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THE WAKE POTENTIALS FROM THE FIELDS ON THE CAVITY BOUNDARY

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The wake potentials of axi-symmetrical structures are expressed, to their leading multipole order, in terms of the e.m.fields on the metallic boundary, with no contribution from the beam pipe. For the longitudinal potential, the expression applies to any type of structure including re-entrant ones or with unequal in and out beam-pipe radii. For the transverse case, however, the beam-pipe radii must be equal. These expressions generalize the usual method of calculating the wake potentials over the cavity gap and extends its applicability to more general structures like collimators, tapers or re-entrant cavities. It should in particular be useful for computer calculations of the wake fields in the time domain.

1. INTRODUCTION

The beam dynamics of high-current circular and linear accelerators is dominated by the effect of the impedances or wake potentials [1] of the metallic structures surrounding the beam trajectory. The longitudinal and transverse wake potentials are defined by

$$W_z(r,\theta,s) \stackrel{\text{def}}{=} -\frac{1}{Q}\int_{-\infty}^{+\infty} dz\ E_z(r,\theta,z,t(z,s)) \tag{1}$$

and

$$W_\perp(r,\theta,s) \stackrel{\text{def}}{=} \frac{1}{Q}\int_{-\infty}^{+\infty} dz\ (E_\perp + v\times B)(r,\theta,z,t(z,s)) \tag{2}$$

where s is the distance behind a given origin $z_0 = vt$ in the exciting bunch, of charge Q, and $t(z,s) = (z+s)/v$. Throughout this paper, we will concentrate exclusively on the ultra-relativistic case where $v = c$ and $\gamma = \infty$ and on structures with circular symmetry around the z-axis. Then, given $r_0 = (r_0, \theta_0 = 0)$ the radial position of the exciting bunch, the potentials obey the following multipole expansion [2]

$$W_z(r,\theta,s) = \sum_{m=0}^{\infty} r^m r_0^m \cos(m\theta)\ \frac{d}{ds} w^{(m)}(s) \tag{3}$$

and

$$W_\perp(r,\theta,s) = \sum_{m=1}^{\infty} m\ r^{m-1} r_0^m (\cos(m\theta)\ \hat{r} - \sin(m\theta)\ \hat{\theta})\ w^{(m)}(s) \tag{4}$$

Keeping only the leading terms of this expansion, i.e. for particles close to the axis, and with a convenient slight change of notations, the potentials reduce to

$$W_z(r,\theta,s) = w_0(s) \qquad (m=0) \tag{5}$$

$$W_\perp(r,\theta,s) = \frac{r_0}{r_0}\, w_1(s) \qquad (m=1) \tag{6}$$

The main consequence of Eqs.(5,6) is that, at the leading order in the multipole expansion, the longitudinal and transverse potentials experienced by a test particle behind the bunch does not depend on its transverse coordinates (r, θ). This property is used in computer codes solving Maxwell's equations in the time domain, like TBCI [3] or ABCI [4], to compute the wake potentials of cavities with side tubes of radius a from the longitudinal and azimuthal potentials evaluated at the tube radius $r = a$. Since the fields E_z, E_θ and B_r are zero on the surface of the tube, supposed infinitely conducting, the integrals giving W_z and W_θ in Eqs.(1,2) reduce to finite range integrals over the gap of the cavity. This method improves the accuracy of the calculation for two reasons:

- it avoids evaluating the wake fields on or near the bunch axis where the fields E_r and B_θ are singular and where, therefore, the calculation of the regular components of the fields suffers the most from numerical noise.

- as shown very clearly by the so-called 'field matching' techniques for solving Maxwell's equations in simple geometries, the cavity is the cause of multiple-scattering of the electromagnetic wave accompanying the charged particle bunch. At any distance from the cavity, there are always scattered waves propagating inside the beam tube and contributing to the wake potentials through the integrals of Eqs.(1,2). Cutting these integrals to a finite range over the beam tubes, which is mandatory in a computer calculation, is therefore an approximation unless one integrates at $r = a$.

However there are cases for which this method cannot be employed, e.g. structures with unequal side-tube radii, like tapers, or structures for which there is no clear line of sight lying on the side tubes, like collimators or re-entrant cavities. In general, all structures for which one cannot draw, within the vacuum or dielectric region, a straight line at constant radius lying on both the external beam-tubes are not suitable for this method.

In this paper, we derive expressions for the wake potentials which enables one to extend this method to structures of arbitrary shape. The wake potentials are given as integrals of e.m. fields over the boundary of the metallic structure such that side tubes give no contribution to the integrals. In fact, one can also chose, when convenient, to integrate over the more general set of "contours" lying on the side tubes and following partly the boundary partly a line at constant radius in the vacuum region. Such contours are illustrated in Fig.2. The only restriction to applying these results is, in the case of the transverse potential, that the side tubes have equal radii.

After introducing some definitions, the longitudinal potential is calculated in Section 2 and the transverse one in Section 3. In both cases, we begin by re-deriving the r-independence of the wake potentials described by Eqs.(5,6) as an intermediate result.

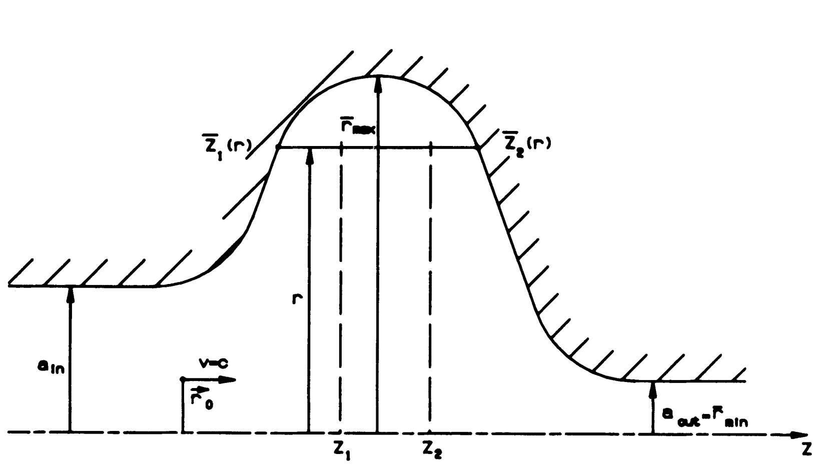

FIGURE 1 A simple cavity : notations.

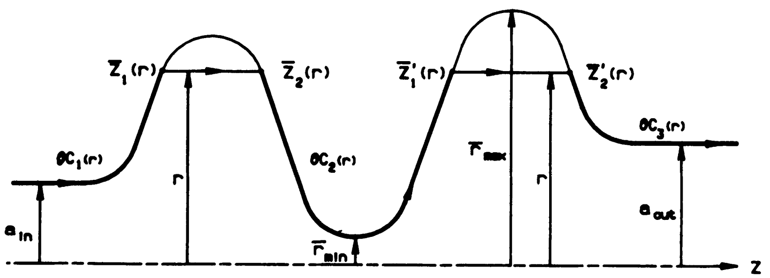

FIGURE 2 A multicell cavity with re-entrant shape. The contour $\partial\mathcal{C}(r)$ is composed of the three contours $\partial\mathcal{C}_1(r)$, $\partial\mathcal{C}_2(r)$ and $\partial\mathcal{C}_3(r)$ indicated in the figure.

2. CALCULATION OF THE LONGITUDINAL POTENTIAL $(m = 0)$

We consider an axi-symmetrical cavity of arbitrary shape with side tubes of radius a_{in} and a_{out} as shown in Fig.1. We denote by $\overline{r}, \overline{z}$ the coordinates of a point on the boundary, and by $\overline{r}_{min}$ and $\overline{r}_{max}$ the minimum and maximum radius of the cavity, including the tubes. We assume that the bunch which excites the wake fields in the cavity has velocity $v = c$, total charge Q and is characterized by a longitudinal charge distribution $\lambda(s)$ normalized to one, i.e.

$$\int_{-\infty}^{+\infty} \lambda(s)\, ds = 1 \tag{7}$$

Concentrating on the longitudinal wake-fields in the mode $m = 0$ for which the azimuthal dependence drops out, it is convenient to decompose the electromagnetic fields as follows

$$
\begin{aligned}
E &= E^{(0)} + E^{(r)} \\
B &= B^{(0)} + B^{(r)}
\end{aligned}
\tag{8}
$$

where $(E^{(0)}, B^{(0)})$ are the e.m. fields which would be generated by the charge distribution in a smooth pipe, given by

$$B_\theta^{(0)}(r,s) = \frac{Q}{2\pi c\epsilon_0 r}\ \lambda(s) \tag{9}$$

$$E_r^{(0)}(r,s) = c\ B_\theta^{(0)}(r,s) \tag{10}$$

the other components being zero. The wake potentials W_z and $W_\perp$ are entirely given by the additional fields $(E^{(r)}, B^{(r)})$, sometimes called 'radiated fields', which are solutions of homogeneous Maxwell's equations and vanish at infinite distance $z = \pm\infty$ in the tubes.

In analogy with the definition of the longitudinal potential given by Eq.(1), we introduce the two following quantities

$$W_z(r,s;z_1,z_2) \stackrel{\text{def}}{=} -\frac{1}{Q}\int_{z_1}^{z_2} dz\ E_z^{(r)}(r,z,t(z,s)) \tag{11}$$

where z_1 and z_2 are longitudinal coordinates such that the line element $[(r,z_1),(r,z_2)]$ lies inside of the cavity (see Fig.1), and

$$\overline{W}_z(r,s) \stackrel{\text{def}}{=} \sum W_z(r,s;\overline{z}_1(r),\overline{z}_2(r)) \tag{12}$$

where $\overline{z}_1(r)$ and $\overline{z}_2(r)$ denote the longitudinal coordinates of the boundary at the radius r and where the sum indicates that one eventually has to add up the contributions of N_r disconnected line elements as shown in Fig.2 for $N_r = 2$. Notice that for small enough r, $\overline{z}_1(r)$ or $\overline{z}_2(r)$ can be infinite. In fact, for $r < \overline{r}_{min}$, both $\overline{z}_1(r)$ and $\overline{z}_2(r)$ are infinite and it is a matter of definition to write

$$W_z(r,s) = \overline{W}_z(r,s) = W_z(r,s;-\infty,+\infty) \tag{13}$$

We now investigate the radial dependence of the generalized functions W_z and $\overline{W}_z$. Using the fact that, for $m = 0$, the radiated fields reduce to $(E_z^{(r)}, E_r^{(r)}, B_\theta^{(r)})$ and that, as already mentionned, they satisfy homogeneous Maxwell's equations, it is easy to derive the following equalities

$$
\begin{aligned}
\partial_r E_z(r,z,t(z,s)) &= \partial_z E_r(r,z,t(z,s)) + \partial_t B_\theta(r,z,t(z,s)) \\
&= \tfrac{d}{dz}(E_r + vB_\theta)(r,z,t(z,s)) - \tfrac{1}{v}\partial_t E_r - v\partial_z B_\theta \\
&= \tfrac{d}{dz}(E_r + vB_\theta)(r,z,t(z,s)) - \tfrac{1}{v\gamma^2}\partial_t E_r
\end{aligned}
\tag{14}
$$

leading to

$$\partial_r W_z(r,s;z_1,z_2) = -\frac{1}{Q}\int_{z_1}^{z_2} dz\ \frac{d}{dz}(E_r^{(r)} + cB_\theta^{(r)})(r,z,t(z,s)) \tag{15}$$

up to terms in $1/\gamma^2$. Therefore, denoting by Δ_{z_1,z_2} the difference operator between z_2 and z_1, one can write in the limit $v \to c$

$$\partial_r W_z(r,s;z_1,z_2) = -\frac{1}{Q}\Delta_{z_1,z_2}(E_r^{(r)} + cB_\theta^{(r)})(r,z,t(z,s)) \tag{16}$$

For $r < \overline{r}_{min}$, this equation shows, by taking $z_1 = -\infty$ and $z_2 = +\infty$ for which the radiated fields vanish, that the longitudinal potential $W_z(r,s)$ does not depend on r, as expressed by Eq.(5).

Turning to $\overline{W}_z$, one gets

$$\frac{d}{dr}\overline{W}_z(r,s) = \sum \left[\partial_r W_z(r,s;\overline{z}_1(r),\overline{z}_2(r)) - \frac{1}{Q}\left(E_z^{(r)}(r,\overline{z}_2(r),t)\,\frac{d\overline{z}_2}{dr}(r)\right.\right.$$
$$\left.\left. -E_z^{(r)}(r,\overline{z}_1(r),t)\,\frac{d\overline{z}_1}{dr}(r)\right)\right] \tag{17}$$

Using Eq.(16), this can be conveniently written as

$$d\overline{W}_z(r,s) = \frac{1}{Q}\sum_{i=1}^{2N_r}\epsilon_i\left[(E_r^{(r)} + cB_\theta^{(r)})dr + E_z^{(r)}d\overline{z}\right](r,\overline{z}_i,t(\overline{z}_i,s)) \tag{18}$$

where i denotes the index of the boundary points at radius r, starting from $z = -\infty$, and $\epsilon_i = (-1)^{i-1}$. Integrating Eq.(18) from $r = 0$, where $\overline{W}_z(r,s) = w_0(s)$, to $r \geq \sup(a_{in}, a_{out})$ gives

$$\overline{W}_z(r,s) - w_0(s) = \frac{1}{Q}\int_{\partial\mathcal{C}(r)}\left[(E_r^{(r)} + cB_\theta^{(r)})d\overline{r} + E_z^{(r)}d\overline{z}\right] \tag{19}$$

where the contour of integration $\partial\mathcal{C}(r)$, as illustrated in Fig.2, is in general composed of several pieces following the boundary of the cavity. By relating, through Eq.(8), the radiated fields to the fields (E,B) which satisfy the metallic boundary condition

$$E_r\ d\overline{r} + E_z\ d\overline{z} = 0 \tag{20}$$

and the fields $(E^{(0)}, B^{(0)})$ given by Eqs.(9,10), this integral simplifies to

$$w_0(s) = \overline{W}_z(r,s) - \frac{c}{Q}\int_{\partial\mathcal{C}(r)} B_\theta(\overline{r},\overline{z},t(\overline{z},s))\ d\overline{r} + \frac{1}{\pi\epsilon_0}\ln\left(\frac{a_{out}}{a_{in}}\right)\ \lambda(s) \tag{21}$$

The important property of the above expression is that the parts of the contour such that $d\overline{r} = 0$, and in particular the side tubes, do not contribute to the integral. Eq.(21) can also be written in terms of the radiated field $B_\theta^{(r)}$:

$$w_0(s) = \overline{W}_z(r,s) - \frac{c}{Q}\int_{\partial\mathcal{C}(r)} B_\theta^{(r)}(\overline{r},\overline{z},t(\overline{z},s))\ d\overline{r} + \frac{1}{2\pi\epsilon_0}\ln\left(\frac{a_{out}}{a_{in}}\right)\ \lambda(s) \tag{22}$$

Moving r to the maximum radius $\overline{r}_{max}$ of the cavity allows to express the wake potential in terms of the fields B_θ on cavity boundary as follows

$$w_0(s) = -\frac{c}{Q}\int_{\partial\mathcal{C}} B_\theta(\overline{r},\overline{z},t(\overline{z},s))\ d\overline{r} + \frac{1}{\pi\epsilon_0}\ln\left(\frac{a_{out}}{a_{in}}\right)\ \lambda(s) \tag{23}$$

where now $\partial\mathcal{C}$ denotes the full contour of the cavity.

3. CALCULATION OF THE TRANSVERSE POTENTIAL ($m = 1$)

The calculation of the transverse potential is parallel to the preceding one, although slightly more complicated. First, some care has to be taken in the definition of the fields $(E^{(0)}, B^{(0)})$ since, unlike in the $m = 0$ case, the smooth-pipe solution depends on the pipe radius. We assume that $(E^{(0)}, B^{(0)})$ refers to the fields in the incoming tube with radius a_{in} in such a way that the radiated fields $(E^{(r)}, B^{(r)})$ vanish for $z \to -\infty$. In the other limit $z \to +\infty$, they tend to the difference between the outgoing and incoming smooth-pipe solutions. Therefore, in both limits, their longitudinal components $E_z^{(r)}, B_z^{(r)}$, as well as their tranverse combination $(E_\perp^{(r)} + v \times B^{(r)})$ vanish for $v \to c$.

We introduce the generalized transverse wake functions

$$W_\perp(r, \theta, s; z_1, z_2) \stackrel{\text{def}}{=} \frac{1}{Q} \int_{z_1}^{z_2} dz \; (E_\perp^{(r)} + v \times B^{(r)})(r, \theta, z, t(z, s)) \tag{24}$$

and

$$\overline{W}_\perp(r, \theta, s) \stackrel{\text{def}}{=} \sum W_\perp(r, \theta, s; \overline{z}_1(r), \overline{z}_2(r)) \tag{25}$$

with the same notations as for Eqs.(11,12). Using the homogeneous Maxwell's equations, one can derive, as was done in Eqs.(14,15) for the longitudinal case, the following identities

$$\left(\frac{1}{r}\partial_r(rW_r) + \frac{1}{r}\partial_\theta W_\theta\right)(r, \theta, s; z_1, z_2) = -\frac{1}{Q}\; \Delta_{z_1, z_2} E_z^{(r)}(r, \theta, z, t(z, s)) \tag{26}$$

$$\left(\frac{1}{r}\partial_r(rW_\theta) - \frac{1}{r}\partial_\theta W_r\right)(r, \theta, s; z_1, z_2) = -\frac{c}{Q}\; \Delta_{z_1, z_2} B_z^{(r)}(r, \theta, z, t(z, s)) \tag{27}$$

neglecting terms of order $1/\gamma^2$ as in Eq.(14). Expressing the azimuthal dependence of the e.m. fields projected in the mode $m = 1$, as follows

$$\begin{array}{lll} E_r = e_r \cos\theta \quad , & E_\theta = e_\theta \sin\theta \quad , & E_z = e_z \cos\theta \\ B_r = b_r \sin\theta \quad , & B_\theta = b_\theta \cos\theta \quad , & B_z = b_z \sin\theta \end{array} \tag{28}$$

one can write

$$\begin{aligned} W_r &= \;\; w_r \cos\theta \\ W_\theta &= -w_\theta \sin\theta \end{aligned} \tag{29}$$

The differential equations (25,26) then lead to

$$\left(\frac{1}{r}\partial_r(rw_r) - \frac{1}{r}w_\theta\right)(r, s; z_1, z_2) = -\frac{1}{Q}\; \Delta_{z_1, z_2} e_z^{(r)}(r, z, t(z, s)) \tag{30}$$

$$\left(\frac{1}{r}\partial_r(rw_\theta) - \frac{1}{r}w_r\right)(r, s; z_1, z_2) = \frac{c}{Q}\; \Delta_{z_1, z_2} b_z^{(r)}(r, z, t(z, s)) \tag{31}$$

Introducing the sum and the difference of w_r and w_θ

$$\begin{aligned} \sigma &= w_r + w_\theta \\ \delta &= w_r - w_\theta \end{aligned} \tag{32}$$

the differential system translates to

$$\partial_r \sigma(r, s; z_1, z_2) = -\frac{1}{Q}\, \Delta_{z_1,z_2} \left(e_z^{(r)} - cb_z^{(r)}\right)(r, z, t(z, s)) \tag{33}$$

$$\frac{1}{r^2}\partial_r (r^2 \delta)(r, s; z_1, z_2) = -\frac{1}{Q}\, \Delta_{z_1,z_2} \left(e_z^{(r)} + cb_z^{(r)}\right)(r, z, t(z, s)) \tag{34}$$

For $r < r_{min}$, with $z_1 = -\infty$ and $z_2 = +\infty$, these are equations for the sum and difference of the actual radial and azimuthal wake potentials. In view of the discussion beginning the section, their right-hand side vanish and the equations are easily solved to yield

$$w_r = w_\theta \equiv w_1 \quad \text{independent of } r \tag{35}$$

reproducing Eq.(6).

Turning to the functions $\overline{\sigma} = \overline{w}_r + \overline{w}_\theta$ and $\overline{\delta} = \overline{w}_r - \overline{w}_\theta$ derived from $\overline{W}_\perp$ defined by Eq.(25), one can show, in the same way as in the longitudinal case, that

$$d\overline{\sigma} = \frac{1}{Q}\left[(e_z^{(r)} - cb_z^{(r)})dr - (e_r^{(r)} - cb_\theta^{(r)} - e_\theta^{(r)} - cb_r^{(r)})d\overline{z}\right] \tag{36}$$

$$d(r^2\overline{\delta}) = \frac{1}{Q}r^2\left[(e_z^{(r)} + cb_z^{(r)})dr - (e_r^{(r)} - cb_\theta^{(r)} + e_\theta^{(r)} + cb_r^{(r)})d\overline{z}\right] \tag{37}$$

Integrating these equations from $r = 0$ where $\overline{\sigma}(r, s) = 2w_1(s)$ and $\overline{\delta} = 0$ to a radius r larger than the tube radii, yields

$$\overline{\sigma}(r, s) - 2w_1(s) = \frac{1}{Q}\int_{\partial\mathcal{C}(r)}\left[(e_z^{(r)} - cb_z^{(r)})d\overline{r} - (e_r^{(r)} - cb_\theta^{(r)} - e_\theta^{(r)} - cb_r^{(r)})d\overline{z}\right] \tag{38}$$

$$r^2\overline{\delta}(r, s) = \frac{1}{Q}\int_{\partial\mathcal{C}(r)} \overline{r}^2\left[(e_z^{(r)} + cb_z^{(r)})d\overline{r} - (e_r^{(r)} - cb_\theta^{(r)} + e_\theta^{(r)} + cb_r^{(r)})d\overline{z}\right] \tag{39}$$

which simplifies again when Eq.(8) is used to translate it into the fields (E, B) satisfying the boundary conditions

$$E_\theta = 0 \quad \text{and} \quad B_r\, d\overline{z} - B_z\, d\overline{r} = 0 \tag{40}$$

and into the smooth-tube solutions $(E^{(0)}, B^{(0)})$ such that

$$E_z^{(0)} = B_z^{(0)} = E_r^{(0)} - cB_\theta^{(0)} = E_\theta^{(0)} + cB_r^{(0)} = 0 \quad (\gamma = \infty) \tag{41}$$

everywhere. One therefore ends up with

$$2w_1(s) = \overline{\sigma}(r, s) - \frac{1}{Q}\int_{\partial\mathcal{C}(r)}\left[e_z d\overline{r} - (e_r - cb_\theta)d\overline{z}\right] \tag{42}$$

$$r^2\overline{\delta}(r, s) = \frac{1}{Q}\int_{\partial\mathcal{C}(r)} \overline{r}^2\left[e_z d\overline{r} - (e_r - cb_\theta)d\overline{z}\right] \tag{43}$$

Unlike what happens in the longitudinal case, the integration over the beam tubes does not drop out from Eq.(42) due to the presence of the term in $d\overline{z}$. It can however be eliminated between Eq.(42) and Eq.(43) but only when the tube have equal radii

$$a_{in} = a_{out} \tag{44}$$

leading to

$$2a^2 w_1(s) = (a^2 - r^2)\overline{w}_r(r,s) + (a^2 + r^2)\overline{w}_\theta(r,s) \quad (45)$$

$$+\frac{1}{Q}\int_{\partial\mathcal{C}(r)} (\overline{r}^2 - a^2)\left[e_z d\overline{r} - (e_r - cb_\theta)d\overline{z}\right] \quad (46)$$

Moving r to $\overline{r}_{max}$, one gets the transverse potential as an integral over the contour $\partial\mathcal{C}$ of the cavity with no contribution from the beam tubes

$$w_1(s) = \frac{1}{2a^2 Q}\int_{\partial\mathcal{C}} (\overline{r}^2 - a^2)\left[e_z d\overline{r} - (e_r - cb_\theta)d\overline{z}\right] \quad (47)$$

If the beam tubes have unequal radii, it is straightforward to eliminate the contribution of either of the tubes leading to the same expressions as in Eqs.(45,46) with a replaced by either a_{in} or a_{out}. For calculations where $s = |z - ct|$ is not too large, one should presumably eliminate the contribution from the outgoing tube since the incoming one is affected by the back-scattering of the incoming wave only over distances of the order of s.

4. CONCLUSION

We have derived expressions of the longitudinal ($m = 0$) and transverse ($m = 1$) wake potentials, for axi-symmetrical metallic structures, in terms of integrals of the electromagnetic fields, at constant distance $|z - ct|$ behind the exciting bunch, which do not involve the value of the fields in the incoming and exiting beam tubes. A family of finite contours of integration can be used, namely the ones involving line elements at constant but not necessarily equal radii, and elements of the metallic boundary of the structure. In particular one can chose to integrate over the contour following the boundary of the structure. In the case of the longitudinal potential, the integral over the boundary involves only the magnetic field B_θ and is such that all boundary elements parallel to the axis, including the beam tubes, drop out from the integral. This result applies to structures of arbitrary shape.

These expressions should be especially useful when implemented in time-domain wake-field computer programs since they generalize and extend the applicability of the usual method which consists in integrating the wake fields, when possible, over the gap of the cavity. In the case of re-entrant cavity shape, for instance, the use of the WAKCOR post-processor of the TBCI program can be avoided by integrating directly the wake fields along the cavity boundary.

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