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CUMULATIVE BEAM BREAKUP WITH SMOOTHLY VARYING PARAMETERS†

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The difference equations for cumulative beam breakup are converted to partial differential equations by identifying slowly varying parameters and neglecting rapidly varying terms. The equation thus obtained leads to a solution for constant parameters that is identical to the transient solution obtained earlier. Solutions are then obtained for slowly varying parameters as a perturbation expansion in powers of the rate of change of these parameters. Analytic results are presented for slowly varying energy, slowly varying charge per bunch, slowly varying deflecting-mode frequency (such as might occur in a relativistic proton linac), and smooth external focusing. Smooth variation of the charge per bunch will suppress the beam-breakup transient if there is a systematic initial displacement. Variation of deflecting-mode frequency will also depress the beam-breakup transient by increasing the value of $1/Q$ to that corresponding to the relative width of the frequency variation. The presence of focusing also depresses the beam breakup, and a threshold parameter for this reduction is presented. All analytic results are shown to be in close agreement with those obtained using simulations.

1. INTRODUCTION

The theory of beam breakup[1–5] has been worked out for identical uncoupled cavities and for a constant beam current. The difference equations for transverse displacement and angle have been solved exactly for a coasting beam, and expressions have been obtained for the steady-state solution where the input beam displacement is constant or modulated at an arbitrary frequency.[5] In addition, an approximate result was obtained for the transient in the absence of external focusing by means of a saddle-point approximation. These results were shown to be in excellent agreement with the numerical simulations for parameters appropriate to a 30-cavity 1300-MHz standing-wave rf linear accelerator structure with a 2.5-MeV, 6.5-A average-current coasting beam.[5]

There are a variety of circumstances under which the parameters may vary

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smoothly with cavity number N and/or bunch number M. These include:

(a) the increase in energy due to acceleration of the beam. This effect has been analyzed by approximating the difference equations by a differential equation that was solved for adiabatic variation of the energy with N.[1,4,6]

(b) the smooth increase of charge per bunch to its final value. The differential-equation method was used to explore this effect,[7] and it was found that the transient for a beam with constant input displacement was greatly reduced, even for relatively rapid charge buildup.

There are additional circumstances under which parameters might be expected to vary slowly and smoothly. These are treated in the present paper and include:

(c) the variation in deflecting-mode frequency with cavity number N. In a relativistic proton linac, for example, the fact that the velocity is not exactly equal to the velocity of light can result in a variation of the deflecting-mode frequency with N due to a systematic variation in cavity geometry.

(d) the presence of external transverse focusing whose strength may vary with energy.

In the present paper the method of treating smooth variation of parameters will be outlined in detail and applied to the above examples.

Notation

The notation is consistent with that used in Ref. 5. Additional symbols are defined as follows:

$$\alpha = \omega\tau + \frac{i\omega\tau}{2Q}$$

$y(N, M) = (L_N/\gamma_N)^{1/2}\xi(N, M)$

$w(N, M)$ = displacement function with rapidly oscillating term removed, defined in Eq. (2.8)

$z(N, M)$ = cavity-excitation function, with rapidly oscillating term removed, defined in Eq. (2.10)

R_{NM} = value of R for bunch M in cavity N

$p(M)r(N)$ = assumed factorization of $L_N R_{NM}/\gamma_N$

$f(M)g(N)$ = assumed factorization of the exponent in transient beam breakup, not including the term involving Q

E, M = generalization of E, M to include varying parameters

K = modification factor for beam breakup with constant initial beam displacement relative to beam breakup with a single displaced beam pulse

$\hat{K}$ = modification of K due to smooth buildup of charge

$w_T(M)$ = function defined in Eq. (5.15), constructed from the simulation, to test the validity of the analysis

ρ, ρ_3, ρ_4 = scaling parameter for beam breakup with external focussing, defined in Eq. (7.31).

2. DIFFERENCE AND DIFFERENTIAL EQUATIONS

The difference equations for the transverse displacement $\xi(N, M)$ and angle $\theta(N, M)$ can be written as

$$\xi(N+1, M) = M_{11}\xi(N, M) + \frac{\gamma_N M_{12}}{\gamma_{N+1}}[\theta(N, M) + \phi(N, M)], \tag{2.1}$$

$$\theta(N+1, M) = M_{21}\xi(N, M) + \frac{\gamma_N M_{22}}{\gamma_{N+1}}[\theta(N, M) + \phi(N, M)], \tag{2.2}$$

where

$$\phi(N, M) = \frac{1}{\gamma_N}\sum_{l=0}^{M-1} s_{M-l}R_{Nl}\xi(N, l), \tag{2.3}$$

$$s_k = \frac{1}{2i}(e^{ik\alpha} - e^{-ik\alpha^*}), \qquad \alpha = \omega\tau + \frac{i\omega\tau}{2Q}. \tag{2.4}$$

Here $\omega/2\pi$ and Q are the frequency and quality factor of the deflecting mode, τ is the time interval between bunches, $mc^2\gamma_N$ is the particle energy at cavity N, and R_{Nl} is a parameter proportional to the charge in the lth bunch and to the ratio of the shunt impedance to the Q of the Nth cavity. The 2×2 matrix M_{ij}, representing the transport and focusing between cavities, will be treated in the "smooth" approximation, even though a more precise treatment using the Courant–Snyder parameters for alternating-gradient focusing is not difficult. In this approximation, one can set

$$M_{11} = M_{22} \cong \cos\mu_N \cong 1 - \frac{\mu_N^2}{2}, \qquad \mu_N \ll 1, \tag{2.5}$$

and $M_{12} = L_N$, the separation between cavities N and $N+1$. Removing the adiabatic damping by setting

$$(\gamma_N/L_N)^{1/2}\xi(N, M) = y(N, M), \tag{2.6}$$

one can write

$$y(N+1, M) - 2\cos\mu_N y(N, M) + y(N-1, M) \cong \sum_{l=0}^{M-1}\frac{L_N R_{Nl}}{2i\gamma_N}[e^{i(M-l)\alpha} - e^{-i(M-l)\alpha^*}]y(N, l),$$

which, for smooth variation with N, reduces to

$$\frac{\partial^2 y}{\partial N^2} + \mu_N^2 y \cong \sum_{l=0}^{M-1}\frac{L_N R_{Nl}}{2i\gamma_N}[e^{i(M-l)\alpha} - e^{-i(M-l)\alpha^*}]y(N, l). \tag{2.7}$$

If we now recognize that $y(N, M)$ will vary like $\exp(\pm iM\alpha)$ for large M, we can go to a phase-amplitude form

$$y(N, M) = \frac{e^{iM\alpha}w(N, M) + e^{-iM\alpha^*}w^*(N, M)}{2}, \tag{2.8}$$

including the possibility of a slow variation of frequency (or α) with N. Neglecting rapidly oscillating terms, we obtain a final pair of equations for the gently varying functions $w(N, M)$ and $z(N, M)$, namely

$$e^{-iM\alpha}\frac{\partial^2}{\partial N^2}(we^{iM\alpha}) + \mu_N^2 w \cong \frac{z}{i}, \qquad \frac{\partial z}{\partial M} \cong \frac{L_N R_{NM}}{2\gamma_N} w, \tag{2.9}$$

where

$$z(N, M) \equiv (L_N/2\gamma_N) \sum_{l=0}^{M-1} R_{Nl} w(N, l) \tag{2.10}$$

is approximated as an integral over l. This will be our starting point for studying the effect of smoothly varying parameters.

3. CONSTANT FREQUENCY, NO FOCUSING, FACTORABLE $R(N, M)$

In the present section we will assume a constant deflecting-mode frequency ($\partial\alpha/\partial N = 0$), and we will neglect focusing ($\mu_N = 0$). If we assume that the energy and transverse shunt impedance may vary smoothly with cavity number N, and that the charge per bunch may vary smoothly with bunch number M, we can use the factored form for LR/γ, namely

$$\frac{L_N R_{NM}}{\gamma_N} = p(M)r(N), \tag{3.1}$$

where $p(M)$ is normalized to 1 as $M \to \infty$. We shall try an exponentional form for $w(N, M)$, namely

$$w(N, M) \sim \exp[f(M)g(N)]. \tag{3.2}$$

In all derivatives of w we will only take those associated with the exponent. This leads readily to

$$f^2 \frac{df}{dM}\left(\frac{dg}{dN}\right)^2 g \cong \frac{1}{2i} p(M)r(N). \tag{3.3}$$

Separation of variables between M and N permits us to write

$$3f^2 \frac{df}{dM} = p(M), \qquad \tfrac{3}{2}\sqrt{g}\frac{dg}{dN} = \tfrac{3}{2}\sqrt{3r(N)/2i}, \tag{3.4}$$

from which we quickly obtain

$$f(M) = \left[\int_0^M p(l)\, dl\right]^{1/3} \tag{3.5}$$

$$g(N) = \tfrac{3}{2}e^{-i\pi/6}\left[\int_0^N \sqrt{r(n)}\, dn\right]^{2/3}. \tag{3.6}$$

The analysis for constant $r(n) \equiv r$, and $p(l) = 1$ has already been carried through,[5,8] with the transient result for a single displaced bunch given by

$$\frac{\xi(N, M)}{\xi_0} \cong \frac{1}{M\sqrt{6\pi}} \mathrm{Re}\left\{\sqrt{E} \exp\left(-\frac{M\omega\tau}{2Q} + iM\omega\tau + \tfrac{3}{2}E\right)\right\}, \tag{3.7}$$

where

$$E = e^{-i\pi/6}(N/\sqrt{r})^{2/3}M^{1/3}. \tag{3.8}$$

For varying parameters, Eq. (3.7) will still be correct, provided one makes the replacements

$$\frac{\sqrt{E}}{M} \exp\left(\tfrac{3}{2}E\right) \to \frac{(\hat{E})^{1/2}}{\hat{M}} \exp\left(\tfrac{3}{2}\hat{E}\right), \tag{3.9}$$

where

$$\hat{M} = \int_0^M p(l)\, dl, \tag{3.10}$$

$$\hat{E} = e^{-i\pi/6}\hat{M}^{1/3}\left[\int_0^N \sqrt{r(n)}\, dn\right]^{2/3}. \tag{3.11}$$

The parameter M in the factor $\exp(-(M\omega\tau/2Q) + iM\omega\tau)$ is unchanged since it comes directly from the factor $\exp(iM\alpha)$ in Eq. (2.8).

4. ACCELERATION

In this section we shall use the general results of the previous section to analyze the effect of acceleration. We shall assume constant charge per bunch, for which $p(l) = 1$ and $f(M) = M^{1/3}$. If we include only the variation of γ with energy, we can write

$$r(n) = \frac{L_N R_N}{\gamma_N} = \frac{LR}{\gamma_i + \gamma' n}, \tag{4.1}$$

where the rate of energy gain corresponds to

$$\gamma' = \frac{\gamma_f - \gamma_i}{N}. \tag{4.2}$$

In this case the N dependent factor is

$$\int_0^N \sqrt{r(n)}\, dn = \frac{2\sqrt{LR}}{\gamma'}(\sqrt{\gamma_f} - \sqrt{\gamma_i}) = \frac{N\sqrt{LR}}{\langle\sqrt{\gamma}\rangle}, \tag{4.3}$$

where

$$\langle\sqrt{\gamma}\rangle = \frac{\sqrt{\gamma_f} + \sqrt{\gamma_i}}{2}. \tag{4.4}$$

Thus, the effect of a linear variation of γ with N leads to a replacement of γ in

Eqs. (3.7) and (3.8) by the square of the average value of $\sqrt{\gamma}$. The corresponding result for other dependence of $r(n)$ on n can readily be obtained from Eq. (3.6).

5. SMOOTH CURRENT BUILDUP

The analysis of the transient, leading to Eq. (3.7), assumes constant charge per bunch. If the charge per bunch were to vary smoothly from zero to its final value, it seems reasonable that the transient peak might be reduced, especially if the buildup time for the charge exceeds the time to reach the transient peak. As we shall see, for a constant initial displacement, the decrease in the peak will be reduced more dramatically, even for rapid buildup of the charge per bunch.

The most convenient analysis for a smooth buildup of charge per bunch starts with the alternative derivation of the beam-breakup transient.[8] After a Laplace-like transform in the variable N, one obtains a power-series expansion in the charge, in which the only significant term is the one in the sum of the phases of the oscillatory terms. The specific form, for constant $r(n) = LR/\gamma$ and no external focusing, is

$$\frac{\xi(N, M)}{\xi_0} = \mathrm{Re}\, e^{iM\alpha} \sum_{j=0}^{\infty} \frac{N^{2j}(LR/\gamma)^j e^{-ij(\pi/2)}}{2^{j-1}(2j)!} \underbrace{\sum_{\alpha>} \sum_{\beta>} \cdots \sum_{>\delta}}_{j-1 \text{ sums}} (1), \tag{5.1}$$

where $\alpha > \beta > \cdots > \delta$ are the ordered bunch numbers for the $j-1$ bunches, in addition to the 0th, which contribute to the jth power of N^2LR/γ. The sums are carried out over a function (1) that is symmetric in the interchange of any two of the indices and is readily shown to be

$$\sum_{\alpha>}^{M} \sum_{\beta} \cdots \sum_{\delta} (1) = \frac{1}{(j-1)!} \left[\sum_{\alpha=0}^{M} (1) \right]^{j-1} \cong \frac{M^{j-1}}{(j-1)!} \tag{5.2}$$

for a constant charge per bunch.

If the charge per bunch varies smoothly according to the function

$$p(l) = 1 - e^{-l/T}, \tag{5.3}$$

where T is the mean buildup "time," each of the factors R in Eq. (5.1) is now accompanied by the factor $p(l)$, $l = \alpha, \beta, \gamma, \ldots, \delta, 1$, where $l = 1$ designates the first bunch [earlier designated by $l = 0$, for which the form in Eq. (5.3) gives $p(0) = 0$]. The factors are again symmetric in the interchange of any two of the indices, and the sum is readily shown to be

$$p(1) \sum_{\alpha>}^{M} p(\alpha) \sum_{\beta>} p(\beta) \cdots \sum_{\delta} p(\delta) \cong \frac{\hat{M}^{j-1}}{(j-1)!}, \tag{5.4}$$

where $\hat{M}$ is defined in Eq. (3.10) and, for the form of $p(l)$ given in Eq. (5.3), is readily shown to be

$$\hat{M} \cong M - T \tag{5.5}$$

for $1 \ll T \ll M$. It is therefore clear from the factor $N^{2j}(LR/\gamma)^j \hat{M}^{j-1}$ that the final result will be a function of the variable $N^{2/3}(LR/\gamma)^{1/3}\hat{M}^{1/3}$ once an overall factor $\hat{M}^{-1}$ is removed. The result in Eq. (3.5) shows that the factor $\hat{E}$ in Eq. (3.11) contains a further replacement of $N(LR/\gamma)^{1/2}$ by the factor

$$N(LR/\gamma)^{1/2} \to \int_0^N \sqrt{r(n)}\, dn. \tag{5.6}$$

Note that M in the factor $e^{iM\alpha}$ in Eq. (5.1) remains unchanged. Our final expression for a single displaced bunch with a smooth current buildup is therefore

$$\xi(N, M) = \frac{p(1)\xi(1)}{\sqrt{6\pi\hat{M}}} \operatorname{Re}\left[(\hat{E})^{1/2} \exp\left(iM\alpha + \tfrac{3}{2}\hat{E}\right)\right], \tag{5.7}$$

where

$$\hat{E} = e^{-i\pi/6} G^{2/3}, \qquad G = \int_0^N \sqrt{r(n)}\, dn. \tag{5.8}$$

If the initial displacement of all bunches is ξ_0, the final displacement will be a sum of terms like Eq. (5.7), specifically

$$\frac{\xi(N, M)}{\xi_0} = \frac{\operatorname{Re}}{\sqrt{6\pi}} \sum_{m=0}^{M-1} \frac{p(m) G^{1/3} \exp\left[i(M-n)\alpha + \tfrac{3}{2} e^{-i\pi/6} G^{2/3} (\hat{M} - m)^{1/3} - i\pi/12\right]}{(\hat{M} - m)^{5/6}} \tag{5.9}$$

The rapidly oscillating factor $e^{-im\alpha}$ in Eq. (5.9) will restrict the significant contributions to the first several values of m, as long as the remaining factors are slowly varying smooth functions of m (which they are). In that case the transient peak will be modified from the single-pulse transient with constant charge per bunch by the factor

$$\hat{K} = p(0) + p(1)e^{i\alpha} + p(2)e^{2i\alpha} + \cdots. \tag{5.10}$$

For constnat $p(l) = 1$, the factor, assuming convergence, is

$$K = \frac{1}{1 - e^{i\alpha}}, \qquad |K| \cong \frac{1}{2\left|\sin\left(\dfrac{\omega\tau}{2}\right)\right|}, \tag{5.11}$$

as previously obtained[8] in the analysis of the approach to the steady state. For $p(l) = 1 - \exp(-l/T) \cong l/T$ for $l \ll T$, the factor $\hat{K}$ is

$$\hat{K} = \frac{1}{1 - e^{i\alpha}} - \frac{1}{1 - e^{i\alpha - 1/T}}, \qquad |\hat{K}| \cong |K|^2/T, \tag{5.12}$$

where the last form is valid for $T \gg 1$. This remarkable result for the overall factor shows that, away from resonance, where the factor $|K|$ is of order 1, there is a reduction in the one-pulse transient by a factor of order $1/T$. The corresponding transient displacement from the steady-state value is given for

$T \gg 1$ by

$$\left| \frac{\xi(N, M) - \xi(N, \infty)}{\xi_0} \right|_{\max} \cong \frac{|K|^2}{T} \frac{(\hat{E})^{1/2}}{\sqrt{6\pi}} \frac{1}{\hat{M}} \times \exp\left(-\frac{M\omega\tau}{2Q} + \frac{3\sqrt{3}}{4} |G^{2/3}| \hat{M}^{1/3} \right), \quad (5.13)$$

with $\hat{M} \cong M - T$ and

$$|\hat{E}| = |G^{2/3}| \hat{M}^{1/3} = \hat{M}^{1/3} \left[\int_0^N \sqrt{r(n)}\, dn \right]^{2/3}. \quad (5.14)$$

Thus, a buildup in charge per bunch with a "time constant" as short as $T = 10$ bunches will reduce the constant offset transient by an order of magnitude. The way in which the charge per bunch builds up will therefore be of particular significance in the presence of systematic initial displacement, such as might occur in the presence of steering errors.

The validity of Eq. (5.13) is clearly demonstrated in Fig. 1, taken from Ref. 7, where we have plotted the logarithm of the approach to steady state,

$$w_T(M) = \ln \left| \frac{\xi(N, M) - \xi(N, \infty)}{\xi_0} \right| + \tfrac{5}{6} \ln (M - T) + \frac{M\omega\tau}{2Q}, \quad (5.15)$$

against $(M - T)^{1/3}$ for the parameters of Ref. 5 and for $T = 0$, 5, 20, and 80. The

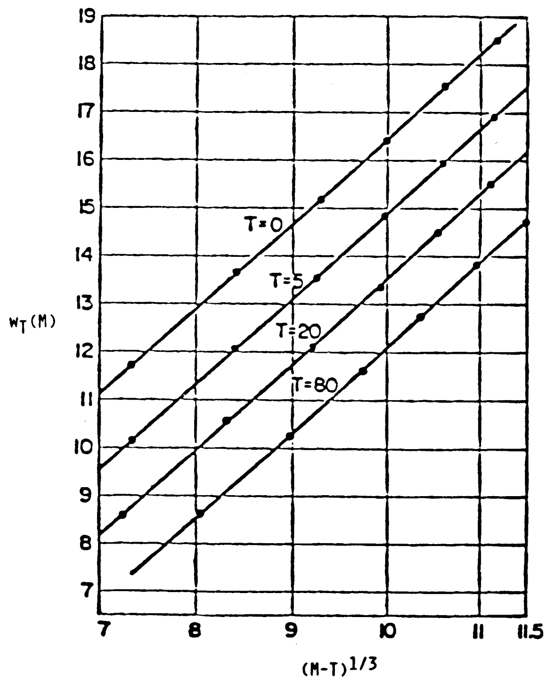

FIGURE 1 Plot of $w_T(M)$ vs $(M - T)^{1/3}$ for $T = 1$, 5, 20, and 80.

results are straight lines, as expected from Eq. (5.13), with the slope and intercept agreeing very closely with those predicted by Eq. (5.13).[7]

6. VARIATION OF DEFLECTING-MODE FREQUENCY (NO FOCUSING)

If one has a linac in which the variation of $\beta = v/c$ is not insignificant, the frequency of the deflecting mode can be expected to vary with γ (or N). In order to assess the importance of this variation in the parameter $\alpha = \omega\tau + i\omega\tau/2Q$, we will assume an exponential form for w and z, namely

$$w(N, M) \cong W \exp[q(N, M)], \qquad z \cong Z \exp[q(N, M)], \tag{6.1}$$

and neglect the variation of all parameters except the $q(N, M)$ in the exponent. This approximation is valid as long as $q(N, M) \gg 1$ in the region of interest. Thus, we have

$$e^{-iM\alpha}\frac{\partial^2}{\partial N^2}(we^{iM\alpha}) \cong W \exp[q(N, M)]\left(\frac{\partial q}{\partial N} + iM\frac{\partial \alpha}{\partial N}\right)^2 \tag{6.2}$$

and

$$\frac{\partial z}{\partial M} = Z\frac{\partial q}{\partial M}\exp[q(N, M)]. \tag{6.3}$$

Equation (2.9) then leads to

$$\left(\frac{\partial q}{\partial N} + iM\frac{\partial \alpha}{\partial N}\right)^2 \frac{\partial q}{\partial M} = \frac{r(N)}{2i}, \tag{6.4}$$

where we now assume R_{NM} is independent of M.

We shall see that solution to Eq. (6.3) can be written as the series

$$q(N, M) = q_0(N)M^{1/3} + q_1(N)M + q_2(N)M^{5/3} + \cdots, \tag{6.5}$$

where each successive term is proportional to one higher power of the small parameter $\alpha' \equiv d\alpha/dN$. We then find, from Eq. (6.4),

$$\begin{aligned} q_0'M^{1/3} &+ (q_1' + \alpha')M + q_2'M^{5/3} + \cdots \\ &= \sqrt{r(N)/2i}\left(\frac{q_0}{3M^{2/3}} + q_1 + \tfrac{5}{3}q_2M^{2/3} + \cdots\right)^{-1/2} \\ &= \sqrt{3r(N)/2i}[q_0^{-1/2}M^{1/3} - \tfrac{3}{2}q_0^{-3/2}q_1M \\ &\quad + (\tfrac{27}{8}q_0^{-5/2}q_1^2 - \tfrac{5}{2}q_0^{-3/2}q_2)M^{5/3} + \cdots]. \end{aligned}$$

Equating terms of the same power of M, we obtain

$$q_0^{1/2}q_0' = \sqrt{3r(N)/2i}, \qquad q_0^{3/2} = (\tfrac{3}{2})^{3/2}e^{-\pi i/4}\int_0^N dn\sqrt{r(n)}, \tag{6.6}$$

$$q_1' + i\alpha' = -(\tfrac{3}{2})^{3/2}q_0^{-3/2}e^{-\pi i/4}\sqrt{r(N)}q_1 = -\frac{G'}{G}q_1, \tag{6.7}$$

where

$$G(N) = \int_0^N dn \sqrt{r(n)} \tag{6.8}$$

and

$$\begin{aligned} q_2' &= e^{-i\pi/4}\sqrt{3r(N)/2}\left(\tfrac{27}{8}q_0^{-5/2}q_1^2 - \tfrac{5}{2}q_0^{-3/2}q_2\right) \\ &= \frac{9}{4}\frac{G'}{G}\frac{q_1^2}{q_0} - \frac{5}{3}\frac{G'}{G}q_2. \end{aligned} \tag{6.9}$$

From Eq. (6.7) we have

$$q_1 G = -i\int_0^N dn\alpha' G = i\int_0^N \Delta\alpha G'(n), \tag{6.10}$$

where we neglect the variation of the term $\omega\tau/2Q$ in α, and where we define $\Delta\alpha(n) = \alpha(n) - \alpha(N)$ such that $\Delta\alpha(N) = 0$.

The contribution of q_1 to the exponent $iM\alpha$ now gives

$$iM\left[\alpha(N) + \frac{\int_0^N dn\, \Delta\alpha G'(n)}{\int_0^N dnG'(n)}\right] = iM\frac{\int_0^N dn\alpha(n)G'(n)}{\int_0^N dnG'(n)}, \tag{6.11}$$

which corresponds to an average frequency determined from the weighting $G'(n)$ and therefore does not affect the amplitude of the transient.

We can now write Eq. (6.9) in the form

$$G^{5/3}q_2' + \tfrac{5}{3}G^{2/3}q_2 = \tfrac{3}{2}e^{i\pi/6}G'q_1^2, \tag{6.12}$$

leading to

$$\begin{aligned} \tfrac{2}{3}e^{-\pi i/6}G^{5/3}q_2 &= -\int_0^N dN'\frac{G'}{G^2}\left[\int_0^{N'} dn\alpha' G\right]^2 \\ &= \frac{\left[\int_0^N dn\alpha' G\right]^2}{G} - 2\int_0^N dN'\alpha'\int_0^{N'} dn\alpha' G \\ &= \frac{\left[\int_0^N dn\, \Delta\alpha G'\right]}{G} + \int_0^N 2\,\Delta\alpha\alpha' G. \end{aligned} \tag{6.13}$$

A final integration by parts in the last term leads to

$$G^{2/3}F_2 = -\tfrac{3}{2}e^{\pi i/6}\left\{\frac{\int_0^N (\Delta\alpha)^2 G'\, dn}{\int_0^N G'\, dn} - \left[\frac{\int_0^N \Delta\alpha G'\, dn}{\int_0^N G'\, dn}\right]2\right\}, \tag{6.14}$$

where the term in braces $\{\}$ is readily interpreted as the rms spread of the frequency

$$\Delta^2_{\rm rms} \equiv \langle(\Delta\alpha)^2\rangle - [\langle\Delta\alpha\rangle]^2, \tag{6.15}$$

weighted according to $G'(n)$.

Our final result for the real part of the exponent $q(N, M)$ is

$$\begin{aligned}\operatorname{Re} q(N, M) &= \operatorname{Re}\,[q_0 M^{1/3} + q_2 M^{5/3} + iM\alpha] \\ &= -\frac{\omega\tau}{2Q}M + \frac{3\sqrt{3}}{4}G^{2/3}M^{1/3} - \frac{3\sqrt{3}}{4}G^{-2/3}\,\Delta^2_{\rm rms}M^{5/3},\end{aligned} \tag{6.16}$$

where we have included the damping term proportional to Q^{-1}.

Equation (6.16) has been derived assuming smooth variation of ω_n and $r(n)$ with n. In Eq. (4.12) of the comparison paper,[8] we derived the result for constant $r(n)$ and arbitrary ω_n, using the approximate series summation method, and applied it to the case of random fluctuation of $\omega_n - \langle\omega\rangle$. The agreement between the two is surprisingly good and leads to the following conclusions:

(1) The replacement of

$$G = N(LR/\gamma)^{1/2} \quad \text{by} \quad \int_0^N \sqrt{r(n)}\,dn \tag{6.17}$$

is undoubtedly valid in Eq. (4.12) of the companion paper and all subsequent formulas involving the fluctuating frequency.

(2) The coefficient of $\exp q(N, M)$ is undoubtedly that in Eq. (3.7), namely $E^{1/2}/M\sqrt{6\pi}$, since that is the conclusion reached in Eqs. (2.5) and (4.10) of the companion paper.[8] Here we used the generalized form of E:

$$E = e^{-i\pi/6}M^{1/3}\left[\int_0^N \sqrt{r(n)}\,dn\right]^{2/3}. \tag{6.18}$$

The conclusions following Eq. (4.12) of the comparison paper now follow. Specifically, the exponent in Eq. (6.16) reaches a maximum value

$$\frac{\bar{p}_2}{\bar{p}} = \frac{2 + 3\lambda}{5}\left(\frac{1+\lambda}{2}\right)^{-3/2} \tag{6.19}$$

at

$$\frac{\bar{M}_2}{\bar{M}} = \left(\frac{1+\lambda}{2}\right)^{-3/2}, \tag{6.20}$$

where $\bar{p}$ and $\bar{M}$ are given in Eqs. (3.3) and (3.4) of the companion paper, and where

$$\lambda^2 = 1 + 15\varepsilon^2, \qquad \varepsilon^2 = Q^2\,\Delta^2_{\rm rms}/\omega^2\tau^2. \tag{6.21}$$

The subscript 2 here refers to the case of smooth variation of the deflecting-mode frequency. (Note that the coefficient of $\Delta^2_{\rm rms}$ in Eq. (6.16) is 3/4 that in Eq. (4.12) of the companion paper.)

We have checked the form of Eq. (6.16) by performing simulations for the parameters used earlier,[5] namely $\omega\tau/2\pi = 24/13$, $Q = 1000$, $r = 2.88 \times 10^{-3}$,

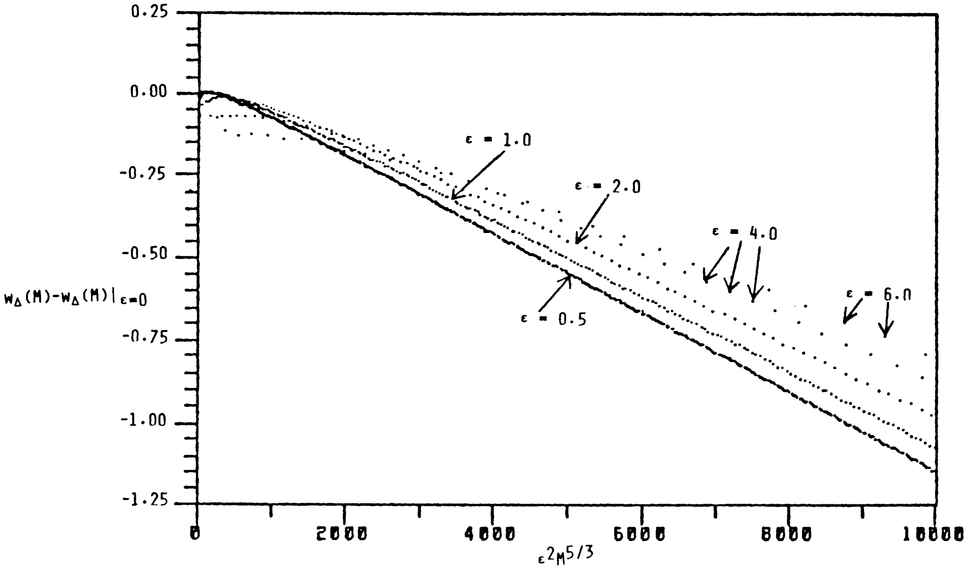

FIGURE 2 $w_\Delta(M) - w_\Delta|_{\varepsilon=0}$ vs $\varepsilon^2 M^{5/3}$ for smooth variation of cavity frequency given in Eq. (6.22).

$N = 30$, and a frequency error of the form

$$\Delta_n = \text{const}\,\frac{n^2(2n-N)(N-n)^2}{2N^5}, \tag{6.22}$$

which is chosen to be relatively independent of the cavities near $n = 0$ and $n = N$, and to have a zero average. In Fig. 2 we have plotted $w_\Delta(M) - w_\Delta(M)|_{\varepsilon=0}$ against $\varepsilon^2 M^{5/3}$, where

$$w_\Delta(M) = \ln\frac{|\xi_{\max}|}{\xi_0} + \tfrac{5}{6}\ln M + \frac{M\omega\tau}{2Q} - \frac{3\sqrt{3}}{4}G^{2/3}M^{1/3}. \tag{6.23}$$

The linear dependence confirms the form of Eq. (6.16), and the slope obtained for small ε is -1.2×10^{-4}, in good agreement with that predicted by Eqs. (6.16) and (6.21). A similar result is obtained from the simulation where the frequency variation is the negative of that in Eq. (6.22).

Let us complete this section by estimating Δ^2_{rms} for a systematic variation of $\Delta\alpha$ with n given by

$$\Delta\alpha(n) = \left[\left(\frac{N_i}{N_i+n}\right)^b - \left(\frac{N_i}{N_f}\right)^b\right]\delta\alpha, \tag{6.24}$$

where $N_f = N_i + N$. The total frequency change is $-\delta\alpha[1-(N_i/N_f)^b]$, where b is a dimensionless geometrical parameter related to the variation of cavity geometry with $\beta \cong 1-(1/2\gamma^2)$. If we assume linear dependence of energy on cavity number, we have

$$r(n) \cong r_i\left(\frac{N_i}{N_i+n}\right), \tag{6.25}$$

corresponding to

$$G'(n) = \text{const}\,(N_i + n)^{-1/2}. \tag{6.26}$$

The result for Δ^2_{rms} in Eqs. (6.14) and (6.15) can easily be obtained. For $N_f \gg N_i$ the final result is

$$\Delta^2_{\text{rms}} \cong \frac{(N_i/N_f)^{1/2}}{4b-1}(\delta\alpha)^2. \tag{6.27}$$

7. EXTERNAL TRANSVERSE FOCUSSING

We shall now explore the effect of external transverse focusing for constant deflecting-mode frequency. This will be done in two limits. The first is when the external transverse focusing force is weak compared to the forces causing beam breakup, and the second is when the beam-breakup forces are a small perturbation on the external focusing force. These limits will be made more quantitative as we proceed. We shall also assume a smooth focusing system. The analysis can readily be extended to quadrupole focusing by including the lattice beta functions if desired.

Our starting point is Eq. (2.9) in the form

$$\frac{\partial^2 w}{\partial N^2} + \mu^2 w = \frac{z}{i}, \qquad \frac{\partial z}{\partial M} = \frac{r}{2} w. \tag{7.1}$$

Using the exponential form $w = W \exp q(N, M)$, and considering only derivatives of the exponent, we obtain

$$\left(\left(\frac{\partial q}{\partial N}\right)^2 + \mu^2\right)\frac{\partial q}{\partial M} \cong \frac{r}{2i}, \tag{7.2}$$

where the terms in μ^2 will be considered as a small perturbation. If we now write

$$q(N, M) = q_0(N)M^{1/3} + q_3(N)M^{-1/3} + \cdots$$

and expand to first order in $q_3(N)$, we find

$$q_0^{1/2} q_0' = \sqrt{3r/2i}, \qquad q_0(N) = \tfrac{3}{2}e^{-i\pi/6}G^{2/3} \tag{7.3}$$

as before, with $G(N) = \int_0^N \sqrt{r}\,dn$, and

$$q_0^{-1/2} q_3' - \tfrac{1}{2} q_0^{-3/2} F_0' F_3 = -\frac{\mu^2}{2q_0^{1/2} q_0'} = -\frac{e^{i\pi/4}\mu^2}{\sqrt{6r}}. \tag{7.4}$$

(The subscript 3 here refers to the case of weak external focusing.) This can readily be integrated to obtain

$$q_3 = -q_0^{1/2}\frac{e^{i\pi/4}}{\sqrt{6}}\int_0^N \frac{\mu^2}{\sqrt{r}}\,dn = -\frac{e^{i\pi/6}}{2}G^{1/3}\int_0^N \frac{\mu^2}{\sqrt{r}}\,dn. \tag{7.5}$$

As a result, the displacement can be written as

$$\frac{\xi(N, M)}{\xi_0} \cong \frac{1}{M\sqrt{6\pi}} \operatorname{Re}\left[e^{-i\pi/12}(MG^2)^{1/6}\right.$$
$$\left.\times \exp\left[iM\omega\tau - \frac{M\omega\tau}{2Q} + \tfrac{3}{2}e^{-i\pi/6}(MG^2)^{1/3} - e^{i\pi/6}(G/M)^{-1/3}H_3/2\right], \quad (7.6)$$

where

$$H_3(N) = \int_0^N \frac{\mu^2}{\sqrt{r}}\, dn. \quad (7.7)$$

The real part of the exponent,

$$p_3(N, M) \cong -\frac{M\omega\tau}{2Q} + \frac{3\sqrt{3}}{4} M^{1/3}G^{2/3} - \frac{\sqrt{3}}{4} M^{-1/3}G^{1/3}H_3, \quad (7.8)$$

reaches a maximum value

$$\frac{\bar{p}_3}{\bar{p}} \cong (2 - \sqrt{1+\rho})\left(\frac{\sqrt{1+\rho}+1}{2}\right)^{1/2} \quad (7.9)$$

at

$$\frac{\bar{M}_3}{\bar{M}} = \left(\frac{\sqrt{1+\rho}+1}{2}\right)^{3/2}, \quad (7.10)$$

where

$$\rho(N) = (\tfrac{4}{3})^{3/2} \frac{H_3(N)}{G(N)} \frac{\omega\tau}{Q}, \quad (7.11)$$

and where $\bar{M}$ and $\bar{p}$ are the location and maximum value of the exponent in the absence of focusing, namely

$$\bar{M} = (\tfrac{3}{4})^{3/4}\left(\frac{Q}{\omega\tau}\right)^{3/2} G, \qquad \bar{p} = \bar{M}\left(\frac{\omega\tau}{Q}\right). \quad (7.12)$$

It now appears that the expansion parameter is ρ, and that our approximation is valid for $\rho \leqslant 1$. It also appears that a value of $\rho \sim 1$ is necessary to reduce the maximum transient exponent significantly. To estimate the corresponding value of μ, we assume constant μ^2/r to obtain

$$\rho \sim (\tfrac{4}{3})^{3/2} \frac{\mu^2}{r} \frac{\omega\tau}{Q} = \left(\frac{\mu N}{\bar{p}}\right)^2. \quad (7.13)$$

Thus, if the maximum exponent in the absence of focusing is ~12 for N cavities, one needs a focusing system that produces ~12 radians of transverse oscillation in N cavities to reduce the transient exponent to approximately 65% of its value in the absence of focusing.

At the opposite extreme, we consider the beam-breakup forces to be a perturbation on the transverse focusing. In this limit the solution of Eq. (7.2) takes the form

$$q(N, M) \cong i\int_0^N \mu\, dn + M^{1/2}q_4(N) + \cdots, \quad (7.14)$$

leading to

$$[\mu^2 + (i\mu + M^{1/2}q_4')^2]\frac{q_4(N)}{2\sqrt{M}} \cong \frac{r}{2i}. \tag{7.15}$$

(The subscript 4 here refers to the case of weak beam breakup in the presence of external focusing.) To lowest order in F_4, this leads to

$$2q_4q_4' \cong \frac{r}{\mu}, \tag{7.16}$$

which can be integrated to obtain

$$F_4(N) = \sqrt{H_4(N)}, \qquad H_4(N) = \int_0^N \frac{r(n)}{\mu(n)}\,dn. \tag{7.17}$$

The real part of the exponent then is

$$p_4(N, M) = -\frac{M\omega\tau}{2Q} + M^{1/2}H_4^{1/2}, \tag{7.18}$$

which reaches a maximum value

$$\bar{p}_4 = \frac{H_4 Q}{2\omega\tau} \tag{7.19}$$

at

$$\bar{M}_4 = H_4\left(\frac{Q}{\omega\tau}\right)^2. \tag{7.20}$$

An alternative derivation of Eq. (7.18) and the corresponding displacement is useful at this point. We shall start with the integral representation[5] in the form

$$\frac{\xi(N, M)}{\xi_0} = \frac{e^{-M\omega\tau/2Q}}{2\pi}\,\mathrm{Re}\int_{-\pi}^{\pi} d\theta \exp(iM\theta + N\sigma), \tag{7.21}$$

where

$$\cosh\sigma(\theta) = \cos\mu + \frac{r\sin\omega\tau}{4(\cos\theta - \cos\omega\tau)}. \tag{7.22}$$

The derivation of Eq. (7.21) requires the contour to pass above the singularities at $\theta = \pm\omega\tau$. For small r we can approximate $\sigma(\theta)$ by

$$\sigma(\theta) \cong i\left(\mu \pm \frac{r}{4\sin\mu(\theta \mp \omega\tau)}\right) \tag{7.23}$$

near the saddle points of Eq. (7.21). The result for the integration through the saddle points at $\theta_s = \pm\omega\tau + i(Nr/4M\sin\mu)^{1/2}$ then turns out to be

$$\frac{\xi(N, M)}{\xi_0} \cong \frac{1}{M\sqrt{8\pi}}\,\mathrm{Re}\left[(M\hat{H}_4)^{1/4}\exp\left[-\frac{M\omega\tau}{2Q} - iM\omega\tau + iN\mu + (M\hat{H}_4)^{1/2}\right],\right. \tag{7.24}$$

where

$$\hat{H}_4 = \frac{rN}{\sin \mu}. \tag{7.25}$$

The similarity between Eqs. (7.25) and (7.17) is clear. For greatest versatility, particularly for $\sin \mu$ near 1, one should use

$$H_4(N) = \hat{H}_4(N) = \int_0^N \frac{r(n)}{\sin[\mu(n)]} dn \tag{7.26}$$

in Eqs. (7.18) to (7.20). The similarity between Eq. (7.24) and Eq. (3.7) is also striking. Apart from factors that relate to the multiplicity of the saddles, E in Eq. (3.7) is replaced by $M^{1/2}H_4^{1/2}$ in Eq. (7.24).

The result in Eq. (7.24) has also been obtained by Yokoya.[9] He starts with the Laplace transform of the displacement with respect to time (bunch number) and derives a differential equation that includes both external transverse focusing and the beam-breakup forces. He then obtains Eq. (7.24) as the inverse transform of a WKB solution of the differential equation, assuming, as we have, that the beam-breakup force is small compared to the focusing force.

What remains is to tie the approximate results in Eqs. (7.9) through (7.12) to those in Eqs. (7.19) and (7.20). This can be most readily done by writing

$$\frac{\bar{p}_4}{\bar{p}} \cong \frac{1}{1.6} \frac{H_4}{G} \left(\frac{Q}{\omega\tau}\right)^{1/2}, \qquad \frac{\bar{M}_4}{\bar{M}} \cong \frac{1}{0.8} \frac{H_4}{G} \left(\frac{Q}{\omega\tau}\right)^{1/2}. \tag{7.27}$$

Assuming constant r and $\sin \mu \cong \mu$, one finds

$$\frac{H_4}{G} \left(\frac{Q}{\omega\tau}\right)^{1/2} \cong \frac{\sqrt{r}}{\mu} \left(\frac{Q}{\omega\tau}\right)^{1/2} = (\tfrac{4}{3})^{3/4} \rho^{-1/2}. \tag{7.28}$$

One then expects a transition as ρ increases from

$$\frac{\bar{p}_3}{\bar{p}} \cong (2 - \sqrt{1+\rho_3}) \left(\frac{\sqrt{1+\rho_3}+1}{2}\right)^{1/2}, \qquad \frac{\bar{M}_3}{\bar{M}} = \left(\frac{\sqrt{1+\rho_3}+1}{2}\right)^{3/2} \tag{7.29}$$

for $\rho_3 \leqslant 1$ to

$$\frac{\bar{p}_4}{\bar{p}} \cong 0.8 \rho_4^{-1/2}, \qquad \frac{\bar{M}_4}{\bar{M}} \cong 1.6 \rho_4^{-1/2} \tag{7.30}$$

for $\rho_4 \geqslant 1$. Here

$$\rho_3 = (\tfrac{4}{3})^{3/2} \frac{H_3}{G} \frac{\omega\tau}{Q}, \qquad \rho_4 = (\tfrac{4}{3})^{3/2} \left(\frac{G}{H_4}\right)^2 \frac{\omega\tau}{Q}, \tag{7.31}$$

with

$$G(N) = \int_0^N \sqrt{r(n)}\, dn, \qquad H_3(N) = \int_0^N \frac{\mu^2(n)}{\sqrt{r(n)}} dn, \qquad H_4(N) = \int_0^N \frac{r(n)}{\sin[\mu(n)]} dn. \tag{7.32}$$

For μ and r independent of n, one has the universal scaling parameter

$$\rho = \rho_3 = \rho_4 = \frac{8}{3\sqrt{3}} \frac{\omega\tau}{Q} \frac{\sin^2\mu}{r}. \tag{7.33}$$

If μ and/or r depend on n, there may be different regions in n in which each of the different approximations leading to Eqs. (7.6) and (7.24) is separately valid. In this case, some sort of hybrid solution is needed, but we shall not try to construct one. Rather we shall assume constant μ^2/r and use numerical simulations to explore the transition between regions.

We have performed simulations for values of sin μ between 0 and 1 with the parameters used earlier,[5] namely $\omega\tau/2\pi = 24/13$, $Q = 1000$, $r = 2.88 \times 10^{-3}$, and $N = 30$. The bunch number $M_{\max}$ at which the transient peak occurs is plotted in Fig. 3 as a function of $\sin^2\mu$. In addition, we show $\bar{M}_3$ and $\bar{M}_4$ from Eqs. (7.29) and (7.27), the low- and high-ρ approximations. Figure 4 contains the corresponding values of the maximum exponent (actually $\ln(|\xi_{\max}|/\xi_0)$ is used from the simulations) as a function of $\sin^2\mu$. The results show clearly the smooth transition from low to high ρ and confirm the validity of the use of the scaling parameter ρ in Eq. (7.33) as a measure of the focusing necessary to reduce beam breakup significantly.

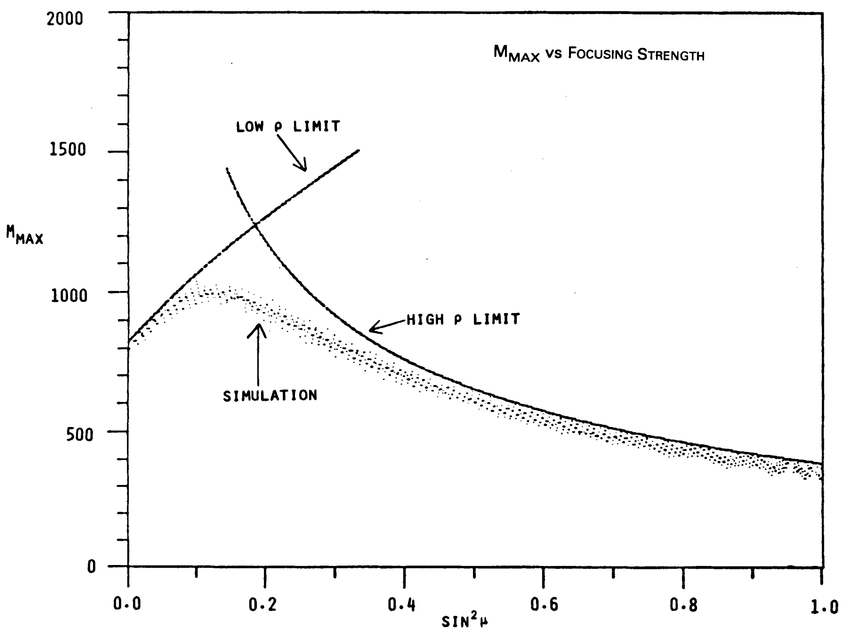

FIGURE 3 $M_{\max}$ vs $\sin^2\mu$ for $\omega\tau/2\pi = 24/13$, $r = 2.88 \times 10^{-3}$, $Q = 1000$, $N = 30$.

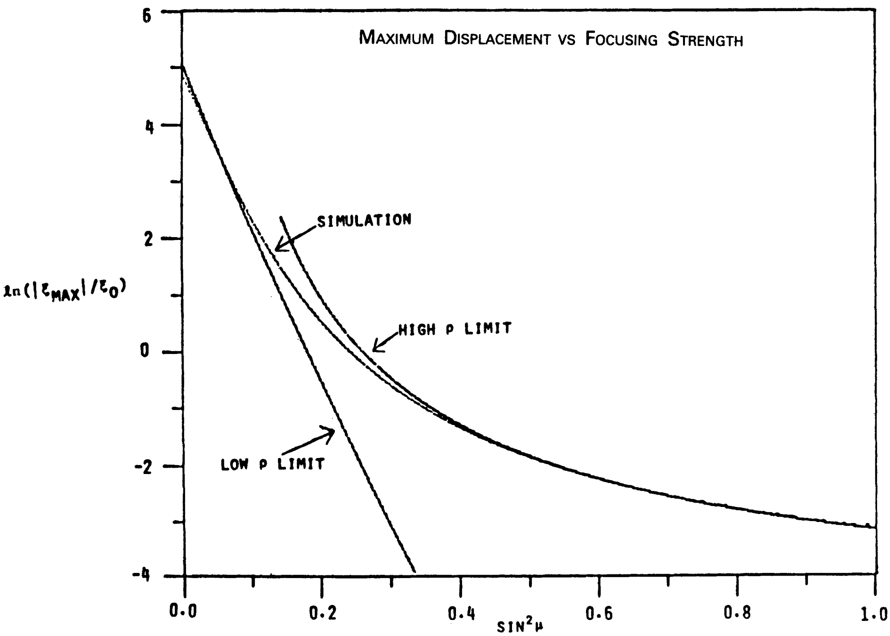

FIGURE 4 Logarithm of maximum displacement vs $\sin^2\mu$ for $\omega\tau/2\pi = 24/13$, $r = 2.88 \times 10^{-3}$, $Q = 1000$, $N = 30$.

8. DISCUSSION

We have started with the difference equations for beam breakup and converted them to two coupled differential equations for the amplitude of the displacement transient and the amplitude of the cavity-excitation fields, using bunch number and cavity numbers as the continuous parameters. The parameters (charge per bunch, deflecting-mode frequency, external focusing strength, $Z_\perp/Q$ for each cavity, and energy γ) are permitted to vary slowly with M or N. We then assume an exponential form for the displacement and excitation-field amplitudes and determine the adiabatic dependence of the exponent on M and N. This analysis yields the correct result in Eq. (3.8) for constant parameters, as expected.

We then briefly address an accelerating beam, and show that a linear variation of γ with N requires replacing the γ in the coasting-beam result by $(\sqrt{\gamma_i} + \sqrt{\gamma_f})^2/4$. The analysis is then applied to a beam in which the charge per bunch builds up rapidly, and we show in Eqs. (5.13) and (5.15) that for an offset beam, the transient approach to equilibrium is strongly damped, even for rapid buildup of charge per bunch. This reduction should therefore be pronounced for a beam with steering errors.

The analysis is then applied to the case of a systematic variation of deflecting-mode frequency with N, such as might occur in a linac that is not yet in the extreme relativistic range. The exponent is expanded to second order in the frequency change, and the additional term in the exponent is shown in Eq. (6.16) to be proportional to the rms width of the frequency variation, in close agreement with the formal result obtained for a randomly fluctuating frequency. The weighting for the calculation for the rms width is shown in Eqs. (6.14) and (6.15) to be $[r(n)]^{1/2}$.

Finally, we explore the effect of external focusing in the two limits where this force is first smaller and then larger than the beam-breakup "defocusing force," and we show in Eqs. (7.8)–(7.11), (7.18)–(7.20), and (7.27)–(7.32) how to estimate this effect on the transition region between these limits.

In summary, we have obtained expressions using the adiabatic approximation in this paper, the transient analysis for a coasting beam and constant parameters,[5] and the approximate summation method of the companion paper.[8] By comparing results using these different methods, we are able to generalize many of the results to regions where the original derivations are not rigorous. Moreover, the simulations show that these generalizations are eminently believable and that the final results can be applied or adapted to a very wide variety of circumstances and conditions.

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