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HIGHER-ORDER SEXTUPOLE RESONANCES IN LOW-EMITTANCE LIGHT SOURCE STORAGE RINGS

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Abstract Large-amplitude single particle motion in low-emittance light source storage rings is strongly influenced by the sextupoles intentionally introduced for the correction of chromatic and geometric aberrations. Tracking studies indicate that in many cases the motion is driven by the higher-order resonances of the sextupoles. By taking a Chasman Green (CG) and a Triple Bend Achromat (TBA) lattice as examples, two approaches are discussed to describe the observed phenomena.

INTRODUCTION

The storage rings we consider have the periodic structure, being composed of identical cells as shown in Fig. 1. To study the particle behavior in such systems, it is useful to make the harmonic analysis in the unit of super periodic cell. The Hamiltonian is expressed as[1,2]

$$H = \nu_x I_x + \nu_y I_y + (2I_x)^{\frac{3}{2}} \sum_{jm} \gamma_j A_{jm} \cos(j\phi_x - m\theta) - 3(2I_x)^{\frac{1}{2}} 2I_y$$
$$\times \sum_m [2B_{1m} \cos(\phi_x - m\theta) + B_{+m} \cos(\phi_+ - m\theta) + B_{-m} \cos(\phi_- - m\theta)], \quad (1)$$

where I_u and ϕ_u (u = x or y) are action and angle variables, $\phi_\pm = \phi_x \pm 2\phi_y$, ν_u is the betatron tune per cell, γ_j equals 3 and 1 for j = 1 and 3, respectively, and $\theta = 2\pi \cdot s/C$ (C: cell length) denotes the time variable. Coefficients such as A_{jm} ($-\infty < m < +\infty$) depend on sextupole strengths and betatron functions and phases at the sextupoles.

As seen from Eq. (1), sextupoles drive resonances $j\nu_x$ = integer, and $\nu_x \pm 2\nu_y$ = integer. The familiar correction scheme to enlarge the dynamic aperture is to identify the resonance and suppress resonance driving terms with additional sextupoles. In case of a CG lattice in its high or middle beta mode which has a mirror symmetry at the center

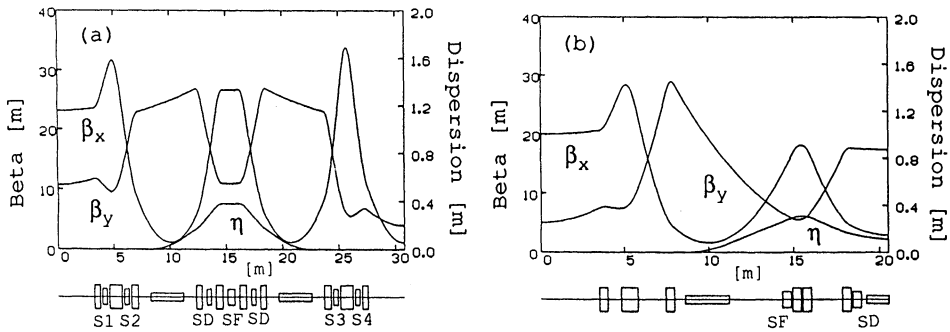
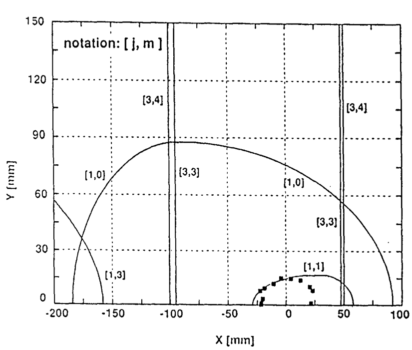


FIGURE 1. Optics functions for (a) CG (hybrid mode) and (b) TBA (high beta mode) lattice. Reflective symmetry about either end.

of the achromatic arc, the double focusing at dipoles with chromaticity correcting sextupoles alone results in the enhancement of a single resonance $\nu_x = 1$. The particle motion of such system can be very well understood in the single resonance approximation. Drastic improvement of the dynamic aperture can be achieved with additional sextupoles in the dispersion free sections by reducing the magnitude of A_{11} and A_{33} as well as the second-order tune shifts with amplitude.[1,3]

PARTICLE TRACKING IN HARMONIC EXPANDED SEXTUPOLE FIELD

Figure 2 illustrates the dynamic aperture of the TBA lattice only with chromaticity correcting sextupoles, together with the boundaries of the stability region predicted in the single resonance approximation.



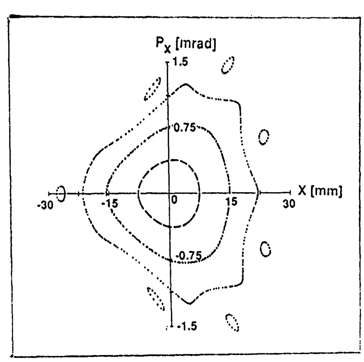


FIGURE 2. (Left) Dynamic aperture of TBA (dark squares) and stable regions of single resonances (solid lines).
FIGURE 3. (Right) Phase space trajectories of TBA close to the 7th-order resonance.

Although the boundary of [j,m] = [1,1] component somewhat simulates the dynamic aperture, the discrepancy is large to conclude that the stability is determined solely by the resonance $\nu_x = 1$. In fact, the phase space trajectories in Fig. 3, obtained from the tracking calculation, indicate the excitation of 7th-order resonance close to the stability limit. Here, we shall make an attempt to explain the observed phenomenon as a "combined effect" of several different resonance driving terms of the lowest order (Eq. (1)), acting simultaneously on the particle motion.

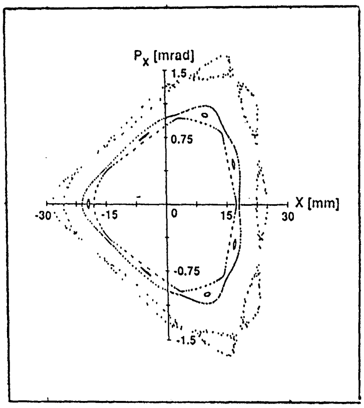
The analysis of such combined harmonics system being clearly beyond the context of single resonance approximation stated ealier, we shall rely on a numerical approach: We analyze the motion by tracking a particle in a "fictitious" sextupole field composed only of selected harmonics. Provided that the results are reproduced in this way, one has identified the harmonics that dominate the dynamics or limit the stability. To this end, we rewrite the Hamiltonian of Eq. (1) in terms of $u = \sqrt{2I_u}\cdot\cos\phi_u$ and $p_u = -\sqrt{2I_u}\cdot\sin\phi_u$ (u = x, y) as

$$H(x,\ p_x,\ y,\ p_y;\ \theta) = \frac{p_x^2}{2\mu_x} + \frac{p_y^2}{2\mu_y} + V(x,\ p_x,\ y,\ p_y;\ \theta), \qquad (2)$$

where $\mu_u = \nu_u^{-1}$. The sextupole part of the potential consists of ten different cubic terms of u and p_u, each having a time-dependent coefficient which is expressed as an infinite sum over the products of harmonics A_{jm} etc. and the trigonometric functions.[2]

Out of the infinite sum over m, our aim is to retain only those specified ones. This requires us to carry out the numerical integration at every point in the ring, since harmonic expanded fields are, by themselves, distributed everywhere. To guarantee the simplecticity of the integration, we adopt the technique of Ref. 4. The dynamic apertures obtained by including a single harmonic are found to agree with analytical results, which serve as a check of the computation.

We show in Fig. 4, the phase space trajectories of the TBA lattice calculated by the fictitious tracking. We naively took account only of four components [j,m] = [1,0],[1,1],[3,3] and [3,4] that limit the stable motion in region close to the actual dynamic aperture in Fig. 2. Nevertheless, figure 4 comprises the major feature of the "real" tracking results: In this particular case, effects of each harmonic are combined to drive the higher-order resonance. It also implies that the four components are most responsible to tune shifts with amplitude, although the excitation of 7th-order resonance at a smaller amplitude indicates that the background contributes to the reduction of tune shift. It is interesting to note the 8th-order resonance with a larger amplitude which is absent in reality.



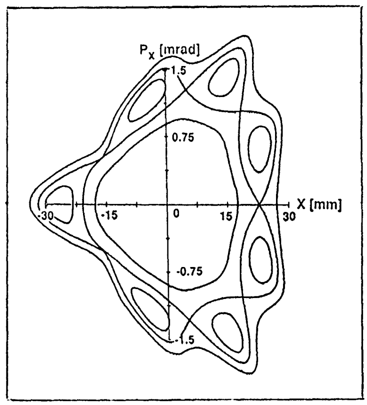


FIGURE 4. (Left) Prediction of fictitious tracking for TBA.
FIGURE 5. (Right) Prediction of PZ theory for TBA.

PERTURBATIVE APPROACH

As an alternative way to describe the higher-order resonances of sextupoles, we take the perturbation approach: We shall work with the standard Poincare-von Zeipel (PZ) theory. Here we simplify our problem to one dimension and analyze merely the horizontal dynamics.

The perturbation theory developed by Poincare and von Zeipel defines a canonical transformation $(\phi, I) \rightarrow (\varphi, J)$ with a generating function[5]

$$S(J, \phi; \theta) \equiv J\cdot\phi + \varepsilon S_1 + \varepsilon^2 S_2 + \varepsilon^3 S_3 + \ldots, \tag{3}$$

where ε is a parameter representing the strength of the perturbation. The idea is to find the functions S_n $(n=1,2, \ldots)$ successively so that the new Hamiltonian becomes independent of angle φ up to $O(\varepsilon^n)$.

In the present study, we derived generating functions up to S_3, thus enabling us to study the perturbative effects of the sextupoles up to fourth order in amplitude. The procedure is as follows; We solve the differential equation satisfied by S_n to obtain S_n in a harmonic expanded form. We then derive a closed form expression for S_n by summing over harmonics analytically, with which higher-order terms of the new Hamiltonian are evaluated. Next we expand them into harmonics again to define coefficients which are the higher-order counterparts of A_{jm} etc. of the lowest-order. Integrals over angles θ and ϕ can be done analytically in the thin sextupole treatment.

Our derivation has the merit that although one requires higher-order terms of the Hamiltonian to be expanded into harmonics for the resonance analysis, each harmonic is given in a closed form which can be readily computed. Formulation is made also to allow a systematic evaluation of higher-order perturbations where one needs to deal with a large number of terms. Details will be given in Ref. 2.

An outcome of the perturbative approach is the following relation of orders of sextupole resonances versus orders of the perturbation:

ε^1 : 1,3. ε^2 : 2,4,6.
ε^3 : 1,3,5,7,9. ε^4 : 2,4,6,8,10,12. etc.

With this in mind, an attempt is made to reproduce the resonances observed in the particle tracking. In combination with the perturbation calculation, single resonance approximation is used. Canonical variables are finally transformed back to the original frame before making the comparison of the phase space structures.

We show in Figure 5, the phase space trajectories of the 7th-order resonance of the TBA lattice obtained from the third-order Hamiltonian. The agreement with the tracking result (Fig. 3) is quite impressive. Another interesting example is found with the TBA by tuning ν_x to a value slightly above 1+1/6: We now observe the 6th-order resonance in the tracking (Fig. 6a). Since this resonance can either be driven as second- or fourth-order perturbation, calculations were performed in two ways. Whereas the second-order result fails to reproduce the phase space structure even topologically (Fig. 6b), the fourth-order calculation achieves fairly good agreement, although it still fails to reproduce the islands (Fig. 6c). An example of 7th-order resonance of the CG lattice is displayed in Fig. 7. We found the

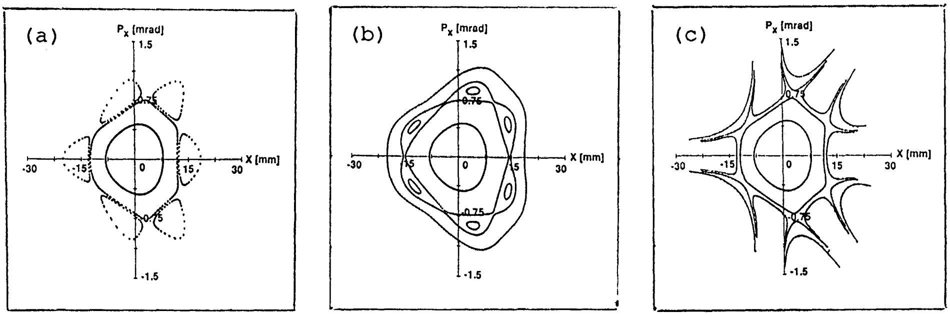


FIGURE 6. Phase space trajectories near 6th-order resonance of TBA. (a) Tracking. (b) 2nd-order and (c) 4th-order PZ theory.

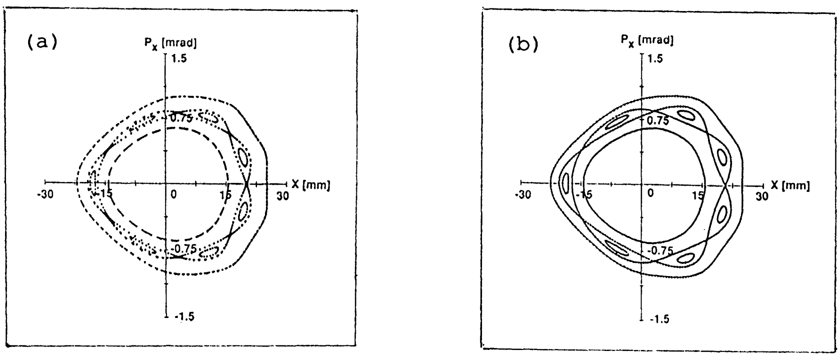


FIGURE 7. Phase space trajectories near 7th-order resonance of CG (hybrid mode). (a) Tracking. (b) 3rd-order PZ theory.

trends that, unlike other modes, hybrid modes of the CG are very much influenced by higher-order resonances after chromatic and geometric corrections, which may be due to the reduction of symmetry. In most cases these resonances limit dynamic apertures in smaller dimensions. We could not reproduce many of these resonances in the present frame work implying that the degrees of perturbation are greater than n = 4.

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