Particle Accelerators, 1990, Vol. 28, pp. 119–124
Reprints available directly from the publisher
Photocopying permitted by license only

SOLVABLE MAP REPRESENTATION OF A NONLINEAR SYMPLECTIC MAP

GOVINDAN RANGARAJAN, ALEX J. DRAGT, AND FILIPPO NERI

Department of Physics and Astronomy, University of Maryland
College Park, MD 20742, U.S.A.

Abstract The evolution of a particle under the action of a beam transport system can be represented by a nonlinear symplectic map $\mathcal{M}$. This map can be factorized into a product of Lie transformations. The evaluation of any given Lie transformation in general requires the summation of an infinite number of terms. There are several ways of dealing with this difficulty: The summation can be truncated, thus producing a map that is nonsymplectic, but still useful for short term tracking. Alternatively, for long term tracking, the Lie transformation can be replaced by some symplectic map that agrees with it to some order and can also be evaluated exactly. This paper shows how this may be done using solvable symplectic maps. A solvable map gives rise to a power series that either terminates or can be summed explicitly. This method appears to work quite well in the various examples that we have considered.

INTRODUCTION

Over the past several years, Lie algebraic techniques have been developed to track particles through beam transport systems[1,2]. These techniques have been incorporated into the computer code MARYLIE[3]. The basic idea is to represent each beam-line element by a Lie algebraic map. Maps representing individual elements can then be concatenated together to produce a single map representing the entire system. When this map is used for long-term tracking, it is approximated by a symplectic map obtained from a generating function. In this paper we describe an alternate symplectic approximation using solvable maps.

BACKGROUND

Let $z = (q_1, q_2, q_3, p_1, p_2, p_3)$ denote the six dimensional phase space location of a particle. Denoting the initial and final locations of the particle by z^i and z^f respectively, the effect of a beam-line element on the trajectory of this particle can be described as follows:

$$z^f = \mathcal{M} z^i. \tag{1}$$

Here $\mathcal{M}$ is a (possibly nonlinear) map that describes the action of the element in question.

Provided synchrotron radiation effects are ignored, the trajectory of a particle through an element is governed by Hamilton's equations of motion. Then it can be shown that the Jacobian M of the map $\mathcal{M}$ representing the element must obey the following symplectic condition:

$$\tilde{M} J M = J \tag{2}$$

where $\tilde{M}$ is the transpose of M and J is an antisymmetric matrix defined as

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{3}$$

Here I is the 3×3 identity matrix. Matrices M satisfying Eq. (2) are called symplectic matrices and the corresponding (possibly nonlinear) maps $\mathcal{M}$ symplectic maps.

It has been shown[1] that $\mathcal{M}$ can be factorized as follows:

$$\mathcal{M} = e^{:f_1:} e^{:f_2:} e^{:f_3:} \ldots \tag{4}$$

Here $:f_m:$ is the Lie operator corresponding to the homogeneous polynomial f_m of degree m and is defined by its action on an arbitrary function g (of phase space variables) as given below:

$$:f_m: g = [f_m, g] \tag{5}$$

where $[f_m, g]$ denotes the Poisson bracket of the functions f_m and g. The exponential of a Lie operator is called a Lie transformation and is given as

$$e^{:f_m:} = 1 + :f_m: + :f_m:^2 /2! + \cdots . \tag{6}$$

where

$$:f_m:^2 \; g = \; [f_m, [f_m, g]] \quad \text{etc.} \tag{7}$$

By working through the definition of a Poisson bracket, it can be easily verified that the homogeneous polynomial defined by $:f_m:^r \; z$ has the degree

$$\deg(:f_m:^r \; z) = r(m-2) + 1. \tag{8}$$

Each one of the Lie transformations in Eq. (4) can be shown to be a symplectic map, and hence the factorization can be stopped at any order without losing symplecticity. Even though the Lie algebraic representation is inherently symplectic in this sense, problems arise when one attempts to use this representation for tracking. For this, we need to evaluate Eq. (1) explicitly, i.e. we need to know the action of each one of the exponential factors on phase space coordinates. Using Eqs. (6) and (8), it is seen that the factor $\exp(: f_1 :)$ produces a phase space translation, and the factor $\exp(: f_2 :)$ produces a linear transformation as illustrated below:

$$\exp(: f_1 :)\, z_a = z_a + k_a \quad a = 1,6 \tag{9}$$

$$\exp(: f_2 :)\, z_a = \sum_{b=1}^{6} R_{ab} z_b \quad a = 1,6 \tag{10}$$

where the k_a's are constants and R is a matrix. Thus we have reduced the evaluation of an infinite number of terms in the Taylor series expansion of these two factors to that of a finite number of terms. Hence they can be easily handled by a computer.

However, this is not possible for the remaining higher order Lie transformations in Eq. (4) which represent the nonlinear content of $\mathcal{M}$. For short term tracking we can truncate the Taylor series expansions of these Lie transformations. This is certainly justifiable from the viewpoint of general perturbation theory since successive terms in this Taylor series expansion give higher and higher order terms. If we restrict ourselves to a small neighborhood around the design orbit, these higher order terms will be small and hence can be neglected. For example, we can neglect all terms that produce contributions of order higher than three to give:

$$\begin{aligned} \exp(:f_3:)\,\exp(:f_4:) &= (1+ :f_3: + :f_3:^2 /2 + \cdots)(1+ :f_4: + \cdots) \\ &\approx 1+ :f_3: + :f_3:^2 /2 + :f_4: \ . \end{aligned} \tag{11}$$

However, this truncated Taylor series method violates the symplectic condition. This non-symplecticity can lead to spurious damping or growth in long term tracking.

Alternatively, these Lie transformations can always be evaluated to machine precision using numerical integration. However, this is very time consuming. It is therefore important to approximate higher order Lie transformations by symplectic maps that can be both evaluated exactly and quickly.

SYMPLECTIFICATION USING SOLVABLE MAPS

Before describing the solvable map method, we briefly describe the generating function method currently used by MARYLIE for symplectic tracking[4]. The basic idea behind this method is as follows. Consider a generating function F_2 of the second kind:

$$F_2(q^i, p^f) = q^i \cdot p^f + P(q^i, p^f) \tag{12}$$

where the superscripts denote initial and final coordinates. Then one chooses the polynomial P such that the following transformations

$$p^i = \partial F/\partial q^i = p^f + \partial P/\partial q^i, \tag{13a}$$

$$q^f = \partial F/\partial p^f = q^i + \partial P/\partial p^f \tag{13b}$$

agree with those given by $\exp(:f_3:)\exp(:f_4:)\ldots\exp(:f_N:)$ through terms of degree $(N-1)$. Since generating functions always produce symplectic maps, one has produced a symplectic approximation.

One drawback of this method is that it involves solving implicit equations (cf. Eq. (13)). Even though this can be overcome in practice by employing Newton's method, it is nevertheless a retreat from the basic philosophy of Lie algebraic perturbation theory which aims to give final coordinates explicitly in terms of initial coordinates. Moreover, in certain examples, this method has been found to either underestimate or overestimate stability boundaries.

We now outline an alternative technique of symplectification using so-called "solvable maps". Solvable maps are generalizations of kick maps. The class of kick maps includes only those symplectic maps for which the Taylor series expansion in Eq. (6) terminates when acting on phase space coordinates[2,5]. The class of solvable maps also includes those symplectic maps for which the Taylor series expansion can be summed explicitly. The Lie transformation $\exp(:aq^{l+2}+bq^{l+1}p:)$ is a simple example of a solvable map that is not a kick. One finds that

$$q' = \exp(:aq^{l+2} + bq^{l+1}p:)\, q = \frac{q}{[1 + lbq^l]^{\frac{1}{l}}} \quad \text{for} \quad l \geq 1, \tag{14a}$$

$$p' = \exp(:aq^{l+2} + bq^{l+1}p:)\, p = \frac{E - a(q')^{l+2}}{b(q')^{l+1}}, \tag{14b}$$

where

$$E = aq^{l+2} + bq^{l+1}p. \tag{15}$$

The basic idea behind the solvable map method is to represent each nonlinear factor $\exp(:f_n:)$ in $\mathcal{M}$ as a product of solvable maps[6]:

$$\exp(:f_n:) = \exp(:g_1:)\,\exp(:g_2:)\ldots\exp(:g_m:) \quad \text{for} \quad n \geq 3. \tag{16}$$

For example, consider the Lie representation of a general third order map in one dimension:

$$\mathcal{M} = \exp(:f_3:)\exp(:f_4:) \tag{17}$$

where

$$f_3 = a_1 q^3 + a_2 q^2 p + a_3 q p^2 + a_4 p^3, \tag{18a}$$

$$f_4 = a_5 q^4 + a_6 q^3 p + \cdots + a_9 p^4. \tag{18b}$$

This can be represented in terms of solvable maps as given below:

$$\begin{aligned}\mathcal{M} = {} & \exp(:b_1 q^3 + b_2 q^2 p:)\exp(:b_3 q p^2 + b_4 p^3:)\times \\ & \exp(:b_5 q^4 + b_6 q^3 p:)\exp(:b_7 q^2 p^2:)\exp(:b_8 q p^3 + b_9 p^4:)\end{aligned} \tag{19}$$

where

$$b_i = a_i \ \text{ for } i = 1,5 \ \text{ and } \ i = 9, \tag{20a}$$

$$b_6 = a_6 - 3a_1 a_3, \tag{20b}$$

$$b_7 = a_7 - 9a_1 a_4/2 - 3a_2 a_3/2, \tag{20c}$$

$$b_8 = a_8 - 3a_2 a_4. \tag{20d}$$

In all the examples studied so far, tracking results obtained using solvable map approximations agree quite well with numerical integration results. For example,the location of stability boundary points obtained using solvable maps agrees to within 15% with points obtained using numerical integration. The degree of agreement quite often is much better. Therefore, this method holds promise as a tracking tool to investigate the dynamic aperture of a beam transport system.

SUMMARY

In this paper, we have proposed a new representation of nonlinear symplectic maps using solvable maps. Good agreement with exact numerical integration has been found in tracking studies for the various examples considered. Therefore, this method promises to be a good compromise combining both speed and accuracy for long term numerical tracking.

REFERENCES

1. For a review of Lie algebraic techniques, see A. J. Dragt, in Physics of High Energy Particle Accelerators, AIP Conf. Proc. No. 87, edited by R. A. Carrigan, F. R. Huson, and M. Month (AIP, New York, 1982), p.147.
2. A. J. Dragt *et. al., Ann. Rev. Nucl. Part. Sci.* 38, 455 (1988).
3. A. J. Dragt *et. al., IEEE Trans. Nucl. Sci.,* NS-32, 2311 (1985).
4. D. Douglas *et. al., IEEE Trans. Nucl. Sci.,* NS-32, 2279 (1985); F. Neri, *Dept. Phys. Tech. Rep.,* Univ. Md. (1986).
5. Recently, J. Irwin has given a compact representation of a nonlinear map in terms of these kick maps, personal communication (1989).
6. In practice, one does not use maps that can be solved only using elliptic and other complicated functions since the cure is as bad as the disease in these cases.