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SYNCHROBETATRON MODE COUPLING AND BEAM BREAKUP INSTABILITY IN STORAGE RINGS

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The strong coupling of coherent synchrobetatron modes of a single bunch in a storage ring, interacting with its very low-Q environment, significantly suppresses the contribution of the eigenmodes of the bunch to the evolution of its coherent oscillations. Such a suppression results in a decrease of the distance in the spectrum of the eigenfrequencies, as well as in the strong influence of the beam breakup phenomenon on the initial development of coherent oscillations of the bunch. The self-consistent modes define the behaviour of coherent oscillations on time intervals that are longer than the periods of synchrotron oscillations of the bunch particles.

KEY WORDS: Synchrobetatron Mode, Mode-Coupling, Beam Breakup Instability, Single-turn Interaction, Storage Ring

1 INTRODUCTION

It is well known that in many cases the possibility of increasing the beam current in a storage ring is limited by coherent interaction of the beam bunches with their environment. If such an interaction is a single-turn interaction, i.e. the bunch wakes decay faster than the revolution period in the ring, the specific features of coherent instabilities significantly depend on the ratio of the bunch coherent frequency shift Ω_m to the frequency of synchrotron oscillations of particles in this bunch ω_c. When this ratio is small, a coherent interaction only slightly distorts the unperturbed spectra and, if $\vec{m}_\perp$, and m_c are the multipole numbers of transverse and synchrotron oscillations (see, for instance, in Ref. 1), the eigenfrequencies of the coherent oscillations occur close to combinations:

$$\omega = \vec{m}_\perp \vec{\omega}_\perp + m_c \omega_c,$$

while the modes of oscillations can be classified using the synchrotron multipole number into betatron, synchrotron and synchrobetatron modes.

However, since coherent frequency shifts increase with the increase in the beam current, it may happen that coherent interaction will couple the synchrotron modes of the bunch and the multipole number m_c will not classify its coherent modes anymore. Physically, this fact corresponds to the development of coherent oscillations during time intervals comparable or faster than the periods of synchrotron oscillations of the bunch particles. Typically the coupling of synchrotron modes of the bunch breaks the stability of the coherent oscillations. In the region of rather moderate mode coupling $|\Omega_m| \simeq \omega_c$

the study of the collective stability of the beam usually requires the solution of the infinite system of the coupled integral equations (see, for instance, in Ref. 2). Its truncation yields both the eigenmodes and eigenfrequencies as well as stability criteria of coherent oscillations. Typically, the study of cases presenting a practical interest needs the employment of numerical methods.

If the interaction couples the synchrotron modes, the following difficulty can be encountered. Suppose, we have found the eigensolutions describing the coherent modes of the beam. If these modes are well defined, i.e. their decrements are smaller than the distance in the spectrum, the analysis of the stability of the beam requires calculation of only the increment of the leading mode. If, however, the interaction is so strong that modes overlap, i.e. their increments exceed the distance between their frequencies, the dependence of the amplitudes of coherent oscillations on time will be determined by the superposition of a large number of eigensolutions. In this case an analysis of the evolution of the initial perturbations of the beam becomes more adequate to the problem. Strong overlapping of the synchrotron modes due to RF potential well flattening has recently been found in Ref. 3.

The case of asymptotically strong mode coupling has been studied in Ref. 4. This region of parameters can be important for bunches in proton or heavy-ion storage rings, working near transition; for electron or positron storage rings with strong flattening of the RF potential well and for so-called quasi-isochronous electron or positron storage rings widely discussed now.[5] In Ref. 4 it was shown that in the region $|\Omega_m| \gg \omega_c$ the linearized Vlasov equation can lose the spectrum of eigenvalues, while the single-bunch instability takes the features specific for so-called beam breakup instability, which previously was studied for bunches in linacs (see in Refs. 6 and 7). In particular, the amplitudes of coherent oscillations in this region can grow by non-exponential laws. Physically, these features are caused by the fact that a development of the instability faster than the period of synchrotron oscillations breaks the feedback coupling of coherent oscillations between the head-on and tail-on particles in the bunch.

In this paper we trace the transformation of the synchrobetatron mode-coupling equations in the asymptotic region $|\omega_c/\Omega_m| \ll 1$ into an equation, which describes the beam breakup instability. We also calculate the leading correction term to this equation, which describes the feedback of head-on and tail-on particles in the beam and therefore yields its eigenmodes in an asymptotic region.

2 GENERAL EQUATIONS

The integral equation describing the coupling of synchrobetatron modes of a single bunch can be obtained as the first approximation of the averaging method after its application to the Vlasov equation, which is linearized near the given stationary state of the beam. If the unperturbed vertical betatron and synchrotron oscillations of a particle near the closed orbit are described by the following formulae:

$$\begin{gathered} z = a_z \cos\psi_z, \quad \theta = \omega_s t + \varphi, \\ \varphi = \varphi_c \cos\psi_c, \quad \dot\varphi = -\omega_c\varphi_c \sin\psi_c, \\ p_z = -\gamma M \omega_z a_z \sin\psi_z, \quad \Delta p = p - p_s, \\ \dot\psi_z = \omega_z(a, \Delta p) = \omega_0(\Delta p)\nu_z(\Delta p, a_z), \quad \dot\psi_c = \omega_c, \\ I_z = \frac{p}{2R_0}\nu_z a_z^2, \end{gathered} \tag{1}$$

both the Hamiltonian, H_0, which describes these oscillations, and the distribution function, f_0, describing the beam without coherent oscillations, by definition, do not depend on the phases of unperturbed oscillations of particles:

$$H = H_0(I_z, \varphi_c), \quad f_0 = F_0(I_z)\rho(\varphi_c). \tag{2}$$

It is well known that Eqs(1) generate the canonical transformation from the variables p_z, z, and $\Delta p, \varphi$ to the action-phase variables of unperturbed oscillations $I_z, \psi_z, \varphi_c^2/2, \psi_c$. Here, we use standard definitions.[8] Subscript s marks the values on the synchronous orbit and therefore, $\omega_s = \omega_0(p_s)$ is the revolution frequency of the synchronous particle, $\Pi = 2\pi R_0$ is the orbit perimeter, M is the mass of the particle, and γ is its relativistic factor.

For the sake of simplicity, we neglect below the Landau damping due to the nonlinearity of the particle synchrotron oscillations and we do the calculations for the most important case, when the betatron coherent oscillations are the dipole ones. Then, for the beam interacting with surrounding electrodes, the Fourier amplitudes of the expansion:

$$f = f_0(I_z, \varphi_c) + \sum_{m_z=\pm1} \sqrt{I_z}\exp(im_z\psi_z) \sum_{m_c=-\infty}^{\infty} \chi_m(\varphi_c)e^{im_c\psi_c - i\omega t} \tag{3}$$

satisfy the following system of integral equations (see, for instance, in Ref 1:

$$(\Delta\omega_m - m_c\omega_c)\chi_m = i\chi_m^{(0)}(\varphi_c) + \rho(\varphi_c)\sum_{n=-\infty}^{\infty} \Omega_{m,n} J_{m_c}(n_1\varphi_c)\chi(n), \tag{4}$$

$$\chi(n) = \sum_{m_c=-\infty}^{\infty} \int_0^{\infty} d\varphi_c' \varphi_c' J_{m_c}(n_1\varphi_c')\chi_m(\varphi_c'). \tag{5}$$

Here, $J_m(x)$ is the Bessel function,[9] $\Delta\omega_m = \omega - m_z\omega_z$, the value

$$n_1 = n + m_z(d\omega_z/d\omega_0) = n + m_z\nu_z + m_z\zeta, \quad \zeta = \frac{d\nu_z}{\partial \ln\omega_0} \tag{6}$$

describes the influence of the ring chromaticity on coherent oscillations, $\chi_m^{(0)}$ yields the initial distribution of the dipole momentum along the bunch, the subscript m means $m \equiv \{m_z, m_c\}$. The value $\Omega_{m,n}$ contributing to Eq. (5) is the coherent frequency shift of the coasting beam, interacting with the same system of electrodes. It is determined by the Green function $G(1,2,\omega)$ of the relevant electrodynamic problem (or, by the beam wake) and for dipole betatron oscillations has the form:[1]

$$\Omega_{m,n} = -\frac{Ne^2R_0}{2pc} m_z g_{m,n}(n\omega_s + m_z\omega_z + m_c\omega_c), \tag{7}$$

where

$$\begin{aligned} g_{m,n}(\omega) = & \frac{1}{4}\frac{\partial^2}{\partial a_{z1}\partial a_{z2}} \left\{ \int_0^\infty \frac{d\psi_{z,1}d\psi_{z,2}}{(2\pi)^2} \exp(-im_z[\psi_{z,1} - \psi_{z,2}])\times \right. \\ & \left. \int_0^\infty \frac{d\theta_1 d\theta_2}{(2\pi)^2} \exp(-in[\theta_1 - \theta_2])G(1,2,\omega) \right\}\Bigg|_{a_{z1},a_{z2}=0}, \end{aligned} \tag{8}$$

and

$$G(1,2,\omega) = \sum_{\alpha,\beta} \frac{v_{\alpha,1}v_{\beta,2}}{c^2} G_{\alpha,\beta}(1,2,\omega), \tag{9}$$

numbers 1 and 2 mark the coordinates of particles; $v_{\alpha,1}$, $v_{\alpha,2}$ are their velocities. We assume that the bunch wake obeys the Coulomb gauge condition:

$$\sum_{\alpha=x,z,\theta} \frac{\partial}{\partial r_\alpha} G_{\alpha,\beta} = 0. \tag{10}$$

Below, we calculate the sums over harmonic numbers using the following summation formula:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} b_n &= \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dn b(n) \exp(-2\pi i l n) \\ &= \int_{-\infty}^{\infty} dn b(n) + \sum_{l\neq 0} \int_{-\infty}^{\infty} dn b(n) \exp(-2\pi i l n). \end{aligned} \tag{11}$$

It is well known (see, for instance, in Ref. 1) that the first term in this equation yields the single-period contribution in the sum over harmonics n, while the second:

$$\sum_{l\neq 0} \int_{-\infty}^{\infty} dn b(n) \exp(-2\pi i l n)$$

yields the contributions due to the periodicity of the function

$$b(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}.$$

For the single-turn interaction of a bunch with its environment, when the bunch wakes decay faster than the particle revolution period, due to Eq. (11) we can replace in Eq. (4) the summation over n by the integration. This results in:

$$(\Delta\omega_m - m_c\omega_c)\chi_m = i\chi_m^{(0)}(\varphi_c) + \rho(\varphi_c) \int_{-\infty}^{\infty} dn \Omega_{m,n} J_{m_c}(n_1\varphi_c)\chi(n). \tag{12}$$

In the linear approximation Eqs. (12) enable the most comprehensive study of the stability of synchrobetatron modes of a single bunch with an arbitrary ratio between coherent frequency shift Ω_m and the frequency of synchrotron oscillations. In the region $|\Omega_m| \ll \omega_c$ Eqs(12) split into the system of independent equations, describing the uncoupled synchrobetatron modes. In the region $|\Omega_m| \sim \omega_c$ the mode coupling becomes significant and the calculation of the beam stability criteria requires the solution of the infinite set of integral equations (12). Except for some simple cases (see, for instance, in the Appendix) this must be done numerically. Finally, in the asymptotic region $|\Omega_m| \gg \omega_c$ the mode coupling becomes so strong that Eqs(12) can lose the spectrum of the eigenvalues (see, for instance, in Ref. 4) and can describe the so-called beam breakup instability with specific features, which are similar to that for bunches in linacs.

3 ASYMPTOTIC EQUATIONS

The asymptotes of Eqs(12), when $\omega_c \to 0$, can be calculated directly. Using Eq. (5) we transform Eqs(12) into the integral equation for azimuthal harmonics of the bunch linear density $\chi(n)$. If, for the sake of simplicity, we assume that the azimuthal length of the device interacting with the bunch is rather short, we may neglect in Eqs(7) and (8) the dependence of $g_{m,n}$ on the subscript n. The resulting equation for $\chi(n)$ can be written in the form:

$$\begin{aligned}\chi(n) = &\sum_{m_c=-\infty}^{\infty} \frac{i}{\Delta\omega_m - m_c\omega_c} \int_0^{\infty} d\varphi_c \varphi_c \chi_m^{(0)}(\varphi_c) J_{m_c}(n\varphi_c) \\ &+ \sum_{m_c=-\infty}^{\infty} \frac{1}{\Delta\omega_m - m_c\omega_c} \int_{-\infty}^{\infty} dn' \rho_m(n,n') w_m(n'-\zeta)\chi(n'),\end{aligned} \tag{13}$$

Here $w_m(n) = \Omega_{m_z}(n)/\omega_c$ and

$$\rho_m(n,n') = \int_0^{\infty} d\varphi_c \varphi_c \rho(\varphi_c) J_{m_c}(n\varphi_c) J_{m_c}(n'\varphi_c). \tag{14}$$

First, let us do the calculations for the homogeneous part of Eq. (13). Using Eq. (11), we replace in Eq. (13) the summation over m_c by the integration and then we separate, in the resulting sum over l, the term with $l = 0$ describing interaction of particles during one period of the synchrotron oscillation. The resulting homogeneous integral equation reads:

$$\chi(n) = \int_{-\infty}^{\infty} dn' [K_{sp}(n,n') + K_{mp}(n,n')] w_m(n'-\zeta)\chi(n'), \tag{15}$$

where the kernel

$$K_{sp}(n,n') = \int_{-\infty}^{\infty} dm_c \frac{\rho_{m_c}(n,n')}{x_m - m_c}, \quad x_m = \frac{\Delta\omega_m}{\omega_c}, \quad \mathrm{Im}\omega > 0 \tag{16}$$

describes single-period effects, while

$$K_{mp}(n,n') = \sum_{l\neq 0}\int_{-\infty}^{\infty} dm_c \frac{\rho_{m_c}(n,n')\exp(-2\pi i l m_c)}{x_m - m_c}, \quad \mathrm{Im}\omega > 0 \tag{17}$$

describes the multi-period effects and therefore, the feedback between the head-on and tail-on parts of the bunch. The kernel K_{mp} is determined by the residue of the integrand in the simple pole $m_c = \Delta\omega_m/\omega_c$. Since $\mathrm{Im}\omega > 0$, and for all particles we have

$$|\psi - \psi'| \leq 2\pi, \tag{18}$$

the integration contour for the terms with $l > 0$ can be looped in the lower half-plane of the complex variable m_c and thus, give no contribution in K_{mp}. The remaining part of the sum in Eq. (17) reads ($T_c = 2\pi/\omega_c$):

$$\begin{aligned} K_{mp}(n,n') = & \quad -2\pi i\,\rho_{x_m}(n,n')\sum_{l=1}^{\infty}\exp(2\pi i l x_m) = \frac{2\pi i}{1-\Lambda}\rho_{x_m}(n,n'), \\ \Lambda = & \quad \exp(-2\pi i x_m) = \exp(-iT_c\Delta\omega_m), \quad \mathrm{Im}\omega > 0. \end{aligned} \tag{19}$$

Respectively, we can rewrite Eq. (15) in the form:

$$\begin{aligned} \chi(n) \; & = \frac{2\pi i}{1-\Lambda}\int_{-\infty}^{\infty} dn'\rho_{x_m}(n,n')w_m(n'-\zeta)\chi(n') + \\ & + \int_{-\infty}^{\infty} dn' K_{sp}(n,n')w_m(n'-\zeta)\chi(n'). \end{aligned} \tag{20}$$

In these equations, the value Λ, by definition, has the physical sense of the amplification coefficient of the amplitude $\chi(n)$ after its transformation over the period of synchrotron oscillations:

$$\chi(n, t + T_c) = \Lambda\chi(n,t).$$

In this sense the first part of Eq. (20) is in general agreement with circuit theory: the output signal of the system with the feedback coupling is inversely proportional to the amplification coefficient.

Eq. (20) indicates that, depending on the value of $|\mathrm{Im}x_m|$, the first and the second terms in the right-hand side have the different weights. In the region $\mathrm{Im}x_m \ll 1$ and therefore, $\delta_m = \mathrm{Im}\omega \ll \omega_c$, the first term in the right-hand side of this equation dominates

$$\chi(n) \simeq \frac{2\pi i}{1-\Lambda}\int_{-\infty}^{\infty} dn'\rho_{x_m}(n,n')w_m(n'-\zeta)\chi(n'), \tag{21}$$

if $|x_m - m_c| \ll 1$. The eigenfrequencies of the coherent modes in this region are close to the harmonics of the synchrotron oscillations:

$$\omega = m_c\omega_c + \Delta\omega_m, \quad |\Delta\omega_m| \ll \omega_c.$$

In some simple cases, the values $\Delta\omega_m$ can be calculated analytically (see, for instance, in Ref. 1). If, for example, $\rho(\varphi_c) = \delta(\varphi_0^2 - \varphi_c^2)$, the value $\rho_{x_m}(n, n')$ reads:

$$\rho_{x_m}(n, n') = J_{m_c}(n\varphi_0)J_{m_c}(n'\varphi_0), \quad x_m \simeq m_c,$$

and therefore $\chi(n) \propto J_{m_c}(n\varphi_0)$. This yields

$$\Delta\omega_m \simeq \int_{-\infty}^{\infty} dn J_{m_c}^2(n\varphi_0)\Omega_{m_z}(n - \zeta).$$

More generally, if α marks the solutions of Eq. (21), the upper limit of the eigenvalues in Eq. (21) can be estimated by the sum:

$$\sum_\alpha (1 - \Lambda)_\alpha \simeq 2\pi i \sum_{m_c=-\infty}^{\infty} \int_{-\infty}^{\infty} dn \rho_{m_c}(n, n) w_m(n - \zeta) =$$

$$= 2\pi i \int_0^{\infty} d\varphi_c \varphi_c \rho(\varphi_c) \int_{-\infty}^{\infty} dn w_m(n - \zeta) \sum_{m_c=-\infty}^{\infty} J_{m_c}^2(n\varphi_c).$$

Using here *

$$\int_0^{\infty} d\varphi_c \varphi_c \rho(\varphi_c) = 1$$

and

$$\sum_{m_c=-\infty}^{\infty} J_{m_c}^2(n\varphi_c) = 1,$$

we can write

$$\sum_\alpha (1 - \Lambda)_\alpha \simeq \frac{2\pi i \Omega_{m_z}}{\omega_c} = 2\pi i \int_{-\infty}^{\infty} dn w_m(n - \zeta). \tag{22}$$

According to these equations, the condition $\text{Im}\omega \ll \omega_c$ holds in the region $|\Omega_{m_z}| \ll \omega_c$, where $|\Lambda_\alpha| \simeq 1$.

The inverse asymptote $|\Lambda| \gg 1$ can be associated with the region $\text{Im}x_m \gg 1$, where, due to the relationship:

$$|\Lambda| = \exp(2\pi \text{Im}x_m) \gg 1$$

the contribution of the first term in the right-hand side of Eq. (20) into eigensolutions $\chi(n)$ is exponentially suppressed. Respectively, in the limit $\omega_c \to 0$ neglecting these small terms, we can write:

$$\chi(n) \simeq \int_{-\infty}^{\infty} dn' K_{sp}(n, n') w_m(n' - \zeta)\chi(n'), \quad \text{Im}x_m \gg 1. \tag{23}$$

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* To avoid confusion we remind the reader that $\rho(\varphi_c)$ is indeed a function of the action variable of the synchrotron oscillations of a particle, and in this sense depends on $\varphi_c^2/2$.

Now, after direct calculation of the integral over m_c in K_{sp} we can write

$$\begin{aligned} K_{sp}(n,n') = & \int_0^\infty d\varphi_c\varphi_c\rho(\varphi_c)\int_{-\infty}^\infty dm_c\frac{J_{m_c}(n\varphi_c)J_{m_c}(n'\varphi_c)}{x_m - m_c}, \quad \mathrm{Im}x_m > 0, \\ = & \int_0^\infty d\varphi_c\varphi_c\rho(\varphi_c)\int_0^{2\pi}\frac{d\psi_1 d\psi_2}{(2\pi)^2}e^{i(n\varphi_1 - n'\varphi_2)}\int_{-\infty}^\infty dm_c\frac{e^{-im_c(\psi_1-\psi_2)}}{x_m - m_c} \\ = & \int_0^\infty d\varphi_c\varphi_c\int_0^{2\pi}\frac{d\psi}{2\pi i}\rho(\varphi_c)e^{-in'\varphi(\psi)}\int_0^\psi du\exp\{in\varphi(\psi-u) + ix_m u\}. \end{aligned} \tag{24}$$

In the region $\mathrm{Im}x_m \gg 1$, due to fast convergency of the integral over u, we can extend the upper limit in this integral to infinity. The substitution in Eq. (24) of $\rho(\varphi_c) = \lambda_0(\varphi)\Phi(\Delta p)$, with $\Phi(\Delta p) = \Phi(-\Delta p)$, yields:

$$\begin{aligned} K_{sp}(n,n') = & \ \omega_c\int\frac{d\Delta p d\varphi}{2\pi}\frac{\rho(\varphi,\Delta p)e^{i(n-n')\varphi}}{\Delta\omega_m - n\omega_0'\Delta p}, \quad \mathrm{Im}\omega \gg \omega_c \\ \simeq & \ \frac{\omega_c}{\Delta\omega_m}\left[\int_{-\infty}^\infty\frac{d\varphi}{2\pi}\lambda_0(\varphi)e^{i(n-n')\varphi} + O\left(\frac{\omega_c^2}{\Delta\omega_m^2}\right)\right]. \end{aligned} \tag{25}$$

The substitution of this kernel into Eq. (23) yields:

$$\begin{aligned} \chi(n) = & \ \frac{1}{\Delta\omega_m}\int_{-\infty}^\infty dn'\Omega_{m_z}(n-\zeta)\chi(n')\int_{-\infty}^\infty\frac{d\varphi}{2\pi}\lambda_0(\varphi)e^{i(n-n')\varphi} \\ = & \int_{-\infty}^\infty\frac{d\varphi}{2\pi}\lambda_0(\varphi)e^{-in\varphi}\int_{-\infty}^\infty d\varphi'\chi(\varphi')W_m(\varphi-\varphi'), \end{aligned} \tag{26}$$

where

$$W_m(\varphi) = \int_{-\infty}^\infty dn\Omega_{m_z}(n)e^{in\varphi} \tag{27}$$

defines the transverse wake function of the bunch. The inverse transformation:

$$\chi(\varphi) = \int_{-\infty}^\infty dne^{in\varphi}\chi(n) \tag{28}$$

converts Eq. (26) into the following integral equation:

$$\chi(\varphi) = \frac{\lambda_0(\varphi)}{\Delta\omega_m}\int_{-\infty}^\infty d\varphi' W_m(\varphi-\varphi')\chi(\varphi'). \tag{29}$$

Due to the causality condition

$$W_m(\varphi) = 0, \quad \varphi > 0,$$

Eq. (29) is an homogeneous Volterra-type integral equation and therefore, has no solutions except the ones. As was shown in, for instance, Ref. 4, non-homogeneous integral equations of this type appear in calculations if one takes into account initial conditions for coherent oscillations. In the approximation considered here ($\omega_c \to 0$) it describes the beam breakup-type instability of a single bunch in a storage ring. As the homogeneous part of the equation has no solutions, the corresponding coherent oscillations of the beam have neither a eigenfrequency spectrum, nor eigenmodes, while their amplitudes can depend on time by essentially non-exponential laws.[4]

Using Eqs. (25) through (29), we can rewrite Eq. (20) in the asymptotic region $|\Lambda| \gg 1$ as follows:

$$\chi(\varphi) = \chi_\Lambda(\varphi) + \frac{\lambda_0(\varphi)}{\Delta\omega_m}\int_\varphi^\infty d\varphi' W_m(\varphi - \varphi')\chi(\varphi'), \tag{30}$$

where

$$\chi_\Lambda(\varphi) = \int_{-\infty}^\infty dn \chi_\Lambda(n) e^{in\varphi}, \tag{31}$$

$$\chi_\Lambda(n) = \frac{2\pi i}{1-\Lambda}\int_{-\infty}^\infty dn' \rho_{x_m}(n, n') w_m(n' - \zeta)\chi(n') \tag{32}$$

In contrast to Eq. (29), the total Eq. (30) is a non-homogeneous integral equation of the Volterra type. Its non-trivial solution can be written using the resolvent kernel of this equation:

$$\chi(\varphi) = \chi_\Lambda(\varphi) + \int_\varphi^\infty R_m(\varphi, \varphi', \Delta\omega_m)\chi_\Lambda(\varphi'). \tag{33}$$

Combining Eqs(32) and (33) we can write the resulting integral equation in the form:

$$\begin{aligned} \Lambda\chi_\Lambda(n) = & \quad -2\pi i \int_{-\infty}^\infty dn' \rho_{x_m}(n, n') w_m(n' - \zeta) \times \\ \times & \quad \left\{\chi_\Lambda(n') + \int_{-\infty}^\infty dn'' R_m(n', n'', \Delta\omega_m)\chi_\Lambda(n'')\right\}, \\ & \quad |\Lambda| \gg 1, \end{aligned} \tag{34}$$

where

$$R_m(n, n', \Delta\omega_m) = \int_{-\infty}^\infty \frac{d\varphi}{2\pi} e^{-in\varphi}\int_\varphi^\infty d\varphi' R_m(\varphi, \varphi', \Delta\omega_m) e^{in'\varphi'}. \tag{35}$$

In some simple cases solution of Eq. (34) can be found in analytic form. We have discussed the model, when $\rho(\varphi_c) = \delta(\varphi_0^2 - \varphi_c^2)$ and therefore,

$$\lambda_0(\varphi) \propto \frac{1}{\sqrt{\varphi_0^2 - \varphi^2}}.$$

For this model we can write:

$$\rho_{x_m}(n, n') = J_{x_m}(n\varphi_0) J_{x_m}(n'\varphi_0).$$

Substituting in Eq. (34) the solution

$$\chi_\Lambda \propto J_{x_m}(n\varphi_0),$$

we obtain the dispersion equation of the problem in the form:

$$\Lambda(x_m) = -2\pi i \int_{-\infty}^{\infty} dn J_{x_m}^2(n\varphi_0) w_m(n-\zeta) - 2\pi i \int_{-\infty}^{\infty} dn w_m(n-\zeta) \times \\ \times J_{x_m}(n\varphi_0) \int_{-\infty}^{\infty} dn' J_{x_m}(n'\varphi_0) R_m(n, n', \Delta\omega_m). \tag{36}$$

This equation can be simplified even more, if the bunch wake $W_m(\varphi)$ is the step-function †:

$$W_m(\varphi) = 2\Omega_{m_z} \begin{cases} 0, & \varphi > 0 \\ 1, & \varphi < 0, \end{cases}$$

where Ω_{m_z} is the coherent frequency shift of the short bunch ($\varphi \to 0$). In this case the resolvent R_m can be easily calculated:[4]

$$R_m(\varphi, \varphi', \Delta\omega_m) = \frac{2\Omega_{m_z}}{\Delta\omega_m} \lambda_0(\varphi) \exp\left\{ \frac{2\Omega_{m_z}}{\Delta\omega_m} \int_{\varphi}^{\varphi'} d\varphi'' \lambda_0(\varphi'') \right\}. \tag{37}$$

Since, for such a wake, the first term in the right-hand side of Eq. (36) vanishes, we can replace Eq. (36) in the asymptotic region by the following:

$$\Lambda(x_m) = 2\Omega_{m_z} \int_{-\infty}^{\infty} \frac{dn}{n} J_{x_m}(n\varphi_0) \int_{-\infty}^{\infty} dn' J_{x_m}(n'\varphi_0) R_m(n, n', \Delta\omega_m).$$

4 DISCUSSION

Eq. (34) is an integral equation of the Fredholm type, and thus definitely has the spectrum of eigenvalues. In this sense the spectrum of eigenmodes of coherent oscillations of a bunch always exists. If the eigenvalues Λ_α are found (the subscript α marks the particular modes), and since the eigenfrequencies of the modes only logarithmically depend on Λ_α (see Eq. (19)):

$$\Delta\omega_{m,\alpha} = \frac{i\omega_c}{2\pi} \ln \Lambda_\alpha, \tag{38}$$

the distance in the spectrum generally decreases, when $|\Lambda|$ increases. From Eq. (38) we can estimate the rate of change of this distance in the asymptotic region $|\Lambda| \gg 1$ as the exponential one:

$$\Delta(\omega_\alpha) = \omega_{\alpha+1} - \omega_\alpha \sim \omega_c \exp(-\delta_{m,\alpha} T_c), \tag{39}$$

where $\delta_{m,\alpha} = \mathrm{Im}\omega_{m,\alpha}$ is the increment of the mode α.

The dependence of the beam dipole momentum on time is determined by the Fourier integrals:

$$\chi(\varphi, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \chi(\varphi), \quad \mathrm{Im}\omega > 0. \tag{40}$$

† However, as is shown in the Appendix, in this particular case the homogeneous part of Eqs(12) can easily be solved directly.

The contributing here Fourier amplitudes $\chi(\varphi)$ must be calculated as the solutions of the non-homogeneous Eq. (13). If we define here

$$\chi^{(0)}(n,\Delta\omega_m)=\sum_{m_c=-\infty}^{\infty}\frac{i}{\Delta\omega_m-m_c\omega_c}\int_0^{\infty}d\varphi_c\varphi_c\chi_m^{(0)}(\varphi_c)J_{m_c}(n\varphi_c), \tag{41}$$

then using Eqs(31) and (33) we can write the amplitudes $\chi(\varphi)$ in the form

$$\chi(\varphi)=\chi_\Lambda(\varphi)+\chi^{(0)}(\varphi,\Delta\omega_m)+\int_\varphi^{\infty}d\varphi' R_m(\varphi,\varphi',\Delta\omega_m)[\chi_\Lambda(\varphi')+\chi^{(0)}(\varphi',\Delta\omega_m)]. \tag{42}$$

As can be seen from Eqs(32) and (42), in the asymptotic region $|\Lambda|\gg 1$ the contributions of eigenmodes in the beam signal will be exponentially small as compared with the contribution from the fast development of initial perturbation $\chi^{(0)}$.

Now, for the sake of convenience, let us select in Eq. (42) the part responsible for the singe-period effects:

$$\chi(\varphi)=\chi_{sp}(\varphi)+\chi_\Lambda(\varphi). \tag{43}$$

Repeating here the calculations, which led from Eq. (15) to Eq. (20) and then to Eq. (25), we can rewrite $\chi^{(0)}(\varphi,\Delta\omega_m)$ in the form:

$$\begin{aligned}\chi^{(0)}(n,\Delta\omega_m)= &\ \frac{2\pi}{\omega_c(1-\Lambda)}\int_0^{\infty}d\varphi_c\varphi_c\chi_{x_m}^{(0)}(\varphi_c)J_{x_m}(n\varphi_c)\\ &+\ i\int\frac{d\Delta p d\varphi}{2\pi}\frac{\chi_m(0)(\varphi,\Delta p)}{\Delta\omega_m-n\omega_0'\Delta p}\\ \equiv&\ \chi_{mp}^{(0)}+\chi_{sp}^{(0)}.\end{aligned} \tag{44}$$

Now we may define

$$\chi_{sp}(\varphi)=\chi_{sp}^{(0)}(\varphi,\Delta\omega_m)+\int_\varphi^{\infty}d\varphi' R_m(\varphi,\varphi',\Delta\omega_m)\chi_{sp}^{(0)}(\varphi',\Delta\omega_m). \tag{45}$$

This part of the solution coincides with that obtained in Ref. 4 and yields the leading term of its asymptotic expansion in powers of ω_c/Ω_{m_z}. As has been found in Ref. 4 the main features of the time-dependence of the function $\chi_{sp}(\varphi,t)$ are determined by the specific singularity of the resolvent kernel R_m as a function of the complex variable $\Delta\omega_m$. The function $\chi_{sp}(\varphi,t)$ is scaled by the value of the coherent frequency shift of the zero-length bunch:

$$\Omega_{m_z}=\int_{-\infty}^{\infty}dn\Omega_{m_z}(n) \tag{46}$$

and generally has the asymptotes:

$$\chi_{sp}(\varphi,t)\sim\begin{cases}\Omega_{m_z}t\mathcal{N}(\varphi), & \Omega_{m_z}t\ll 1,\\ \exp[\sqrt{\Omega_{m_z}t\mathcal{N}(\varphi)}], & \Omega_{m_z}t\ll 1,\end{cases} \tag{47}$$

where

$$\mathcal{N}(\varphi) = \int_{\varphi}^{\infty} d\varphi' \lambda_0(\varphi'). \tag{48}$$

The multi-period part of $\chi(\varphi)$ in the asymptotic region $|\Lambda| \gg 1$ satisfies the equation:

$$\begin{aligned} \Lambda\chi_\Lambda(n) = \quad & \tilde{\chi}^{(0)}_{mp} - 2\pi i \int_{-\infty}^{\infty} dn' \rho_{x_m}(n, n') w_m(n' - \zeta) \times \\ & \times \left\{ \chi_\Lambda(n') + \int_{-\infty}^{\infty} dn'' R_m(n', n'', \Delta\omega_m) \chi_\Lambda(n'') \right\}, \end{aligned} \tag{49}$$

where

$$\begin{aligned} \tilde{\chi}^{(0)}_{mp} = \quad & -2\pi i \int_{-\infty}^{\infty} dn' \rho_{x_m}(n, n') w_m(n' - \zeta) \times \\ & \times \left\{ \chi^{(0)}_{mt}(n') + \int_{-\infty}^{\infty} dn'' R_m(n', n'', \Delta\omega_m) \chi^{(0)}_{mt}(n'') \right\}, \end{aligned} \tag{50}$$

and

$$\chi^{(0)}_{mp}(n, \Delta\omega_m) \simeq -\frac{2\pi}{\omega_c \Lambda} \int_0^{\infty} d\varphi_c \varphi_c \chi^{(0)}_{x_m}(\varphi_c) J_{x_m}(n\varphi_c). \tag{51}$$

The dependence of the multi-period part $\chi_\Lambda(t)$ on time can generally be represented by its expansion in eigenmodes:

$$\chi_{mp}(\varphi, t) = \sum_{\alpha} A_\alpha \chi_{\Lambda,\alpha}(\varphi) \exp(-i\Delta\omega_\alpha t). \tag{52}$$

By Eq. (38) the increments in this series are proportional to ω_c. Therefore, we may conclude that in the asymptotic region $\Omega_{m_z} \gg \omega_c$ the initial behaviour ($\omega_c t \leq 1$) of transverse coherent oscillations of a single bunch will be mainly determined by the beam breakup phenomenon. A self-consistent behaviour of oscillations might be expected on rather longer time intervals $\omega_c t \geq 1$. To this end, if the oscillations are unstable due to single-period effects, the calculation of the self-consistent spectrum of the beam in the asymptotic region can contradict the assumed application of the perturbation theory.

Although we did the calculations in the asymptotic region, it is clear that the contribution from the beam breakup phenomenon in the evolution of the dipole momentum of the beam can be significant even in the region of the moderate mode coupling $\omega_c \leq |\Omega_m|$. However in this case the relevant resolvent kernels $R_m(\varphi, \varphi', \Delta\omega_m)$ should be calculated taking into account the higher order corrections in powers of $\omega_c/\Delta\omega_m$ (see in Eq. (25)).

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A SOLVABLE MODEL WITH MODE – COUPLING

The homogeneous part of the infinite set of Eqs(12):

$$(\Delta\omega_m - m_c\omega_c)\chi_m = \rho(\varphi_c)\int_{-\infty}^{\infty} dn\,\Omega_{m_z}(n)J_{m_c}(n_1\varphi_c)\times \sum_{m_c=-\infty}^{\infty}\int_0^{\infty} d\varphi_c'\varphi_c' J_{m_c}(n_1\varphi_c')\chi_m(\varphi_c') \quad (A.1)$$

can be solved exactly for the simplified model, where

$$\rho(\varphi_c) = \delta(\varphi_0^2 - \varphi_c^2), \qquad \Omega_{m_z}(n) = \frac{i\Omega_{m_z}}{\pi(n + i\Delta)}, \quad (A.2)$$

and $\zeta = 0$. In the region $|\Omega_{m_z}| \ll \omega_c$ the quantity Ω_{m_z} defines the coherent frequency shift of the betatron mode ($m_c = 0$). Since, in this model,

$$\chi_m(\varphi_c) = C_m\delta(\varphi_0^2 - \varphi_c^2), \quad (A.3)$$

we replace Eqs(A.1) by an equivalent system of the algebraic equations:

$$(\Delta\omega_m - m_c\omega_c)C_m = \frac{i\Omega_{m_z}}{\pi}\sum_{m_c'=-\infty}^{\infty} C_{m'}\int_{-\infty}^{\infty}\frac{dn}{n+i\Delta}J_{m_c}(n\varphi_0)J_{m_c'}(n\varphi_0), \quad (A.4)$$

or defining

$$x = \frac{\Delta\omega_m}{\omega_c}, \quad w = \frac{\Omega_{m_z}}{\omega_c},$$

$$(x - m_c)C_m = \frac{iw}{\pi}\sum_{m_c'=-\infty}^{\infty} C_{m'}\int_{-\infty}^{\infty}\frac{dn}{n+i\Delta}J_{m_c}(n)J_{m_c'}(n). \quad (A.5)$$

Now, using

$$J_m(0)=\delta_{m,0},\quad J_m(-n)=(-1)^m J_m(n),$$
$$J_{-m}(n)=(-1)^m J_m(n)$$

and Ref. 9

$$\frac{\pi}{2}(m^2-m'^2)\int_0^{\infty}\frac{dn}{n}J_m(n)J_{m'}(n)=\sin\left[\frac{\pi}{2}(m-m')\right],$$

we transform the system of Eqs(A.5) into the following:

$$(x-w)C_0=iW\sum_{k=0}^{\infty}\frac{C^-_{2k+1}}{(2k+1)^2},\tag{A.6}$$

$$C^-_{2k+1}=iW\frac{2x}{x^2-(2k+1)^2}\left[\frac{C_0}{(2k+1)^2}+\sum_{p=1}^{\infty}\frac{C^+_{2p}}{(2k+1)^2-4p^2}\right],\tag{A.7}$$

$$C^+_{2p}=iW\frac{2x}{x^2-4p^2}\sum_{k=0}^{\infty}\frac{C^-_{2k+1}}{(2k+1)^2-4p^2}.\tag{A.8}$$

Here $W=(4w/\pi^2)$ and

$$C^+_{2k}=C_{2k}+C_{-2k},\quad C^-_{2k+1}=C_{2k+1}-C_{-(2k+1)}.$$

Substituting Eq. (A.8) into Eq. (A.7), we rewrite Eq. (A.7) in the form:

$$C^-_{2k+1}=iW\frac{2x}{x^2-(2k+1)^2}\left[\frac{C_0}{(2k+1)^2}+\sum_{k'=0}^{\infty}S_{k,k'}C^-_{2k'+1}\right],\tag{A.9}$$

$$S_{k,k'}=\sum_{p=1}^{\infty}\frac{2iWx}{[(2k+1)^2-4p^2][x^2-4p^2][(2k'+1)^2-4p^2]}.$$

The sum over p in this equation can be easily calculated using the identity:

$$\sum_{p=1}^{\infty}\frac{1}{\alpha^2-4p^2}=\frac{\pi}{4\alpha}\left[\cot\frac{\pi\alpha}{2}-\frac{2}{\pi\alpha}\right],$$

which results in:

$$S_{k,k'}=-\frac{1}{2x^2(2k+1)^2(2k'+1)^2}+\frac{\pi}{4x}\frac{\cot(\pi x/2)}{[x^2-(2k+1)^2][x^2-(2k'+1)^2]}.$$

Using this expression instead of Eq. (A.9) we obtain

$$\begin{aligned}C^-_{2k+1}= & -\frac{2W^2}{(2k+1)^2[x^2-(2k+1)^2]}\frac{w}{x-w}\sum_{k'=0}^{\infty}\frac{C^-_{2k'+1}}{(2k'+1)^2}\\ & -W^2\frac{\pi x\cot(\pi x/2)}{[x^2-(2k+1)^2]^2}\sum_{k'=0}^{\infty}\frac{C^-_{2k'+1}}{x^2-(2k'+1)^2}.\end{aligned}\tag{A.10}$$

Solutions of these equations obviously read:

$$C^{-}_{2k+1} = \frac{A(x)}{(2k+1)^2[x^2-(2k+1)^2]} + \frac{B(x)}{[x^2-(2k+1)^2]^2}. \qquad (A.11)$$

The substitution of Eq. (A.11) in Eq. (A.10) yields the dispersion equation

$$x = w\left\{1 - 2W^2\left(F_1(x) - \frac{W^2\pi x\cot(\pi x/2)}{1+W^2\pi x\cot(\pi x/2)F_3(x)}F_2^2(x)\right)\right\}, \qquad (A.12)$$

where

$$F_1(x) = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^4[x^2-(2k+1)^2]},$$

$$F_2(x) = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2[x^2-(2k+1)^2]^2}, \qquad (A.13)$$

$$F_3(x) = \sum_{k=0}^{\infty}\frac{1}{[x^2-(2k+1)^2]^3}.$$

The functions F_1, F_2 and F_3 can be easily calculated using the identity:

$$\tan\frac{\pi x}{2} = \frac{4x}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2-x^2},$$

which yields

$$F_1(x) = \frac{\pi^4}{96x^2} + \frac{\pi^2}{8x^4} - \frac{\pi}{4x^5}\tan\frac{\pi x}{2},$$

$$F_2(x) = \frac{\pi^2}{8x^4}\left(1 + \frac{1}{2\cos^2(\pi x/2)} - \frac{3}{\pi x}\tan\frac{\pi x}{2}\right),$$

$$F_3(x) = -\frac{\pi}{32x^3}\left\{\left(\frac{3}{x^2} + \frac{\pi^2}{2\cos^2(\pi x/2)}\right)\tan\frac{\pi x}{2} - \frac{3\pi}{2x\cos^2(\pi x/2)}\right\}.$$

In the region $|x| \ll 1$ (and, correspondingly $|w| \ll 1$) these functions have the asymptotes:

$$F_3 \simeq F_1 \simeq -F_2 \simeq -\frac{\pi^6}{960}.$$

Substituting these values into Eq. (A.12) we can estimate the first-order correction to the coherent frequency shift of the betatron mode ($m_c = 0$) in the case of weak mode coupling:

$$x \simeq w\left(1 + \frac{\pi^2}{30}w^2\right).$$

In the region of the strong synchrobetatron mode coupling $|w| \geq 1$ the dispersion equation must be solved numerically.