tau} + x'\vec{e}_x + z'\vec{e}_z. \tag{31}$$

Let us assume here and further on, that the equilibrium orbit is bit-planar, since either radial curvature K_x or vertical K_z is equal to zero ($K_x \cdot K_z = 0$). Let us retain in the equation for $l' = |\frac{d\vec{r}}{ds}|$ the terms not higher than second order in deviation from the equilibrium orbit x, x', z, z'. Then, one obtains:

$$l' \simeq 1 + K_x x + K_z z + \frac{x'^2}{2} + \frac{z'^2}{2}, \tag{32}$$

In this approximation it is easy to find the expression for velocity:

$$\vec{\beta} = \frac{1}{c}\frac{d\vec{r}}{dt} = \frac{\beta}{l'}\vec{r}\,' \simeq \beta\frac{(1 + K_x + K_z)\vec{\tau} + x'\vec{e}_x + z'\vec{e}_z}{1 + K_x x + K_z z + x'^2/2 + z'^2/2},$$

or

$$\vec{\beta} = \beta\left[(1 - \frac{x'^2}{2} - \frac{z'^2}{2})\vec{\tau} + (1 - K_x x - K_z z)\cdot(x'\vec{e}_x + z'\vec{e}_z)\right]. \tag{33}$$

Let us now express the relativistic factor γ and velocity β in terms of relativistic deviation p_σ of particle energy $\mathcal{E}$ from its equilibrium value $\mathcal{E}_0$:

$$\gamma = \frac{\mathcal{E}}{mc^2} = \frac{\mathcal{E}_0 + (\mathcal{E} - \mathcal{E}_0)}{mc^2} = \frac{\mathcal{E}_0}{mc^2}(1 + \frac{\mathcal{E} - \mathcal{E}_0}{mc^2}) = \gamma_0(1 + p_\sigma) \tag{34}$$

and

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \simeq 1 - \frac{1}{2\gamma^2}. \tag{35}$$

Let us transform the equation (30). Substituting the expression for $\vec{S}$ in the introduced frame ($\vec{e}_x$, $\vec{e}_z$, $\vec{\tau}$) in the form

$$\vec{S} = S_x\vec{e}_x + S_z\vec{e}_z + S_\tau\vec{\tau}$$

one can obtain:

$$\vec{S}' = \left[\vec{W}\vec{S}\right], \tag{36}$$

where the derivative in the left part refers to components of vector $\vec{S}$ only (but not to vectors $\vec{e}_x$, $\vec{e}_z$, $\vec{\tau}$!), and $\vec{W}$:

$$\vec{W} = \frac{l'}{c\beta}\vec{W}_0 - K_z\vec{e}_x + K_x\vec{e}_z. \tag{37}$$

The expression (29) for $\vec{W}_0$ can be rewritten in another form (for $\vec{E}=0$, that means that only the magnetic system of collider is considered, whereas the cavities and other elements with electrical fields are not taken into account):

$$\vec{W}_0 = -\frac{1+a\gamma}{\gamma}\frac{\mathcal{E}_0}{mc^2}\frac{ec\vec{H}}{\mathcal{E}_0} + \frac{a\gamma}{1+\gamma}\vec{\beta}\left(\frac{\mathcal{E}_0}{mc^2}\vec{\beta}\frac{ec\vec{H}}{\mathcal{E}_0}\right)$$

or

$$\vec{W}_0 = -\gamma_0 c\left[\frac{1+a\gamma}{\gamma}\vec{B} - \frac{a\gamma}{1+\gamma}(\vec{\beta}\vec{B})\vec{\beta}\right], \tag{38}$$

where $\vec{B}=\left(e\vec{H}\right)/(\mathcal{E}_0)$ and the ratio $\gamma_0=\mathcal{E}_0/mc^2$ are introduced. Let us express the values $\frac{1}{\gamma}$ and $\frac{\gamma}{1+\gamma}$ and velocity β, which are included in (38), in terms of p_σ:

$$\begin{aligned}
\frac{1}{\gamma} &= \frac{1}{\gamma_0(1+p_\sigma)} \simeq \frac{1-p_\sigma+p_\sigma^2}{\gamma_0}, \\
\frac{\gamma}{1+\gamma} &\simeq \frac{1-\gamma_0+p_\sigma\gamma_0+\gamma_0^2}{\gamma_0^2}, \\
\beta &\simeq 1-\frac{1-2p_\sigma+3p_\sigma^2}{2\gamma_0^2}.
\end{aligned} \tag{39}$$

Substituting the expressions (33) and (39) into (38) and the obtained result together with (32) into (37), one can find the expressions for $\vec{W}$ components. It is necessary to substitute there the decompositions of $\vec{B}$ components in the form of series in powers of x, z (naturally not more than of the second order):

$$\begin{aligned}
B_x =& B_{0x} + \left(q-\frac{1}{2}B'_{0s}\right)x + gz - \frac{1}{2}(m_z+K_xq+K_zg+B''_{0x})x^2 + \\
&+\frac{1}{2}m_z z^2 + m_x xz, \\
B_z =& B_{0z} - \left(q+\frac{1}{2}B'_{0s}\right)z + gx - \frac{1}{2}(m_x-K_zq+K_xg+B''_{0z})z^2 + \\
&+\frac{1}{2}m_x x^2 + m_z xz, \\
B_s =& B_{0s} + B'_{0x}x + B'_{0z}z + \frac{1}{2}\left(q'-\frac{1}{2}B''_{0s}\right)x^2 - \frac{1}{2}\left(q'+\frac{1}{2}B''_{0s}\right)z^2 + \\
&+(g'-K_zB'_{0x}-K_xB'_{0z})xz.
\end{aligned} \tag{40}$$

The following values, which characterize the magnetic field, are introduced there:

$$K_{x,z} = \pm \frac{eH_{0x,0z}}{\mathcal{E}_0},$$

$$g = \frac{e}{\mathcal{E}_0}\frac{\partial H_z}{\partial x} = \frac{e}{\mathcal{E}_0}\frac{\partial H_x}{\partial z}, q = \frac{e}{2\mathcal{E}_0}\left(\frac{\partial H_x}{\partial x} - \frac{\partial H_z}{\partial z}\right), \tag{41}$$

$$m_{x,z} = \frac{e}{2\mathcal{E}_0}\frac{\partial^2 H_{x,z}}{\partial x \partial z} = \frac{e}{2\mathcal{E}_0}\frac{\partial^2 H_{z,x}}{\partial x^2, \partial z^2}$$

and the values of all quantities on the right sides are taken on the equilibrium orbit.

Thus, one obtains the final expressions for components of $\vec{W}$ (including the zero, first and second orders on x, z, σ, $p_x = x' - (eH_{0s})/(2\mathcal{E}_0)\cdot z$, $p_z = z' + (eH_{0s})/(2\mathcal{E}_0)\cdot x$, p_σ):

$$\begin{aligned}
W_x = &- (B_{0x} + K_z) - \left(a\gamma_0 + \frac{a}{2\gamma_0} + \frac{1}{2\gamma_0^2}\right) B_{0x} + \\
&+ (1 + a\gamma_0)\left(\frac{1}{2}B'_{0s} + q\right) x + a(\gamma_0 - 1)B_{0s}p_x + \\
&+ \left[\frac{1}{2}a(\gamma_0 - 1)B_{0s}^2 - (1 + a\gamma_0)(B_{0x}K_z + g)\right] z + B_{0x}p_\sigma + \\
&+ \frac{1}{2}(1 + a\gamma_0)(gK_z - qK_x + m_z + B''_{0x})x^2 + a\gamma_0 B'_{0x}xp_x - \\
&- (1 + a\gamma_0)(gK_x + qK_z + m_x)xz - (\frac{1}{2}B'_{0s} - q)xp_\sigma + \\
&+ \frac{1}{2}(a\gamma_0 - 1)B_{0x}p_x^2 + a\gamma_0 B'_{0z}p_x z + a\gamma_0 B_{0z}p_x p_z - \\
&- (1 + a\gamma_0)\left(gK_z + \frac{1}{2}m_z\right) z^2 + (B_{0x}K_z + g)zp_\sigma - \\
&- \frac{1}{2}(1 + a\gamma_0)B_{0x}p_z^2 - B_{0x}p_\sigma^2;
\end{aligned} \tag{42}$$

$$
\begin{aligned}
W_z =& -(B_{0z} - K_x) - \left(a\gamma_0 + \frac{a}{2\gamma_0} + \frac{1}{2\gamma_0^2}\right) B_{0z} - \\
& - \left[\frac{1}{2}a(\gamma_0 - 1)B_{0s}^2 + (1 + a\gamma_0)(B_{0z}K_x + g)\right] x + \\
& + (1 + a\gamma_0)\left(\frac{1}{2}B'_{0s} + q\right) z + a(\gamma_0 - 1)B_{0s}p_z + B_{0z}p_\sigma - \\
& - (1 + a\gamma_0)\left(gK_x + \frac{1}{2}m_x\right) x^2 + \\
& + (1 + a\gamma_0)(qK_x - gK_z - m_z)xz + a\gamma_0 B'_{0x}xp_z + \\
& + (B_{0z}K_x + g)xp_\sigma - \frac{1}{2}(1 + a\gamma_0)B_{0z}p_x^2 + a\gamma_0 B_{0x}p_x p_z + \\
& + \frac{1}{2}(1 + a\gamma_0)(gK_x + qK_z + m_x + B''_{0z})z^2 + a\gamma_0 B'_{0z}zp_z - \\
& - \left(\frac{1}{2}B'_{0s} + q\right) Zp_\sigma + \frac{1}{2}(a\gamma_0 - 1)B_{0Z}p_z^2 - B_{0z}p_\sigma^2;
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
W_\tau =& -\left(1 + a + \frac{1}{2\gamma_0^2}\right) B_{0s} - (1 + a)B'_{0x}x - (1 + a)B'_{0z}z + a(\gamma_0 - 1)B_{0x}p_x + \\
& + a(\gamma_0 - 1)B_{0z}p_z + (1 + a)B_{0s}p_\sigma - \frac{1}{2}\left[(2a\gamma_0 + 1)B_{0s}^3 + \left(q' - \frac{1}{2}B''_{0s}\right)\right] x^2 + \\
& + a\gamma_0\left(q - \frac{1}{2}B'_{0s}\right) xp_x - g'xz + \left[a\gamma_0 g + \frac{1}{2}(2a\gamma_0 + 1)B_{0s}^2\right] xp_z + \\
& + B'_{0x}xp_\sigma - \frac{1}{2}(2a\gamma_0 + 1)B_{0s}p_x^2 - \left[\frac{1}{2}(2a\gamma_0 + 1)B_{0s}^2 - a\gamma_0 g\right] p_x z - \\
& - \frac{1}{2}\left[\frac{1}{4}(2a\gamma_0 + 1)B_{0s}^3 - \left(q' + \frac{1}{2}B''_{0s}\right)\right] z^2 - a\gamma_0\left(q + \frac{1}{2}B'_{0s}\right) zp_z - \\
& - \frac{1}{2}(2a\gamma_0 + 1)B_{0s}p2_z + B'_{0z}zp_\sigma - B_{0s}p_\sigma^2.
\end{aligned}
\tag{44}
$$

6. THE ALGORITHM OF THE NONLINEAR SPIN MOTION CALCULATION

In the practical calculation of spin motion, different approaches are possible. In one of them (A) the operators — matrices of spin transformation are calculated and then the spin vector is successively "pulled" through each element of the collider magnetic structure. In the other one (B), for calculation of one-turn transformation of the spin vector, the addition of matrices of all elements is performed after determining each of them. One can calculate the one-turn transformation only after that. Let us describe the succession of actions for each of the approaches. It is necessary to note that in both cases for each collider element one must:

- calculate the vector $\vec{W}$ in agreement with section 5;
- calculate the integrals (6) and (7) using the orbital Hamiltonian for this element.

Then the procedures are different.

(A)The operator $\mathcal{M}$ of spin transformation is calculated (formulas (8)) for a current element and the spin vector at its entrance is "pulled" through this element. Then the operator $\mathcal{M}$ of the next element is calculated and so on.

(B)The "total" frequency of spin precession in the current element is calculated (formulae (13)). Then its orbital part is transformed and orbital transformations are added (formulas (25)). After that, the matrices of spin transformation in this element are calculated with the help of formulas (21) – (23). At last, the "total" matrices to current collider azimuth are determined (formulas (28)).

7. CONCLUSIONS

Thus, the obtained results allow to create the computer code by the optimal method in each specific case (one- and many turns spin dynamics, the dynamical and equilibrium degree of polarization and so on).

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