CAVITY PHASE NOISE IN PROTON STORAGE RINGS AND ITS INFLUENCE ON COUPLED SYNCHRO-BETATRON OSCILLATIONS

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In the following report we use the Fokker-Planck equation to investigate the influence of cavity phase noise in proton storage rings on fully coupled synchro-betatron oscillations. The paper deals only with the special case of white noise in a stationary linearized bucket. The equations derived are valid for arbitrary velocity of the protons (below and above transition energy). Noise limits are obtained for the HERA proton ring.

KEY WORDS: RF phase noise, storage rings, synchro-betatron oscillations

1 INTRODUCTION

The influence of cavity noise on particle motion has already been studied in several papers.[1,2,3] In this work we generalize these papers by including the transverse oscillation modes and consider the combined system of longitudinal and transverse motion by a simultaneous treatment of synchrotron and betatron oscillations, taking into account all kinds of coupling (synchro-betatron coupling and coupling of the betatron oscillations by skew quadrupoles and solenoids). This subject has already been treated in a previous paper,[4] using a dispersion formalism (including amplitude fluctuations). The aim of this report is to repeat these studies for phase fluctuations within the framework of the fully coupled 6-dimensional formalism and to provide some numerical results for the HERA proton ring. The concept to be used in this report is well known from radiation theory.[5]

In detail, the work is organized as follows:

We start with the fully coupled 6-dimensional description of the particle motion using the coordinates x, p_x, z, p_z, σ, $p_\sigma = (1/\beta_0^2) \cdot \eta$ with $\eta = \Delta E/E_0$ which allows to handle the external magnetic forces in a consistent canonical manner and which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities. This description is summarized in section 2 and leads to a derivation of the stochastic equations of motion, taking into account the influence of cavity noise.

The linear equations of motion, neglecting stochastic terms induced by the cavity noise, (i.e. the unperturbed problem) are studied in section 3 by defining the 6-dimensional transfer matrix and by investigating the eigenvalue spectrum of the revolution matrix. Furthermore, action-angle variables for the coupled orbital motion are introduced.

The perturbed problem, taking into account the cavity noise is presented in section 4. We are then able to derive the stochastic equations of motion in terms of the orbital action-angle variables (section 5) which form the basis for a Fokker-Planck treatment of stochastic particle motion.

The Fokker-Planck equation for orbital motion under the influence of cavity noise is written down in section 6 and a solution of this equation is shown in section 7.

The equations so derived are valid for arbitrary velocity of the protons (below and above transition energy).

The numerical results for the HERA proton ring are presented in section 8 and the summary may be found in section 9.

Finally we remark that several parts of this report are taken over from Ref. .[4] But since that paper may not be universally available we mention them here again in order to present a complete description of the theory.

2 EQUATIONS OF MOTION

To investigate cavity noise in proton storage rings we begin with the derivation of the equations of motion. We will use the same variables as those in Ref. 8:

$$x,\ z,\ \sigma = s - v_0 \cdot t \ \text{ and } \ p_\sigma = \frac{1}{\beta_0^2} \cdot \eta \tag{2.1}$$

(v_0 =design speed= $c\beta_0$) with

$$\eta = \Delta E / E_0 \tag{2.2}$$

by introducing as usual:[9]

a) the closed design orbit (a piecewise flat path of a particle with constant energy E_0) which will in the following be described by the vector $\vec{r}_0(s)$ where s is the length along this ideal orbit;

b) an orthogonal coordinate system ('dreibein') accompanying the particles which travels along the design orbit and comprises:[10]

the unit tangent vector	$\vec{e}_s(s) = \frac{d}{ds}\vec{r}_0(s) \equiv \vec{r}_0{}'(s)$;
a unit vector	$\vec{e}_x(s)$ perpendicular to $\vec{e}_s$ in the horizontal plane
and the unit vector	$\vec{e}_z(s) = \vec{e}_s(s) \times \vec{e}_x(s)$.

The Serret-Fresnet formulae for the dreibein $(\vec{e}_s, \vec{e}_x, \vec{e}_z)$ read as

$$\frac{d}{ds}\,\vec{e}_x(s) = +K_x(s)\cdot\vec{e}_s(s)\ ; \tag{2.3a}$$

$$\frac{d}{ds}\,\vec{e}_z(s) = +K_z(s)\cdot\vec{e}_s(s)\ ; \tag{2.3b}$$

$$\frac{d}{ds}\,\vec{e}_s(s) = -K_x(s)\cdot\vec{e}_x(s) - K_z(s)\cdot\vec{e}_z(s) \tag{2.3c}$$

with the assumption that

$$K_x(s)\cdot K_z(s) = 0$$

(piecewise no torsion) and where $K_x(s), K_z(s)$ designate the curvatures in the x-direction and in the z-direction respectively.

In this natural coordinate system an arbitrary orbit-vector $\vec{r}(s)$ can be written in the form

$$\vec{r}(x,z,s) = \vec{r}_0(s) + x(s)\cdot\vec{e}_x(s) + z(s)\cdot\vec{e}_z(s)\ . \tag{2.4}$$

Thus x and z describe the amplitude of transverse motion (betatron oscillations), while $\sigma = s - v_0\cdot t$ and $p_\sigma \equiv (1/\beta_0^2)\cdot\eta$ describe the longitudinal (synchrotron) oscillation. The quantity σ defines the longitudinal separation of particles from the centre of the bunch and η describes the energy deviation of the particle.

Starting then from the Hamiltonian for the motion of a charged particle in an electromagnetic field:

$$\hat{\mathcal{H}}(X_1,X_2,X_3;P_1,P_2,P_3,t) = c\cdot\left\{\vec{\pi}^2 + m_0^2c^2\right\}^{1/2} + e\phi \tag{2.5}$$

with

$$\vec{\pi} = \vec{P} - \frac{e}{c}\vec{A} \quad \text{(kinetic momentum vector)} \tag{2.6}$$

where X_1, X_2, X_3 and P_1, P_2, P_3 are canonical orbital position and momentum variables in a fixed Cartesian coordinate system ($\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$), and introducing the length s along the design orbit as the independent variable (instead of the time t), one can construct the Hamiltonian of the orbit motion with respect to the variables (2.1) by a succession of canonical transformations and a scale transformation.[6,7,8]

Choosing a gauge with $\phi = 0$ (e.g. Coulomb gauge), one thus obtains (see Eq. A.10 in Appendix A ; we are writing here x, p_x, z, p_z, σ, p_σ instead of $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$):

$$\begin{aligned}\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) &= p_\sigma - (1+\hat{\eta})\cdot[1+K_x\cdot x+K_z\cdot z]\times\\ &\qquad\left\{1-\frac{(p_x-\frac{e}{p_0\cdot c}A_x)^2+(p_z-\frac{e}{p_0\cdot c}A_z)^2}{(1+\hat{\eta})^2}\right\}^{1/2}\\ &\quad - [1+K_x\cdot x+K_z\cdot z]\cdot\frac{e}{p_0\cdot c}A_s\end{aligned} \tag{2.7}$$

where $\hat{\eta}$ is defined in Appendix A.

The corresponding canonical equations read as:

$$\frac{d}{ds}\,\vec{y} = -\underline{S}\cdot\frac{\partial\mathcal{H}}{\partial\vec{y}} \tag{2.8}$$

with

$$\vec{y}^{\,T} = (x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma) \tag{2.9}$$

where the matrix $\underline{S}$ is given by

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix}\ ;\quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\ . \tag{2.10}$$

Since $\mathcal{H}$ contains the transverse coordinates $x,\ p_x,\ z,\ p_z$ as well as the longitudinal coordinates $\sigma,\ p_\sigma$ we are thus able to handle synchrotron oscillations (longitudinal motion) and betatron oscillations (transverse motion) simultaneously.

In order to utilize this Hamiltonian, the electric field $\vec{\epsilon}$ and the magnetic field $\vec{B}$ or the corresponding vector potential,

$$\vec{A} = \vec{A}(x,z,s), \tag{2.11}$$

for the cavities and for commonly occurring types of accelerator magnets must be given. Once $\vec{A}$ is known the fields $\vec{\varepsilon}$ and $\vec{B}$ may be found using the relations:

$$\vec{\epsilon} \;=\; -\text{grad}\ \phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\ ; \tag{2.12a}$$

$$\vec{B} \;=\; \text{curl}\ \vec{A}\ . \tag{2.12b}$$

Expressed in the variables $x,\ z,\ s,\ \sigma$, Eqs. (2.12a, b) become (with $\phi = 0$):

$$\vec{\varepsilon} = \beta_0\cdot\frac{\partial}{\partial\sigma}\,\vec{A} \tag{2.13}$$

and

$$B_x = \frac{1}{(1+K_x\cdot x+K_z\cdot z)} \times\left\{\frac{\partial}{\partial z}\,[(1+K_x\cdot x+K_z\cdot z)\cdot A_s] - \frac{\partial}{\partial s}A_z\right\}\ ; \tag{2.14a}$$

$$B_z = \frac{1}{(1+K_x\cdot x+K_z\cdot z)} \times\left\{\frac{\partial}{\partial s}A_x - \frac{\partial}{\partial x}\,[(1+K_x\cdot x+K_z\cdot z)\cdot A_s]\right\}\ ; \tag{2.14b}$$

$$B_s = \frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\ . \tag{2.14c}$$

We assume that the ring consists of quadrupoles, skew quadrupoles, bending magnets, solenoids and cavities. Then the vector potential $\vec{A}$ can be written as:[4]

$$\begin{aligned}\frac{e}{p_0 \cdot c} A_s &= -\frac{1}{2}[1 + K_x \cdot x + K_z \cdot z] + \frac{1}{2} g \cdot (z^2 - x^2) + N \cdot xz \\ &\quad -\frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] ; \qquad (2.15a)\end{aligned}$$

$$\frac{e}{p_0 \cdot c} A_x = -H \cdot z \; ; \quad \frac{e}{p_0 \cdot c} A_z = +H \cdot x \qquad (2.15b)$$

(h = harmonic number) with the following abbreviations:

$$g = \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} ; \qquad (2.16a)$$

$$N = \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0} ; \qquad (2.16b)$$

$$H = \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot B_s(0,0,s) \; ; \qquad (2.16c)$$

$$K_x = +\frac{e}{p_0 \cdot c} \cdot B_z(0,0,s) \; ; \quad K_z = -\frac{e}{p_0 \cdot c} \cdot B_x(0,0,s) \; . \qquad (2.16d)$$

In detail, one has:

a)	$g \neq 0$;	$N = K_x = K_z = H = V = 0$:	quadrupole;
b)	$N \neq 0$;	$g = K_x = K_z = H = V = 0$:	skew quadrupole;
c)	$K_x^2 + K_z^2 \neq 0$;	$g = N = H = V = 0$:	bending magnet;
d)	$H \neq 0$;	$g = N = K_x = K_z = V = 0$:	solenoid;
e)	$V \neq 0$;	$g = K_x = K_z = N = H = 0$:	cavity.

Thus the Hamiltonian takes the form:

$$\begin{aligned}\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) &= p_\sigma - (1+\hat{\eta}) \cdot [1 + K_x \cdot x + K_z \cdot z] \times \\ &\quad \left\{1 - \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{(1+\hat{\eta})^2}\right\}^{1/2} \\ &\quad +\frac{1}{2} \cdot [1 + K_x \cdot x + K_z \cdot z]^2 \\ &\quad -\frac{1}{2} \cdot g \cdot (z^2 - x^2) - N \cdot xz \\ &\quad +\frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] . \qquad (2.17)\end{aligned}$$

To introduce the cavity noise we may write:

$$\varphi = \varphi_0 + \delta\varphi \qquad (2.18)$$

where $\delta\varphi$ describes the phase noise:

$$\delta\varphi = \lambda_{PH} \cdot \xi(s) \ . \tag{2.19}$$

We assume Gaussian white noise, i.e. noise described by a stationary stochastic process with

$$< \xi(s) > \; = \; 0 \ ; \tag{2.20a}$$

$$< \xi(s) \cdot \xi(s') > \; = \; \delta(s - s') \ . \tag{2.20b}$$

One may visualize $\xi(s)$ as a random sequence of small positive and negative pulses in equal proportions i.e. flat spectrum, while the underlying 'mechanism' which causes the fluctuations does not change in course of time.

Here the cavity phase φ_0 is determined by the stability condition for synchrotron motion:[4,7]

$$\begin{cases} \varphi_0 = 0 & \text{above 'transition' ;} \\ \varphi_0 = \pi & \text{below 'transition' ;} \end{cases}$$

(stationary buckets).

Now, since

$$|p_x + H \cdot z| \; \ll \; 1 \ ;$$

$$|p_z - H \cdot x| \; \ll \; 1$$

the square root term in (2.17) may be expanded in series, so that in practice the particle motion can be conveniently calculated to various orders of approximation.

Furthermore we can write (see Appendix A):

$$\hat{\eta} \equiv f(\eta) \; = \; p_\sigma - \frac{1}{\gamma_0^2} \cdot \frac{1}{2} p_\sigma^2 \pm \cdots \ . \tag{2.21}$$

From (2.21) one obtains to the first order:

$$p_\sigma \; \approx \; \frac{\Delta p}{p_0} \ .$$

In the following we shall use a series expansion of the Hamiltonian up to second order in the variables x, p_x, z, p_z, σ, p_σ. Then we obtain from (2.17):

$$\mathcal{H} \; = \; \mathcal{H}_0 + \mathcal{H}_1 \tag{2.22}$$

with

$$\mathcal{H}_0 \; = \; \frac{1}{\gamma_0^2} \cdot \frac{1}{2} p_\sigma^2 - [K_x \cdot x + K_z \cdot z] \cdot f(\eta)$$

$$
\begin{aligned}
&+\frac{1}{2}\cdot\left\{[p_x+H\cdot z]^2+[p_z-H\cdot x]^2\right\}\\
&+\frac{1}{2}\cdot\left\{(K_x^2+g)\cdot x^2+(K_z^2-g)\cdot z^2-2N\cdot xz\right\}\\
&-\frac{1}{\beta_0^2}\cdot\frac{1}{2}\sigma^2\cdot\frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi_0\\
&-\frac{1}{\beta_0^2}\cdot\sigma\cdot\frac{eV(s)}{E_0}\cdot\sin\varphi_0\ ; \qquad (2.23a)
\end{aligned}
$$

$$
\mathcal{H}_1 = -\frac{1}{\beta_0^2}\cdot\sigma\cdot\frac{eV(s)}{E_0}\cdot\cos\varphi_0\cdot\delta\varphi \qquad (2.23b)
$$

(constant terms in the Hamiltonian, i.e. no influence in the motion, are dropped).

Using (2.8), one obtains the linearized equations of motion in the form:

$$
\frac{d}{ds}\vec{y} = \underline{A}\cdot\vec{y}+\delta\vec{c}_{PH} \qquad (2.24)
$$

with

$$
\vec{y}=\begin{pmatrix} x\\ p_x\\ z\\ p_z\\ \sigma\\ p_\sigma \end{pmatrix} ; \qquad \delta\vec{c}_{PH} = \frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot\cos\varphi_0\cdot\lambda_{PH}\cdot\xi(s)\cdot\begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix} \qquad (2.25)
$$

and

$$
\underline{A} = \qquad (2.26)
$$

$$
\begin{pmatrix}
0 & 1 & H & 0 & 0 & 0\\
-[K_x^2+g+H^2] & 0 & N & H & 0 & K_x\\
-H & 0 & 0 & 1 & 0 & 0\\
N & -H & [K_z^2-g+H^2] & 0 & 0 & K_z\\
-K_x & 0 & -K_z & 0 & 0 & 1/\gamma_0^2\\
0 & 0 & 0 & 0 & \frac{1}{\beta_0^2}\frac{eV(s)}{E_0}\frac{2\pi h}{L}\cos\varphi_0 & 0
\end{pmatrix}
$$

which represent a system of stochastic differential equations describing coupled synchro-betatron oscillations of protons under the influence of cavity noise.

Note that the linear (transverse) betatron oscillations and the longitudinal motion are coupled by the term

$$
-[K_x\cdot x+K_z\cdot z] \qquad (2.27)
$$

appearing in (2.26) which depends on the curvature of the orbit in the bending magnets.

3 THE UNPERTURBED PROBLEM

In order to study the particle's motion under the influence of phase fluctuations it is reasonable to neglect in Eq. (2.24) in a first approximation the small term $\delta\vec{c}_{PH}$ and to consider only the 'unperturbed problem':

$$\frac{d}{ds}\vec{y} = \underline{A}\cdot\vec{y}\,. \tag{3.1}$$

The (small) perturbation described by $\delta\vec{c}_{PH}$ will then be treated in a second step with perturbation theory.

Since Eq. (3.1) is linear and homogeneous, the solution can be written in the form:

$$\vec{y}(s) = \underline{M}(s, s_0)\cdot\vec{y}(s_0) \tag{3.2}$$

which defines the transfer matrix $\underline{M}(s, s_0)$ of the motion.

In order to obtain more information about the particle's motion and to set up the Fokker-Planck equation we now look for the eigenvalue spectrum of the revolution matrix:

$$\begin{aligned}\underline{M}(s_0+L, s_0)\,\vec{v}_\mu(s_0) &= \lambda_\mu\cdot\vec{v}_\mu(s_0)\,; \\ (\mu &= 1,2,3,4,5,6)\end{aligned} \tag{3.3}$$

and require that the stability condition

$$|\lambda_\mu| \leq 1 \tag{3.4}$$

be satisfied.

Since the equations of motion can be written in canonical form (see Eq. 2.22), i.e. $x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma$ are canonical variables, the transfer matrix is symplectic:[13]

$$\underline{M}^T(s, s_0)\cdot\underline{S}\cdot\underline{M}(s, s_0) = \underline{S} \tag{3.5}$$

(with $\underline{S}$ given by Eq. 2.10) and the eigenvectors $\vec{v}_\mu(s_0)$ occur in pairs

$$(\vec{v}_k,\ \vec{v}_{-k})\ ;\ k = I,\ II,\ III \tag{3.6a}$$

with the reciprocal eigenvalues $\lambda_k,\ \lambda_{-k}$ satisfying :[11]

$$\lambda_k\cdot\lambda_{-k} = 1 \tag{3.6b}$$

($k = I,\ II,\ III$ refers to the three 2-dimensional subspaces (x, p_x), (z, p_z), (σ, p_σ) of the full six dimensional phase space).

Thus the stability condition (3.4) can be written as:[9]

$$|\lambda_k| = |\lambda_{-k}| = 1\,;\quad \lambda_{-k} = \lambda_k^*$$

$$\Longrightarrow\quad \left\{\begin{array}{l}\lambda_k = e^{-i\cdot 2\pi Q_k}\,;\quad Q_k\ \text{real}\,; \\ \vec{v}_{-k}(s_0) = \vec{v}_k^{\,*}(s_0)\end{array}\right. \tag{3.7}$$

so that all eigenvalues must lie on a unit circle. With the normalization condition for the $\vec{v}_k(s_0)$:

$$\vec{v}_k^{+}(s_0) \cdot \underline{S} \cdot \vec{v}_{-k}(s_0) \quad = \quad i$$

we then obtain from (3.5) the orthogonality relations:

$$\begin{cases} \vec{v}_k^{+}(s_0) \cdot \underline{S} \cdot \vec{v}_k(s_0) = -\vec{v}_{-k}^{+}(s_0) \cdot \underline{S} \cdot \vec{v}_{-k}(s_0) = i \; ; \\ \vec{v}_\mu^{+}(s_0) \cdot \underline{S} \cdot \vec{v}_\nu(s_0) = 0 \;\; \text{otherwise} \; ; \end{cases} \tag{3.8}$$

$$(k = I, II, III) \; .$$

The vectors $\vec{v}_{\pm k}(s_0)$ are the eigenvectors of the revolution matrix $\underline{M}(s_0 + L, s_0)$ with starting point s_0. The eigenvectors of the revolution matrix $\underline{M}(s + L, s)$ with starting point s can finally be obtained by operating with the transfer matrix $\underline{M}(s, s_0)$:[11]

$$\vec{v}_{\pm k}(s) \quad = \quad \underline{M}(s, s_0)\, \vec{v}_{\pm k}(s_0) \; ; \tag{3.9a}$$

$$\underline{M}(s + L, s)\, \vec{v}_{\pm k}(s) \quad = \quad \lambda_{\pm k} \cdot \vec{v}_{\pm k}(s) \tag{3.9b}$$

whereby the eigenvalues remain unchanged:

$$\lambda_{\pm k}(s) = \lambda_{\pm k}(s_0) \equiv \lambda_{\pm k} \; . \tag{3.9c}$$

The vectors $\vec{v}_{\pm k}(s)$ defined by (3.9a) also fulfill the same orthogonality relations (3.8) as $\vec{v}_{\pm k}(s_0)$.

Putting

$$\vec{v}_\mu(s) = \vec{\hat{v}}_\mu(s) \cdot e^{-i \cdot 2\pi Q_\mu \cdot (s/L)} \tag{3.10}$$

the factor $\vec{\hat{v}}_\mu(s)$

$$\vec{\hat{v}}_\mu(s) = \vec{v}_\mu(s) \cdot e^{+i \cdot 2\pi Q_\mu \cdot (s/L)}$$

is seen to be a periodic function with period L:[11]

$$\vec{\hat{v}}_\mu(s + L) \quad = \quad \vec{\hat{v}}_\mu(s) \; . \tag{3.11}$$

(Note that the orthogonality relations (3.8) are also valid for the 'Floquet-vectors' $\vec{\hat{v}}_\mu$.)

The general solution of the equation of motion (3.1) is a linear combination of the special solutions (3.10) and can be therefore written in the form

$$\vec{y}(s) \quad = \quad \sum_{k=I,II,III} \Big\{ A_k \cdot \vec{\hat{v}}_k(s) e^{-i \cdot 2\pi Q_k \cdot (s/L)} + A_{-k} \cdot \vec{\hat{v}}_{-k}(s) e^{+i \cdot 2\pi Q_{-k} \cdot (s/L)} \Big\} . \tag{3.12}$$

Using these results we can now introduce a new set of canonical variables[5,14] by writing for the coefficients A_k, A_{-k} $(k = I, II, III)$ in Eq. (3.12):

$$A_k = \sqrt{J_k} \cdot e^{-i[\Phi_k - 2\pi Q_k \cdot s/L]} ; \tag{3.13a}$$

$$A_{-k} = \sqrt{J_k} \cdot e^{+i[\Phi_k - 2\pi Q_k \cdot s/L]} . \tag{3.13b}$$

Then Eq. (3.12) takes the form:

$$\vec{y}(s) = \sum_{k=I,II,III} \sqrt{J_k} \cdot \left\{ \vec{v}_k(s) \cdot e^{-i\Phi_k} + \vec{v}_{-k}(s) \cdot e^{+i\Phi_k} \right\} . \tag{3.14}$$

Taking the partial derivatives

$$\frac{\partial \vec{y}}{\partial \Phi_k} \text{ and } \frac{\partial \vec{y}^{\,T}}{\partial J_k}$$

with the orthogonality relations of the Floquet vectors one obtains the equations:

$$\frac{\partial \vec{y}^{\,T}}{\partial J_k} \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial \Phi_l} = -\frac{\partial \vec{y}^{\,T}}{\partial \Phi_l} \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial J_k} = \delta_{kl} ; \tag{3.15a}$$

$$\frac{\partial \vec{y}^{\,T}}{\partial J_k} \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial J_l} = \frac{\partial \vec{y}^{\,T}}{\partial \Phi_k} \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial \Phi_l} = 0 \tag{3.15b}$$

which can be combined into the matrix form[14]

$$\underline{\mathcal{J}}^T \cdot \underline{S} \cdot \underline{\mathcal{J}} = \underline{S} \tag{3.16}$$

with $\underline{\mathcal{J}}$ the Jacobian matrix

$$\underline{\mathcal{J}} = \left(\frac{\partial \vec{y}}{\partial \Phi_I}, \frac{\partial \vec{y}}{\partial J_I}, \frac{\partial \vec{y}}{\partial \Phi_{II}}, \frac{\partial \vec{y}}{\partial J_{II}}, \frac{\partial \vec{y}}{\partial \Phi_{III}}, \frac{\partial \vec{y}}{\partial J_{III}} \right) \tag{3.17}$$

being a 6×6-matrix just written as a row of column vectors $(\partial \vec{y}/\partial \Phi_I)$ etc.

Equation (3.16) proves that Eq. (3.14) represents a canonical transformation

$$x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma \longrightarrow \Phi_I,\ J_I,\ \Phi_{II},\ J_{II},\ \Phi_{III},\ J_{III} \tag{3.18}$$

and that Φ_k, J_k $(k = I, II, III)$ are indeed canonical variables which can now be interpreted as action-angle variables since

$$\frac{dJ_k}{ds} = 0 ; \tag{3.19a}$$

$$\frac{d\Phi_k}{ds} = \frac{2\pi}{L} Q_k . \tag{3.19b}$$

These variables may also be used to describe the orbital motion.

4 THE PERTURBED PROBLEM

The general solution of the unperturbed equation of motion (3.1) can be written in the form

$$\vec{y}(s) = \sum_{k=I,II,III} \{A_k \cdot \vec{v}_k(s) + A_{-k} \cdot \vec{v}_{-k}(s)\}$$

with A_k, A_{-k} being constants of integration ($k = I,\ II,\ III$).

In order to solve the perturbed problem (2.24) we now make the following 'ansatz' (variation of constants):

$$\vec{y}(s) = \sum_{k=I,II,III} \{A_k(s) \cdot \vec{v}_k(s) + A_{-k}(s) \cdot \vec{v}_{-k}(s)\} \ . \tag{4.1}$$

Inserting (4.1) into (2.24) one obtains:

$$\sum_{k=I,II,III} \left\{A'_k(s) \cdot \vec{v}_k + A'_{-k}(s) \cdot \vec{v}_{-k}\right\} = \delta\vec{c}_{PH} \ . \tag{4.2}$$

Using the orthogonality conditions (3.8) and the relation (2.25) one gets from (4.2) for $k = I, II, III$:

$$\begin{aligned} A'_k(s) &= -i \cdot \vec{v}_k^{\,+}(s)\underline{S} \cdot \delta\vec{c}_{PH} \\ &= +i \cdot \lambda_{PH} \cdot \xi(s) \cdot \frac{eV(s)}{E_0} \cdot \cos\varphi_0 \cdot [v_{k5}(s)]^* \ ; \end{aligned} \tag{4.3a}$$

$$A'_{-k}(s) = [A'_k(s)]^* \ . \tag{4.3b}$$

5 STOCHASTIC EQUATIONS FOR THE ACTION-ANGLE VARIABLES $J_k(s)$ AND $\Phi_k(s)$

Representing $A_k(s)$ in the form (3.13a), we obtain for the derivative $A'_k(s)$ ($k = I,\ II,\ III$):

$$\begin{aligned} A'_k(s) &= \frac{1}{2} \cdot \frac{J'_k}{\sqrt{J_k}} \cdot e^{-i \cdot [\Phi_k - 2\pi Q_k \cdot s/L]} - i \cdot \left[\Phi'_k - \frac{2\pi}{L} Q_k\right] \cdot A_k \\ &= A_k \cdot \left\{\frac{1}{2} \cdot \frac{J'_k}{J_k} - i \cdot \left[\Phi'_k - \frac{2\pi}{L} Q_k\right]\right\} \end{aligned}$$

and it follows for the derivatives $J'_k(s)$ and $\Phi'_k(s)$ of the action - angle variables J_k and Φ_k:[5]

$$J_k'(s) = \frac{d}{ds}\,[A_k(s)\cdot A_{-k}(s)] = 2\cdot\Re e\,\{A_k'(s)\cdot A_{-k}(s)\}\ ; \tag{5.1a}$$

$$\begin{aligned}\Phi_k'(s) - \frac{2\pi}{L}\cdot Q_k &= -\frac{A_k'(s)\cdot A_{-k}(s)}{i\cdot J_k(s)} + \frac{1}{2}\cdot\frac{A_k'(s)\cdot A_{-k}(s) + A_k(s)\cdot A_{-k}'(s)}{i\cdot J_k(s)}\\ &= -\frac{1}{J_k(s)}\cdot\Im m\,\{A_k'(s)\cdot A_{-k}(s)\}\ .\end{aligned} \tag{5.1b}$$

Here the terms $(A_k' \cdot A_{-k})$ appearing in (5.1a, b) are given by:

$$A_k'(s)\cdot A_{-k}(s) = Z_k(s) \tag{5.2}$$

with

$$\begin{aligned}Z_k(s) &= i\cdot\lambda_{PH}\cdot\xi(s)\cdot\frac{eV(s)}{E_0}\cdot\cos\varphi_0\cdot[v_{k5}(s)]^*\cdot A_{-k}(s)\\ &= i\cdot\lambda_{PH}\cdot\xi(s)\cdot\frac{eV(s)}{E_0}\cdot\cos\varphi_0\cdot\sqrt{J_k}\cdot e^{i\,\cdot\,\Phi_k(s)}\cdot[\hat{v}_{k5}(s)]^*\end{aligned} \tag{5.3}$$

where we have used Eqs. (3.10) and (3.13a, b).

If we write then $J_k'(s)$ and $\Phi_k'(s)$ in the form:

$$J_k'(s) = K_J^{(k)}(\Phi_l, J_l) + Q_J^{(k)}(\Phi_l, J_l)\cdot\xi(s)\ ; \tag{5.4a}$$

$$\Phi_k'(s) = K_\Phi^{(k)}(\Phi_l, J_l) + Q_\Phi^{(k)}(\Phi_l, J_l)\cdot\xi(s) \tag{5.4b}$$

we obtain:

$$\begin{aligned}K_J^{(k)} &= 0\ ;\\ Q_J^{(k)} &= i\cdot\frac{eV(s)}{E_0}\cdot\cos\varphi_0\cdot\sqrt{J_k}\\ &\qquad\times\left\{[\hat{v}_{k5}(s)]^*\cdot e^{i\,\cdot\,\Phi_k} - \hat{v}_{k5}(s)\cdot e^{-i\,\cdot\,\Phi_k}\right\}\cdot\lambda_{PH}\ ;\end{aligned} \tag{5.5a}$$

$$\begin{aligned}K_\Phi^{(k)} &= \frac{2\pi}{L}\cdot Q_k\ ;\\ Q_\Phi^{(k)} &= -\frac{eV_(s)}{E_0}\cdot\cos\varphi_0\cdot\frac{1}{2\sqrt{J_k}}\\ &\qquad\times\left\{[\hat{v}_{k5}(s)]^*\cdot e^{i\,\cdot\,\Phi_k} + \hat{v}_{k5}(s)\cdot e^{-i\,\cdot\,\Phi_k}\right\}\cdot\lambda_{PH}\ .\end{aligned} \tag{5.5b}$$

The stochastic equations (5.4a, b) for the variables J_k and Φ_k ($k = I, II, III,$) together with (5.5a, b) are now the basis for a Fokker-Planck treatment of stochastic particle motion in storage rings.

6 THE FOKKER-PLANCK EQUATION FOR CAVITY NOISE

The Fokker-Planck equation for the phase space density function W (in the Stratonowich-version) now reads as:[15]

$$\begin{aligned}\frac{\partial W}{\partial s} &= \sum_{k=I,II,III}\left\{-\frac{\partial}{\partial J_k}[D_J^{(k)}\cdot W]-\frac{\partial}{\partial \Phi_k}[D_\Phi^{(k)}\cdot W]\right\}\\ &+\sum_{k,l=I,II,III}\left\{\frac{1}{2}\frac{\partial^2}{\partial J_k\partial J_l}[Q_J^{(k)}\cdot Q_J^{(l)}\cdot W]+\frac{\partial^2}{\partial J_k\partial \Phi_l}[Q_J^{(k)}\cdot Q_\Phi^{(l)}\cdot W]\right.\\ &\left.\qquad+\frac{1}{2}\frac{\partial^2}{\partial \Phi_k\partial \Phi_l}[Q_\Phi^{(k)}\cdot Q_\Phi^{(l)}\cdot W]\right\}\end{aligned}\tag{6.1}$$

with

$$D_J^{(k)} = K_J^{(k)}+\tilde{K}_J^{(k)}\ ;\tag{6.2a}$$

$$D_\Phi^{(k)} = K_\Phi^{(k)}+\tilde{K}_\Phi^{(k)}\tag{6.2b}$$

where the quantities $\tilde{K}_J^{(k)}$ and $\tilde{K}_\Phi^{(k)}$ contain the drift terms:

$$\tilde{K}_J^{(k)} = \frac{1}{2}\sum_{l=I,II,III}\left\{\frac{\partial Q_J^{(k)}}{\partial J_l}\cdot Q_J^{(l)}+\frac{\partial Q_J^{(k)}}{\partial \Phi_l}\cdot Q_\Phi^{(l)}\right\}\ ;\tag{6.3a}$$

$$\tilde{K}_\Phi^{(k)} = \frac{1}{2}\sum_{l=I,II,III}\left\{\frac{\partial Q_\Phi^{(k)}}{\partial J_l}\cdot Q_J^{(l)}+\frac{\partial Q_\Phi^{(k)}}{\partial \Phi_l}\cdot Q_\Phi^{(l)}\right\}\ .\tag{6.3b}$$

Introducing the time t by the relation $ds = v_0\cdot dt$ and using a long time scale[15] which is comparable with the oscillation time of synchro-betatron motion we make an average around several circumferences, which we indicate by a bracket $\langle\quad\rangle$, and then write the Fokker-Planck equation in the form:

$$\begin{aligned}\frac{1}{v_0}\frac{\partial W}{\partial t} &= \sum_{k=I,II,III}\left\{-\frac{\partial}{\partial J_k}[\langle D_J^{(k)}\rangle\cdot W]-\frac{\partial}{\partial \Phi_k}[\langle D_\Phi^{(k)}\rangle\cdot W]\right\}\\ &+\sum_{k,l=I,II,III}\left\{\frac{1}{2}\cdot\frac{\partial^2}{\partial J_k\partial J_l}[\langle Q_J^{(k)}\cdot Q_J^{(l)}\rangle\cdot W]+\frac{\partial^2}{\partial J_k\partial \Phi_l}[\langle Q_J^{(k)}\cdot Q_\Phi^{(l)}\rangle\cdot W]\right.\\ &\left.\qquad+\frac{1}{2}\cdot\frac{\partial^2}{\partial \Phi_k\partial \Phi_l}[\langle Q_\Phi^{(k)}\cdot Q_\Phi^{(l)}\rangle\cdot W]\right\}\end{aligned}\tag{6.4}$$

whereby oscillating terms of the integrand due to the (linear) s-dependence of the angle variables Φ_k:

$$\Phi_k(s) = \Phi_k(s_0)+(s-s_0)\cdot\frac{2\pi}{L}Q_k$$

(see Eq. 3.19b) may be neglected since they are approximately averaged away by integration.

Equation (5.5a, b) leads to:

$$\langle K_J^{(k)} \rangle = 0 ; \tag{6.5a}$$

$$\langle K_\Phi^{(k)} \rangle = \frac{2\pi}{L} \cdot Q_k \tag{6.5b}$$

and

$$\langle (Q_J^{(k)})^2 \rangle = 2J_k \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \cdot \left[\frac{eV(\tilde{s})}{E_0} \cdot \lambda_{PH} \right]^2 ; \tag{6.6a}$$

$$\langle (Q_\Phi^{(k)})^2 \rangle = \frac{1}{2J_k} \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \cdot \left[\frac{eV(\tilde{s})}{E_0} \cdot \lambda_{PH} \right]^2 ; \tag{6.6b}$$

$$\langle Q_J^{(k)} \cdot Q_J^{(l)} \rangle = 0 \text{ for } k \neq l ; \tag{6.7a}$$

$$\langle Q_\Phi^{(k)} \cdot Q_\Phi^{(l)} \rangle = 0 \text{ for } k \neq l ; \tag{6.7b}$$

$$\langle Q_J^{(k)} \cdot Q_\Phi^{(l)} \rangle = 0 . \tag{6.7c}$$

Furthermore one has, taking into account (6.3a, b):

$$\begin{aligned} \tilde{K}_J^{(k)} &= \left[\frac{eV(s)}{E_0} \cdot \lambda_{PH} \right]^2 \cdot |\hat{v}_{k5}|^2 \\ &= \left[\frac{eV(s)}{E_0} \cdot \lambda_{PH} \right]^2 \cdot |v_{k5}|^2 ; \end{aligned} \tag{6.8a}$$

$$\begin{aligned} \tilde{K}_\Phi^{(k)} &= \frac{i}{4J_k} \cdot \left[\frac{eV(s)}{E_0} \cdot \lambda_{PH} \right]^2 \\ &\times \left\{ \left[(\hat{v}_{k5})^* \right]^2 e^{i \cdot 2\Phi_k} - [\hat{v}_{k5}]^2 e^{-i \cdot 2\Phi_k} \right\} \end{aligned} \tag{6.8b}$$

and thus

$$\langle \tilde{K}_J^{(k)} \rangle = \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \cdot \left[\frac{eV(\tilde{s})}{E_0} \cdot \lambda_{PH} \right]^2 ; \tag{6.9a}$$

$$\langle \tilde{K}_\Phi^{(k)} \rangle = 0 . \tag{6.9b}$$

From (6.2a, b), (6.5a, b) and (6.8a, b) we then get:

$$\langle D_J^{(k)} \rangle = \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \cdot \left[\frac{eV(\tilde{s})}{E_0} \cdot \lambda_{PH} \right]^2 ; \tag{6.10a}$$

$$\langle D_\Phi^{(k)} \rangle = \frac{2\pi}{L} \cdot Q_k . \tag{6.10b}$$

Introducing the constants

$$M_k = v_0 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \left[\frac{eV(\tilde{s})}{E_0} \cdot \lambda_{PH} \right]^2 ; \tag{6.11a}$$

$$b_k = 2\pi \cdot \frac{v_0}{L} \cdot Q_k \tag{6.11b}$$

we may finally write:

$$\begin{aligned}
\frac{\partial W}{\partial t} &= \sum_{k=I,II,III} \left\{ -\frac{\partial}{\partial J_k} [M_k \cdot W] - \frac{\partial}{\partial \Phi_k} [b_k \cdot W] \right. \\
&\qquad \left. + \frac{1}{2} \cdot \frac{\partial^2}{\partial J_k^2} [2J_k \cdot M_k \cdot W] + \frac{1}{2} \cdot \frac{\partial^2}{\partial \Phi_k^2} \left[\frac{1}{2J_k} \cdot M_k \cdot W \right] \right\} \\
&= \sum_{k=I,II,III} \left\{ -\frac{\partial}{\partial J_k} \left[-M_k \cdot J_k \cdot \frac{\partial W}{\partial J_k} \right] \right. \\
&\qquad \left. - \frac{\partial}{\partial \Phi_k} \left[b_k \cdot W - \frac{M_k}{4J_k} \cdot \frac{\partial W}{\partial \Phi_k} \right] \right\} .
\end{aligned} \tag{6.12}$$

This equation determines the charge-distribution of the particles in a bunch and can serve for further studies.

Remark

The starting equation (2.24) of our Fokker-Planck treatment of cavity noise can be obtained from the Hamiltonian (2.22) and is thus fully symplectic. As a result, the Fokker-Planck equation (6.12) contains no damping terms. Non-symplectic terms in the equations of motion (2.24) would lead to an imaginary part of the tune shifts Q_k ($k = I, II, III$) inducing a damping of the oscillation modes. (For more details see Refs. 4, 5.)

7 SOLUTION OF THE FOKKER-PLANCK EQUATION

In order to solve the Fokker-Planck equation (6.12) we make the ansatz:

$$W = w_I(J_I, \Phi_I) \cdot w_{II}(J_{II}, \Phi_{II}) \cdot w_{III}(J_{III}, \Phi_{III}) \tag{7.1}$$

Then Eq. (6.12) simplifies to:

$$\frac{\partial w_k}{\partial t} = -\frac{\partial}{\partial J_k} \left[-M_k \cdot J_k \cdot \frac{\partial w_k}{\partial J_k} \right] - \frac{\partial}{\partial \Phi_k} \left[b_k \cdot w_k - \frac{M_k}{4J_k} \cdot \frac{\partial w_k}{\partial \Phi_k} \right] . \tag{7.2}$$

Since the phase diffusion term

$$\frac{M_k}{4J_k} \cdot \frac{\partial^2 w_k}{\partial^2 \Phi_k}$$

superposed on the phase advance term

$$-b_k \cdot \frac{\partial w_k}{\partial \Phi_k}$$

will lead to a uniform distribution of the phase Φ_k in $[0, 2\pi]$,[15] we may write:

$$w_k(J_k, \Phi_k) = \frac{1}{2\pi} \cdot \hat{w}_k(J_k) \tag{7.3}$$

and obtain (dropping the subscript k):

$$\frac{\partial \hat{w}}{\partial t} = \frac{\partial}{\partial J}\left[M \cdot J \cdot \frac{\partial \hat{w}}{\partial J}\right] . \tag{7.4}$$

Note that the constant b_k no longer appears in (7.4) and we are left only with the coefficient M_k.

The solution of Eq. (7.4) is given by[16]:

$$\hat{w}(J, t) = \int_0^\infty dJ_0 \cdot K(J, J_0; t) \cdot \rho(J_0) \tag{7.5}$$

($\rho(J_0)$ =arbitrary function of J_0) with

$$K(J, J_0; t) = \frac{1}{M \cdot t} \cdot \exp\left[-\frac{(J + J_0)}{M \cdot t}\right] \cdot I_0\left(\frac{2\sqrt{J \cdot J_0}}{M \cdot t}\right) \tag{7.6}$$

where $I_0(\,)$ is the zeroth-order modified Bessel function.

$K(J, J_0; t)$ is the fundamental solution of Eq. (7.4) and by definition satisfies the initial condition

$$\lim_{t \to 0+} K(J, J_0; t) = \delta(J - J_0) \tag{7.7}$$

($\delta(J - J_0)$ = Dirac delta function) . It then follows that $\rho(J_0)$ is just the initial distribution existing at $t = 0$:

$$\hat{w}(J, 0) = \rho(J) . \tag{7.8}$$

The distribution $\hat{w}(J, t)$ can be characterized by the moments

$$m_n(t) = \int_0^\infty dJ \cdot J^n \cdot \hat{w}(J, t) . \tag{7.9}$$

Using the relation:

$$\int_0^\infty dx \cdot e^{-\alpha \cdot x} \cdot I_0(2\sqrt{\beta x}) = \frac{1}{\alpha} \cdot e^{\beta/\alpha} \tag{7.10}$$

and the integrals which may be obtained by differentiating both sides of (7.10) with respect to α, one is able to calculate the moments corresponding to the fundamental solution (7.6). For $m_1(t)$ and $m_2(t)$ one gets for example:

$$m_1(t) = J_0 + M \cdot t \tag{7.11a}$$

and

$$m_2(t) = J_0^2 + 4MJ_0 \cdot t + 2M^2 \cdot t^2 \tag{7.11b}$$

(see also Ref. 2).

Thus the time development of the distribution function $\hat{w}(J,t)$ and of the moments $m_n(t)$ is characterized by the coefficient $M \equiv M_k$ $(k = I, II, III)$ which may be interpreted as the reciprocal of the rise time constant τ_k:

$$\tau_k = \frac{1}{M_k} . \tag{7.12}$$

Remark

From Eq. (4.3a) we obtain by integration, using (2.20a, b):

$$A_k(s) = A_k(s_0) + i \cdot \lambda_{PH} \cdot \int_{s_0}^{s} d\tilde{s} \cdot \xi(\tilde{s}) \cdot \frac{eV(\tilde{s})}{E_0} \cdot \cos\varphi_0 \cdot v_{k5}^*(\tilde{s}) \tag{7.13}$$

and

$$\begin{aligned} < |A_k(s)|^2 > - < |A_k(s_0)|^2 > &= \lambda_{PH}^2 \cdot \int_{s_0}^{s} ds' \cdot \int_{s_0}^{s} ds'' \cdot \frac{e^2 \cdot V(s')V(s'')}{E_0^2} \\ &\quad \times v_{k5}(s')\, v_{k5}^*(s'') \cdot < \xi(s') \cdot \xi(s'') > \\ &= \int_{s_0}^{s} ds' \cdot |v_{k5}(s')|^2 \cdot \left[\frac{eV(s')}{E_0} \cdot \lambda_{PH}\right]^2 . \end{aligned} \tag{7.14}$$

Hence we get for the emittance growth rate of the kth mode $(k = I,\ II,\ III)$:

$$\begin{aligned} \frac{d}{dt} < \epsilon_k > &= 2 \cdot \frac{d}{dt} < J_k > \\ &= 2 \cdot \frac{d}{dt} < |A_k(s)|^2 > \\ &\approx 2 \cdot \frac{< |A_k(s+L)|^2 > - < |A_k(s)|^2 >}{(L/v_0)} \\ &= 2 \cdot \frac{v_0}{L} \int_{s}^{s+L} ds' \cdot |v_{k5}(s')|^2 \cdot \left[\frac{eV(s')}{E_0} \cdot \lambda_{PH}\right]^2 \\ &\equiv 2 \cdot M_k \end{aligned} \tag{7.15}$$

(see Eq. 4.53 in Ref. 4). Thus the role of the constant $\tau_k = (1/M_k)$ as the growth time constant becomes obvious.

Note that this relation can also directly be obtained from the Fokker-Planck equation (7.4), since one has by partial integration:

$$\begin{aligned}\frac{d}{dt} < J > &= \frac{d}{dt}\int_0^\infty dJ \cdot J \cdot \hat{w}(J,t) = \int_0^\infty dJ \cdot J \cdot \frac{\partial \hat{w}}{\partial t} \\ &= \int_0^\infty dJ \cdot J \cdot \frac{\partial}{\partial J}\left[M \cdot J \cdot \frac{\partial \hat{w}}{\partial J}\right] = -M \cdot \int_0^\infty dJ \cdot J \cdot \frac{\partial \hat{w}}{\partial J} \\ &= +M \cdot \int_0^\infty dJ \cdot \hat{w} = M \ .\end{aligned}$$

8 APPLICATION TO HERA

It is well known that cavity phase noise can limit the proton beam life time in HERA at top energy where the beam must be stored for many hours, whereas at 40 GeV noise should only produce a small emittance growth since the beam stays at this energy only for about 20 minutes.

Using the previously developed theory we shall set phase noise limits for the two different RF systems of the HERA proton storage ring. In order to achieve this we shall assume that a longitudinal emittance growth less then 10% can be tolerated for the 52 MHz system, whereas for the 208 MHz system at top energy (820 GeV) we shall assume growth of less than 10% in 24 hours.

To calculate the noise levels one needs the equations (7.11a) (or equivalently 7.15) which give the mean emittance increase with time and (6.11a) that defines the growth rates. After some algebra we obtain (assuming a uniform distribution of $V(s)$ around the circumference or using a Fourier expansion of $V(s)$ and taking into account only the 0^{th} component of the Fourier series):

$$M_k = \lambda_{PH}^2 \cdot \nu_0 \cdot \left[\frac{1}{L}\frac{e\hat{V}}{E_0}\right]^2 \cdot \sum_{n=1}^{N} |v_{k5}(s_n)|^2 \ .$$

Introducing the quantity

$$< |v_{k5}|^2 > = \frac{1}{N}\sum_{n=1}^{N} |v_{k5}(s_n)|^2$$

the above formula may also be written as:

$$M_k = \lambda_{PH}^2 \cdot \nu_0 \cdot \left[\frac{N}{L}\frac{e\hat{V}}{E_0}\right]^2 \cdot < |v_{k5}|^2 > \tag{8.1}$$

with

$$N \cdot \hat{V} = \int_{s_0}^{s_0+L} d\tilde{s} \cdot V(\tilde{s})$$

(total circumferential voltage) .

From Eq. (7.15) we finally have:

$$\frac{d}{dt} < \epsilon_k > = 2 \cdot M_k$$

$$= 2 \cdot \lambda_{PH}^2 \cdot \nu_0 \cdot \left[\frac{N}{L} \frac{e\hat{V}}{E_0} \right]^2 \cdot < |v_{k5}|^2 > \quad (8.2)$$

which is the relation to be used in our estimations.

We have also assumed white noise, as it is evident from Eq. (2.20a,b) and in this case a very simple relation between the constant λ_{PH} and the spectral density $G_f(\omega)$ exists:

$$G_f = \frac{\lambda_{PH}^2}{2\pi\nu_0} . \quad (8.3)$$

8.1 40 GeV case – 52 MHz system

At this energy the total circumferential voltage $U \equiv N\hat{V} = 26.4\,\text{kV}$, the transverse emittances $\epsilon_{x,z} = 16.9\,\pi\text{rad}\,\mu\text{m}$, the bunch area is $\epsilon_\sigma = 0.175$ eVs or equivalently 0.061 rad (using the relation $\epsilon\,(\text{eVs}) = m_0c^2/\omega_{rf}\ \epsilon\,(\text{rad})$ with $m_0c^2 = 0.938\,\text{GeV}$) or 56 mm (using m=(R/h) rad), the harmonic number $h = 1100$ and for the eigenvectors v_{i5} with $i = I,\ II,\ III$ (or respectively horizontal, vertical and longitudinal subspace) one calculates:

$$< |v_{I5}|^2 > = 1.3 \cdot 10^{-2}\ \text{m} \quad \text{for the horizontal mode ;}$$
$$< |v_{II5}|^2 > = 2.7 \cdot 10^{-3}\ \text{m} \quad \text{for the vertical mode ;}$$
$$< |v_{III5}|^2 > = 1.4 \cdot 10^{+3}\ \text{m} \quad \text{for the longitudinal mode .}$$

From those numbers with $L = 6336\,\text{m}$ and using Eq. (8.1) one obtains the following growth rates:

$$M_I = 4.3 \cdot 10^{-14}\, |\lambda_{PH}|^2\ \text{s}^{-1} ;$$
$$M_{II} = 8.9 \cdot 10^{-15}\, |\lambda_{PH}|^2\ \text{s}^{-1} ;$$
$$M_{III} = 4.6 \cdot 10^{-9}\, |\lambda_{PH}|^2\ \text{s}^{-1} .$$

Using Eq. (8.2) and the above discussed assumptions on the longitudinal emittance growth, with $(\epsilon_\sigma)_{initial} = 0.056$ m, one may easily verify that

$$\lambda_{PH} < 23\ \text{rad m}^{1/2}$$

and translating this number to familiar units one has that

$$\lambda_{PH} < 1.3\ \text{mrad/Hz}^{1/2} .$$

Finally using Eq. (8.3) one obtains that

$$G_f < -65 \text{ dB/Hz} .$$

Using the specified value of λ_{PH} ($= 23$ rad $\mathrm{m}^{1/2}$) and Eq. (8.2) the horizontal emittance will increase within 20 minutes about 0.05 πrad μm, or 0.3% whereas the vertical emittance will increase approximately 0.01 μm πrad or 0.07%.

8.2 820 GeV case – 208 MHz system

At this energy $U \equiv N\hat{V} = 2.4\,\mathrm{MV}$, $\epsilon_{x,z} = 20\,\pi\mathrm{rad}\,\mu\mathrm{m}$, the bunch area is $\epsilon_\sigma =$ 0.3 eVs (or 0.42 rad or 95 mm), h= 4400 and the eigenvectors v_{i5} are:

$$\begin{aligned} <|v_{I5}|^2> &= 6.7 \cdot 10^{-4} \text{ m} ; \\ <|v_{II5}|^2> &= 7.0 \cdot 10^{-5} \text{ m} ; \\ <|v_{III5}|^2> &= 0.4 \cdot 10^{+3} \text{ m} . \end{aligned}$$

Using those numbers and Eq. (8.1) one obtains for the growth rates:

$$\begin{aligned} M_I &= 2.7 \cdot 10^{-14}\, |\lambda_{PH}|^2 \text{ s}^{-1} ; \\ M_{II} &= 4.5 \cdot 10^{-15}\, |\lambda_{PH}|^2 \text{ s}^{-1} ; \\ M_{III} &= 1.6 \cdot 10^{-8}\, |\lambda_{PH}|^2 \text{ s}^{-1} . \end{aligned}$$

Using as before the Eq. (8.2) and the above discussed assumptions on the longitudinal emittance growth, one may easily verify that

$$\lambda_{PH} < 1.5 \text{ rad m}^{1/2}$$

and translating this number to familiar units one has that

$$\lambda_{PH} < 0.084 \text{ mrad/Hz}^{1/2} .$$

Finally from Eq. (8.3) one obtains that

$$G_f < -90 \text{ dB/Hz} .$$

Using the specified value of λ_{PH} ($= 1.5$ rad $\mathrm{m}^{1/2}$) and Eq. (8.2) the horizontal emittance will increase within 24 hours about 0.02 μm πrad or 0.1% whereas the vertical emittance will increase approximately 0.002 μm πrad or 0.01%.

9 SUMMARY

We have investigated the influence of cavity phase fluctuations on the motion of charged particles in storage rings using the Fokker-Planck equation.

The motion was described in terms of the fully coupled six-dimensional formalism with the canonical variables x, p_x, z, p_z, σ, $p_\sigma = (1/\beta_0^2) \cdot \eta$.

With this set of variables we were then in a position to treat the betatron and synchrotron oscillations simultaneously in a canonical manner, i.e. to provide an analytic technique which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities.

In this paper we have (for simplicity) neglected a shift of the six-dimensional closed orbit induced by magnetic dipole fields. A technique to handle this effect may for instance be found in Refs. 10, 12. The formalism developed remains valid. Furthermore, we have restricted our investigations to proton rings. But it is easy to extend these considerations to electron beams. In this case it is necessary to introduce a cavity phase $\neq (0, \pi)$ and additional nonsymplectic terms due to stochastic radiation effects have to be taken into account in the equations of motion which lead to a damping of the oscillation modes.

In order to derive the Fokker-Planck equation (canonical) action-angle variables were introduced taking into account a coupling of the betatron oscillations by skew quadrupoles and solenoids and the linear coupling between the betatron and the synchrotron oscillations induced by a non-vanishing dispersion in the cavities.

The Fokker-Planck equation was solved for phase noise in the cavities for which the emittance grows linearly with time. Keeping the noise below the indicated limits for both systems should not be a problem. However it must be remembered that only phase noise was considered here.

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REFERENCES

1. D. Boussard, G. Dome, C. Graziani, The influence of the RF Noise on the lifetime of bunched to proton beams, *Proc. XIth International Conference on High Energy Accelerators,* Geneva, p. 620, (1980).
2. S. Krinsky and J.M. Wang, Bunch diffusion due to RF Noise, *Part. Accel.* **12** (1982) 107.
3. G. Dome, Theory of RF Acceleration and RF Noise, CERN Accelerator School, General Accelerator Physics, (CERN, Geneva, Switzerland, 11-21 October 1983), CERN 84-15 (1984).
4. G. Ripken, E. Karantzoulis, A Fokker-Planck-Treatment of Coupled Synchro-Betatron Motion in Proton Storage Rings under the Influence of Cavity Noise, DESY 90-135.
5. D.P. Barber, K. Heinemann, H. Mais, G. Ripken, A Fokker-Planck-Treatment of Stochastic Particle Motion within the Framework of a fully coupled 6-dimensional Formalism for Electron-Positron Storage Rings including Classical Spin Motion in linear Approximation, DESY 91-146.
6. G. Ripken, Non-linear canonical equations of coupled synchro- betatron motion and their solution within the framework of a non-linear 6-dimensional (symplectic) tracking program for ultra-relativistic protons, DESY 85-84.
7. D.P. Barber, G. Ripken, F. Schmidt, A non-linear canonical formalism for the coupled synchro-betatron motion of protons with arbitrary energy, DESY 87-36.
8. D.P. Barber, K. Heinemann, G. Ripken, A Canonical Eight-Dimensional Formalism for Linear and Non-Linear Classical Spin-Orbit Motion in Storage Rings, DESY 91-47.

9. E. Courant and H. Snyder, Theory of the Alternating-Gradient Synchrotron, *Annals of Physics* **3** (1958) 1.
10. H. Mais and G. Ripken, Theory of Spin-Orbit Motion in Electron-Positron Storage Rings. Summary of Results, DESY 83-62.
11. H. Mais and G. Ripken, Theory of Coupled Synchro-Betatron Oscillations (I), DESY M-82-05.
12. H. Mais and G. Ripken, Spin-Orbit Motion in a Storage Ring in the Presence of Synchrotron Radiation using a Dispersion Formalism, DESY 86-29.
13. F. Willeke, G. Ripken, Methods of Beam Optics, DESY 88-114.
14. G. Ripken, F. Willeke, On the Impact of Linear Coupling on Nonlinear Dynamics, DESY 90-001.
15. J. M. Jowett, Introductory Statistical Mechanics For Electron Storage Rings, SLAC-PUB-4033 (1986).
16. H. Risken, *The Fokker-Planck Equation*, Second Edition, Springer Verlag 1989.
C.W. Gardiner, *Handbook of Stochastic Methods*, Second Edition, Springer Verlag 1985.
17. F. Schmidt, F. Willeke, F. Zimmermann, Comparison of methods to prove long term stability in proton storage rings, DESY-HERA 91-07 (1991). Published in *Part. Accel.* **35** (1991) 249.

APPENDIX A: The Hamiltonian

A.1 The Starting Hamiltonian

Using the variables

$$x,\ z,\ \sigma = s - v_0 \cdot t \ \text{ and } \ p_\sigma \equiv \eta \tag{A.1}$$

(v_0 =design speed= $c\beta_0$) with

$$\eta = \Delta E / E_0 \tag{A.2}$$

and choosing a gauge with $\phi = 0$ (e.g. Coulomb gauge), the Hamiltonian for orbital motion takes the form:[4,7]

$$\begin{aligned}\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) &= p_\sigma - \beta_0\sqrt{(1+p_\sigma)^2 - \left(\frac{m_0c^2}{E_0}\right)^2}\cdot[1+K_x\cdot x + K_z\cdot z] \\ &\times\left\{1 - \frac{(p_x - \beta_0\cdot\frac{e}{E_0}A_x)^2 + (p_z - \beta_0\cdot\frac{e}{E_0}A_z)^2}{\beta_0^2\cdot\left[(1+p_\sigma)^2 - (\frac{m_0c^2}{E_0})^2\right]}\right\}^{1/2} \\ &- [1+K_x\cdot x + K_z\cdot z]\cdot\beta_0\cdot\frac{e}{E_0}A_s\ .\end{aligned} \tag{A.3}$$

Defining:

$$(1+\hat{\eta}) = \frac{1}{\beta_0}\sqrt{(1+\eta)^2 - \left(\frac{m_0c^2}{E_0}\right)^2} = \frac{1}{\beta_0}\cdot\frac{p\cdot c}{E_0} = \frac{p}{p_0}\ ; \tag{A.4a}$$

$$\hat{\eta} = \frac{p}{p_0} - 1 = \frac{p-p_0}{p_0} = \frac{\Delta p}{p_0} \tag{A.4b}$$

($p = m_0\gamma v$) with

$$\hat{\eta} \equiv f(\eta) = \frac{1}{\beta_0}\sqrt{(1+\eta)^2 - \left(\frac{m_0c^2}{E_0}\right)^2} - 1 \equiv \frac{\Delta p}{p_0}$$

$$= f(0) + f'(0) \cdot \eta + f''(0) \cdot \eta^2 \pm \cdots$$
$$= \frac{1}{\beta_0^2} \cdot \eta - \frac{1}{\beta_0^4 \cdot \gamma_0^2} \cdot \frac{1}{2} \eta^2 \pm \cdots . \qquad \text{(A.5)}$$

and using then the relations

$$\frac{e}{E_0} A_x = \frac{p_0 \cdot c}{E_0} \cdot \frac{e}{p_0 \cdot c} A_x = \beta_0 \cdot \frac{e}{p_0 \cdot c} A_x ; \qquad \text{(A.6a)}$$
$$\frac{e}{E_0} A_z = \frac{p_0 \cdot c}{E_0} \cdot \frac{e}{p_0 \cdot c} A_z = \beta_0 \cdot \frac{e}{p_0 \cdot c} A_z ; \qquad \text{(A.6b)}$$
$$\frac{e}{E_0} A_s = \frac{p_0 \cdot c}{E_0} \cdot \frac{e}{p_0 \cdot c} A_s = \beta_0 \cdot \frac{e}{p_0 \cdot c} A_s \qquad \text{(A.6c)}$$

this Hamiltonian can be written as:

$$\begin{aligned} \mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma; s) &= p_\sigma - \beta_0^2 \cdot (1 + \hat{\eta}) \cdot [1 + K_x \cdot x + K_z \cdot z] \\ &\times \left\{ 1 - \frac{(p_x - \beta_0^2 \cdot \frac{e}{p_0 \cdot c} A_x)^2 + (p_z - \beta_0^2 \cdot \frac{e}{p_0 \cdot c} A_z)^2}{\beta_0^4 \cdot (1 + \hat{\eta})^2} \right\}^{1/2} \\ &- [1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0^2 \cdot \frac{e}{p_0 \cdot c} A_s . \end{aligned} \qquad \text{(A.7)}$$

A.2 Scale Transformation

Instead of p_σ, p_x, p_z we now introduce the variables

$$\tilde{p}_\sigma = \frac{1}{\beta_0^2} \cdot p_\sigma \equiv \frac{1}{\beta_0^2} \cdot \eta ; \qquad \text{(A.8a)}$$
$$\tilde{p}_x = \frac{1}{\beta_0^2} \cdot p_x ; \qquad \text{(A.8b)}$$
$$\tilde{p}_z = \frac{1}{\beta_0^2} \cdot p_z ; \qquad \text{(A.8c)}$$

via a scale transformation:

$$x,\ z,\ \sigma,\ p_x,\ p_z,\ p_\sigma \quad \Longrightarrow \quad \tilde{x} = x,\ \tilde{z} = z,\ \tilde{\sigma} = \sigma,\ \tilde{p}_x,\ \tilde{p}_z,\ \tilde{p}_\sigma ; \qquad \text{(A.9a)}$$

$$\mathcal{H} \quad \Longrightarrow \quad \tilde{\mathcal{H}} = \frac{1}{\beta_0^2} \cdot \mathcal{H} . \qquad \text{(A.9b)}$$

Then the new Hamiltonian is given by:

$$\tilde{\mathcal{H}}(\tilde{x}, \tilde{p}_x, \tilde{z}, \tilde{p}_z, \tilde{\sigma}, \tilde{p}_\sigma; s) = \frac{1}{\beta_0^2} \cdot \mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma; s)$$

$$= \tilde{p}_\sigma - (1+\hat{\eta}) \cdot [1 + K_x \cdot \tilde{x} + K_z \cdot \tilde{z}] \times \left\{ 1 - \frac{(\tilde{p}_x - \frac{e}{p_0 \cdot c} A_x)^2 + (\tilde{p}_z - \frac{e}{p_0 \cdot c} A_z)^2}{(1+\hat{\eta})^2} \right\}^{1/2} - [1 + K_x \cdot \tilde{x} + K_z \cdot \tilde{z}] \cdot \frac{e}{p_0 \cdot c} A_s \quad \text{(A.10)}$$

with

$$\begin{aligned} \hat{\eta} \equiv f(\eta) &= f(0) + f'(0) \cdot \eta + f''(0) \cdot \eta^2 \pm \cdots \\ &= \frac{1}{\beta_0^2} \cdot \eta - \frac{1}{\beta_0^4 \cdot \gamma_0^2} \cdot \frac{1}{2} \eta^2 \pm \cdots \\ &= \tilde{p}_\sigma - \frac{1}{\gamma_0^2} \cdot \frac{1}{2} \tilde{p}_\sigma^2 \pm \cdots . \end{aligned} \quad \text{(A.11)}$$

The corresponding canonical equations read:

$$\frac{d}{ds} \tilde{x} = + \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_x} ; \quad \frac{d}{ds} \tilde{p}_x = - \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{x}} ; \quad \text{(A.12a)}$$

$$\frac{d}{ds} \tilde{z} = + \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_z} ; \quad \frac{d}{ds} \tilde{p}_z = - \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{z}} ; \quad \text{(A.12b)}$$

$$\frac{d}{ds} \tilde{\sigma} = + \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_\sigma} ; \quad \frac{d}{ds} \tilde{p}_\sigma = - \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{\sigma}} . \quad \text{(A.12c)}$$