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CUMULATIVE BEAM BREAKUP WITH RANDOMLY FLUCTUATING PARAMETERS†

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An alternate derivation of the transient solution for cumulative beam breakup is presented. This solution, which is identical to the solution derived earlier for constant parameters, permits the analysis for randomly fluctuating parameters. Analytical results are obtained for random initial displacement, fluctuating charge per bunch, and for a distribution of deflecting-mode frequencies such as might occur because of construction tolerances. These results are shown to be in close agreement with simulations. Fluctuating charge per bunch will enhance beam breakup if the beam has a systematic initial displacement. The distribution of deflecting-mode frequencies helps to suppress beam breakup, especially for cavities with large Q. In fact the relative width of the distribution of frequencies corresponds to a reduced value of Q, as might have been expected.

1. INTRODUCTION

The theory of beam breakup[1–5] has been worked out for identical uncoupled cavities and for a constant beam current. The difference equations for transverse displacement and angle have been solved exactly for a coasting beam, and expressions have been obtained for the steady-state solution where the input-beam displacement is constant or modulated at an arbitrary frequency.[5] In addition, an approximate result was obtained for the transient in the absence of external focusing by means of a saddle-point approximation. These results were shown to be in excellent agreement with the numerical simulations for parameters appropriate to a 30-cavity 1300-MHz standing-wave rf linear-accelerator structure with a 2.5-MeV, 6.5-A average-current coasting beam.[5]

The dominant feature of the transient result is amplification of the transverse displacement in the absence of external focusing, corresponding to the real exponent‡

$$e_t = \frac{3\sqrt{3}}{4} r^{1/3} N^{2/3} M^{1/3}, \tag{1.1}$$

† Supported by the U.S. Department of Energy.
‡ This is the result in Eq. (71) of Ref. 5.

where N is the cavity number and M is the bunch number. Here

$$r \equiv \frac{LR}{\gamma} = \frac{eI c\tau}{2W}\left(\frac{Z_\perp T^2}{LQ}\right)L^2, \tag{1.2}$$

where I is the beam current for particles of energy W, L is the distance between cavity centers, τ^{-1} is the bunch frequency, and

$$\frac{Z_\perp T^2}{LQ} = \frac{2cZ_0}{\omega}\frac{\left|\int \frac{\partial E}{\partial x} e^{-i\omega z/c}\, dz\right|^2}{L\int E^2\, dv} \tag{1.3}$$

is the usual ratio of transverse shunt impedance to quality factor Q. Here $Z_0 = 120\pi$ ohms is the impedance of free space. The form of Eq. (1.3) is used to make explicit the linear dependence of $Z_\perp$ on cavity length. We assume no space between the cavities, which are each of length L.

It can be shown[5] that for an accelerated beam, one can replace W in Eq. (1.2) by

$$W \to \frac{W_f}{4} = \frac{W'NL}{4}, \tag{1.4}$$

where the final and initial energies satisfy $W_f \gg W_i$, and where W' is the rate of energy gain per meter. Our result for the exponent in Eq. (1.1) can then be written as

$$e_t \cong \frac{3^{3/2}}{2^{5/3}}\left(\frac{ecZ_\perp T^2}{LQW'}\right)^{1/3}(Izt)^{1/3}, \tag{1.5}$$

where $z = NL$ is the accelerator length and $t = M\tau$ is the pulse duration.

Equation (1.5) corresponds to the expression obtained at SLAC for a traveling-wave linac. Experimental studies[1,2] of the dependence of starting current on pulse duration and accelerator length give approximate confirmation of the form of Eq. (1.5), with the conclusion that beam breakup occurs when e_t is between 15 and 20, corresponding to an amplification of some stimulus by a factor of order 10^7 to 10^8. Efforts to identify the initial noise stimulus have been inconclusive.

It is apparent that the large amplification in Eqs. (1.1) or (1.5) depends on coherent oscillations of all the cavities. A distribution in frequency of the transverse deflecting mode in the cavities, which would be expected from construction tolerances, is expected to lower the effective exponent. In fact, SLAC increased the starting currents significantly by intentionally detuning many of the cavities.[1,2] What is not clear is whether the initial construction tolerances were large enough to modify the result in Eq. (1.5) and thereby imply a reduced amplification of a larger initial stimulus.

In the present paper, we derive the effect of a distribution in transverse frequencies. In the process we treat random variation of other parameters, such as the initial displacement and the charge per beam bunch as well. In order to take into account random variation of parameters, we find it necessary to rederive the transient result using an approximate summation technique, since the analytic solution obtained earlier is only valid for constant transverse frequency and constant charge per bunch.

1.1. Notation

The notation is consistent with that used in Ref. 5. Additional symbols are defined as follows:

$r = M_{12}R/\gamma$
$M_{12} = L$ in the absence of external focusing.
$e_t =$ the real part of the exponent for transient beam breakup, not including the term involving Q
$\phi(N, M) =$ the angular deflection experienced by the M bunch due to the excitation of the N cavity
$\sigma =$ rms width of the transient pulse amplitude
$p, p(M) =$ the real part of the total exponent for transient beam breakup
$\bar{M} =$ value of M at which $p(M)$ is a maximum
$\bar{p} =$ maximum value of $p(M)$
$\Delta_n, \Delta(n) =$ deviation of $\omega\tau$ in cavity n from its average value
$G = Nr^{1/2}$
$\varepsilon = Q\Delta_{\rm rms}/\omega\tau$
$w(M), w_1(M) =$ function defined in Eq. (4.18), constructed from the simulation, to test the validity of the analysis
$\delta_m =$ relative charge fluctuation of bunch m
$\zeta(\delta) =$ correction factor to the maximum of the single pulse transient due to charge fluctuation

The following typographical corrections to Ref. 5 are noted:

Eq. (52) and Eq. (B-16) $\quad -\dfrac{i}{2\pi} \to \dfrac{1}{2\pi}$

Eq. (52) and Eq. (B-16) $\quad \sinh N\sigma \to \dfrac{\sinh N\sigma}{\sinh \sigma}$

Eq. (64) $\quad N \to \dfrac{N}{\sqrt{4\pi}}$

Eq. (A-4) $\quad (N-r)! \to r!\,(N-r)!$

Eq. (B-14) $\quad -\dfrac{1}{2\pi} \to \dfrac{1}{2\pi i}$

Eq. (B-14) $\quad \sinh(N\sigma(u)) \to \dfrac{\sinh(N\sigma(u))}{\sinh(\sigma(u))}$

2. ALTERNATIVE ANALYSIS OF TRANSIENT GROWTH

Our starting point for a coasting beam is the difference equations[5]

$$\xi(N+1, M) = M_{11}\xi(N, M) + M_{12}[\theta(N, M) + \phi(N, M)], \tag{2.1}$$

$$\theta(N+1, M) = M_{21}\xi(N, M) + M_{22}[\theta(N, M) + \phi(N, M)], \tag{2.2}$$

where $\xi(N, M)$ and $\theta(N, M)$ are the displacement and angle of the Mth bunch as it enters the Nth cavity and

$$\phi(N, M) = \frac{R}{\gamma}\sum_{l=0}^{M-1} s_{M-l}\xi(N, l) \tag{2.3}$$

is the angular deflection experienced by the Mth bunch in traversing the Nth cavity. Here

$$s_k = e^{-k\omega\tau/2Q}\sin k\omega\tau. \tag{2.4}$$

As in Ref. 5, we obtain a solution in powers of $r = M_{12}R/\gamma$ by analogy to the Laplace transform in the variable N. The result for a single initially displaced bunch ($M' = 0$), in the absence of transverse focusing ($\mu = 0$, $M_{12} = L$), is[6]

$$\frac{\xi(N, M)}{\xi_0} = \sum_{j=0}^{N} (S^j)_{M0} r^j \frac{(N+j)!}{(N-j)!\,(2j)!}, \tag{2.5}$$

where the matrix element S_{pq} is given by

$$S_{pq} = s_{p-q} = \begin{Bmatrix} e^{-(p-q)\omega\tau/2Q}\sin[(p-q)\omega\tau], & p > q \\ 0, & p \leqslant q \end{Bmatrix}. \tag{2.6}$$

The M0 element of S^j is then given by

$$\begin{aligned}(S^j)_{M0} &= \sum_{\alpha>\beta>\cdots>\delta}^{M}\sum^{M}\cdots\sum^{M} S_{M\alpha}S_{\alpha\beta}\cdots S_{\delta 0} \\ &= e^{-M\omega\tau/2Q}\sum_{\alpha>\beta>\cdots>\delta}^{M}\sum\cdots\sum \sin[(M-\alpha)\omega\tau]\sin[(\alpha-\beta)\omega\tau]\cdots\sin[\delta\omega\tau].\end{aligned} \tag{2.7}$$

The first two terms in the product can be combined to give

$$\tfrac{1}{2}\cos[(M+\beta-2\alpha)\omega\tau] - \tfrac{1}{2}\cos[(M-\beta)\omega\tau].$$

Subsequent summation over α for $M \gg 1$ will cause the first term to average to zero while the second term accumulates. Accordingly, we expect the only surviving term, after trigonometric combination of the factors in Eq. (2.7), to be the one independent of $\alpha, \beta, \ldots, \delta$, corresponding to the *sum* of all phase angles, specifically

$$\begin{aligned}(S^j)_{M0} &\cong \frac{e^{-M\omega\tau/2Q}\cos(M\omega\tau - j(\pi/2))}{2^{j-1}}\sum_{\alpha>\beta>\cdots>\delta}^{M}\sum\cdots\sum(1) \\ &\cong \mathrm{Re}\,\frac{e^{-(M\omega\tau/2Q)+iM\omega\tau-ij(\pi/2)}M^{j-1}}{2^{j-1}(j-1)!}.\end{aligned} \tag{2.8}$$

For the parameters of interest, the dominant terms in the sum in Eq. (2.5) come from values of j that satisfy

$$1 \ll j \ll N \ll M. \tag{2.9}$$

Equation (2.5) can therefore be written as

$$\frac{\xi(N, M)}{\xi_0} \cong \frac{e^{-M\omega\tau/2Q}}{M} \operatorname{Re} e^{iM\omega\tau} \sum_{j=0}^{\infty} \frac{E^{3j}}{2^{j-1}(j-1)!\,(2j)!}, \tag{2.10}$$

where

$$E = r^{1/3} M^{1/3} N^{2/3} e^{-i\pi/6}. \tag{2.11}$$

The sum in Eq. (2.10) cannot be readily evaluated as it stands. However, since the important contributions come from $j \gg 1$, we can use Stirling's approximation to show that

$$\frac{(3j)!}{2^{j-1}(j-1)!\,(2j)!} \cong \left(\frac{3j}{\pi}\right)^{1/2} \left(\frac{3}{2}\right)^{3j}, \tag{2.12}$$

leading to

$$\frac{\xi(N, M)}{\xi_0} \cong \frac{e^{-M\omega\tau/2Q}}{M} \operatorname{Re} e^{iM\omega\tau} \sum_{j=0}^{\infty} \left(\frac{3j}{\pi}\right)^{1/2} \frac{(\frac{3}{2}E)^{3j}}{(3j)!}. \tag{2.13}$$

The $(3j)^{1/2}$ term in the sum can be similarly absorbed into the factorial term in the sum, leading to $\Gamma(3j + \frac{1}{2})$ as a replacement for $(3j)!$ The final result for $|E| \gg 1$ is the same as that obtained by setting $j = E/2$, representing the "saddle" of the sum, in the $j^{1/2}$ term. One then can write

$$3 \sum_{j=0}^{\infty} \frac{(\frac{3}{2}E)^{3j}}{(3j)!} = \exp\left[\tfrac{3}{2}E\right] + \exp\left[\tfrac{3}{2}E\left(\frac{-1 + i\sqrt{3}}{2}\right)\right] + \exp\left[-\tfrac{3}{2}E\left(\frac{1 + i\sqrt{3}}{2}\right)\right], \tag{2.14}$$

where the first term dominates for the particular phase of E chosen in Eq. (2.11). We then obtain

$$\frac{\xi(N, M)}{\xi_0} \cong \frac{e^{-M(\omega\tau/2Q)}}{M\sqrt{6\pi}} \operatorname{Re}\left[\sqrt{E}\, e^{iM\omega\tau + (3/2)E}\right], \tag{2.15}$$

in *exact* agreement with Eq. (69) of Ref. 5.

For a constant initial displacement the approach to equilibrium has the same form as Eq. (2.15), except that the bracket [ ] has an additional factor K whose amplitude is given by

$$|K| = \frac{1}{2\,|\sin(\omega t/2)|}. \tag{2.16}$$

Figure 1 shows a simulation for a constant initial displacement of 1 mm, with the parameters $r = 2.88 \times 10^{-3}$, $\omega\tau/2\pi = 24/13$, $Q = 1000$, and $N = 30$. As discussed in Ref. 5, the agreement with Eqs. (2.15), (2.16) is excellent.

The above derivation is less rigorous than that using the saddle-point approximation derived from the integral representation,[5] particularly because it uses approximate forms for a series in which there is some cancellation of the terms due to the factor $\exp(-ij\pi/2)$ in Eq. (2.10). Nevertheless, this derivation can be readily extended to include fluctuating parameters, in contrast to the one in Ref. 5.

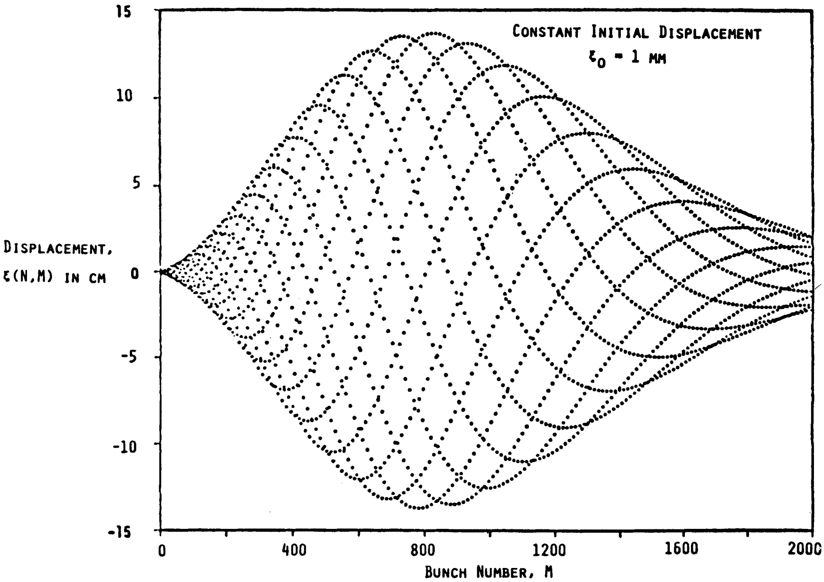

FIGURE 1 Displacement vs M for $\omega\tau/2\pi = 24/13$, $r = 2.88 \times 10^{-3}$, $Q = 1000$, $N = 30$, $\xi_0 = 1$ mm.

3. RANDOM INITIAL DISPLACEMENT

For completeness, we include the solution for random initial displacement. This has been treated in greater detail[7] starting with the transient for a single displaced pulse given in Eq. (2.15). The final result for the rms displacement is given by

$$\left(\frac{\xi_{\text{rms}}}{\xi^0_{\text{rms}}}\right)^2 \cong \frac{|\xi_{\max}|^2}{\xi_0^2}\frac{\sigma\sqrt{\pi}}{2}, \tag{3.1}$$

where σ is the rms width of the transient pulse amplitude and where $|\xi_{\max}|/\xi_0$ is the maximum amplitude of Eq. (2.15). It can be shown that the rms width is given by

$$\sigma \cong (3/\bar{p})^{1/2}, \tag{3.2}$$

where

$$\bar{p} = \bar{M}(\omega\tau/Q) \tag{3.3}$$

is the maximum value of the real part of the exponent in Eq. (2.15), occurring at†

$$\bar{M} = \left(\frac{3}{4}\right)^{3/4} N\left(\frac{Q}{\omega\tau}\right)^{3/2} r^{1/2}. \tag{3.4}$$

† If one takes into account the $M^{-5/6}$ dependence of the coefficient multiplying the exponential in Eq. (2.15), the actual maximum shifts to

$$M^{\max} \cong \bar{M} - (5/2)(Q/\omega\tau). \tag{3.4a}$$

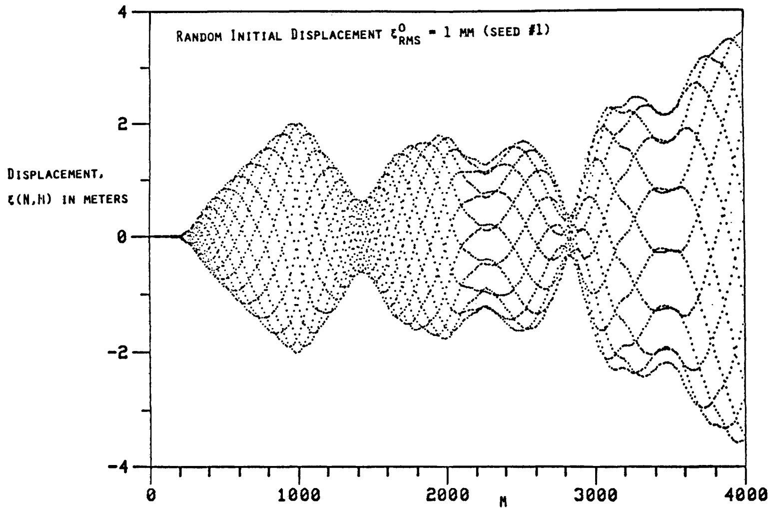

FIGURE 2 Displacement vs M for $\omega\tau/2\pi = 24/13$, $r = 2.88 \times 10^{-3}$, $Q = 1000$, $N = 30$, $\xi^0_{\rm rms} = 1$ mm.

In terms of these parameters the rms displacement is

$$\left(\frac{\xi_{\rm rms}}{\xi^0_{\rm rms}}\right)^2 \cong \frac{1}{6\bar{M}}\left(\frac{\bar{p}}{\pi}\right)^{1/2} e^{2\bar{p}}. \tag{3.5}$$

The same result can be derived by Fourier analyzing the random initial displacement and using an earlier result for a modulated beam.[5] The major contributions come from the frequency bands of width

$$\Delta(\omega' t) \simeq \frac{2}{3}\frac{\omega\tau}{Q}(\bar{p})^{-1/2} \tag{3.6}$$

around the resonances at

$$\omega'\tau = \pm\left(\omega\tau - \frac{\omega\tau}{2Q\sqrt{3}}\right)(\mathrm{mod}\, 2\pi), \tag{3.7}$$

and the final result is identical with that in Eq. (3.5).

Numerical simulations[7] confirm the validity of Eq. (3.5). An example of such a simulation, with the same parameters as in Fig. 1, is shown in Fig. 2, where the distribution in initial displacement is Gaussian, with $\xi^0_{\rm rms} = 1$ mm. In this case, the predicted value of $\xi_{\rm rms}/\xi^0_{\rm rms}$ is about 2800.

4. FLUCTUATION OF DEFLECTING-MODEL FREQUENCY

We can now proceed to include the variation of the deflecting-mode frequency with cavity number. If we write each $\omega\tau$ in Eq. (2.7) as

$$\omega\tau = \bar{\omega}\tau + \Delta, \tag{4.1}$$

the combination with the sum of phases will still be expected to dominate. This will now have the form

$$S^j_{M0} \cong \mathrm{Re}\, \frac{e^{-M(\omega\tau/2Q)+iM\bar{\omega}\tau-ij(\pi/2)}}{2^{j-1}}$$
$$\times \sum_{\alpha>\beta>\cdots>\delta}^{M} \sum \cdots \sum \exp\{i[M-\alpha)\Delta_a + (\alpha-\beta)\Delta_b \cdots + \delta\Delta_d]\}, \quad (4.2)$$

were $\Delta_a, \Delta_b, \ldots, \Delta_d$ are for cavity numbers $a, b, \ldots, d$. [There are j cavities and $j-1$ sums that enter into Eq. (4.2).]

We will now expand the exponential to second order to simplify the sums over $\alpha, \beta, \ldots, \delta$. This leads to the typical terms

$$1,\ i\Delta_c(\beta-\gamma),\ -\Delta_b\Delta_c(\alpha-\beta)(\beta-\gamma),\ -\frac{\Delta_b^2}{2}(\alpha-\beta)^2, \quad (4.3)$$

where sums need to be taken over all combinations of lower-case Greek indices. When these sums are approximated by integrals for large M we find

$$\begin{aligned}
\sum_{\alpha>\beta>\cdots\delta}^{M} \sum \cdots \sum 1 &\cong \frac{M^{j-1}}{(j-1)!} \\
\sum_{\alpha>\beta>\cdots\delta}^{M} \sum \cdots \sum (\alpha-\beta) &\cong \frac{M^j}{j!} \\
\sum_{\alpha>\beta>\cdots\delta}^{M} \sum \cdots \sum (\alpha-\beta)(\gamma-\delta) &\cong \frac{M^{j+1}}{(j+1)!}, \quad \begin{Bmatrix} \alpha \neq \gamma \\ \beta \neq \delta \end{Bmatrix} \\
\sum_{\alpha>\beta>\cdots\delta}^{M} \sum \cdots \sum (\alpha-\beta)^2 &\cong \frac{2M^{j+1}}{(j+1)!}.
\end{aligned} \quad (4.4)$$

The sum in Eq. (4.2) can now be written, for $j \gg 1$, as

$$S^j_{M0} \simeq \frac{M^{j-1}}{2^{j-1}(j-1)!} \mathrm{Re}\, e^{-(M\omega\tau/2Q)+iM\bar{\omega}\tau-ij(\pi/2)}$$
$$\times \left[1 + \frac{iM}{j} \sum_{c_1}^{c_j} \Delta_c - \frac{M^2}{2j(j+1)} \sum_{b_1}^{c_j} \sum_{\substack{c_1 \\ b\neq c}}^{c_j} \Delta_b\Delta_c - \frac{M^2}{j(j+1)} \sum_{c_1}^{c_j} \Delta_c^2\right]. \quad (4.5)$$

Equation (4.5) must now be averaged over all possible configurations of j frequency errors among N cavities. Assuming that the weighting $(N+j)!/(2j)!\,(N-j)!$ in Eq. (2.5) is the same for all configurations,† we have

$$\frac{1}{j}\sum_{c_1}^{c_j} \Delta_c = \frac{1}{N}\sum_{1}^{N} \Delta_n,\quad \frac{1}{j}\sum_{c_1}^{c_j} \Delta_c^2 = \frac{1}{N}\sum_{1}^{N} \Delta_n^2,\quad \sum_{b_1}^{c_j} \sum_{\substack{c_1 \\ b\neq c}}^{c_j} \Delta_b\Delta_c = \frac{j^2}{N^2}\sum_{1}^{N} \sum_{\substack{1 \\ n\neq n'}}^{N} \Delta_n\Delta_{n'}. \quad (4.6)$$

† This assumption appears to be only approximately correct.

The bracket [ ] in Eq. (4.5) must now be written as the product of a term with unit amplitude, corresponding to a frequency shift, and an amplitude reduction, i.e.,

$$[1 + ia\Delta - b\Delta^2] \cong e^{ia\Delta}\left[1 - \Delta^2\left(b - \frac{a}{2}\right)\right], \tag{4.7}$$

accurate to order Δ^2. The expected value of the amplitude reduction for large j then becomes, using Eq. (4.6),

$$\left[1 - \frac{M^2}{j}\left(1 - \frac{j}{2N}\right)\Delta_{\rm rms}^2\right], \tag{4.8}$$

where $\Delta_{\rm rms}^2$ is the mean square of the width of the distribution, given by

$$\Delta_{\rm rms}^2 = \langle\Delta^2\rangle - \langle\Delta\rangle^2. \tag{4.9}$$

For a symmetric distribution of the frequency fluctuation, we have $\langle\Delta\rangle = 0$. Since the discussion following Eq. (2.13) implies that one can set $j \cong E/2$ in terms like that in Eq. (4.8), the displacement in Eq. (2.15) can be rewritten, for $j \ll N$, as

$$\frac{\xi(N, M)}{\xi_0} = \frac{1}{M\sqrt{6\pi}} \operatorname{Re} \sqrt{E} \exp\left(iM\omega\tau - M\frac{\omega\tau}{2Q} + \tfrac{3}{2}E - \frac{2\Delta_{\rm rms}^2 M^2}{E}\right), \tag{4.10}$$

where we have written the term involving $\langle\Delta_{\rm rms}^2\rangle$ as an exponential, and where E is given in Eq. (2.11). If one writes

$$E = G^{2/3}M^{1/3}e^{-i\pi/6}, \qquad G = Nr^{1/2}, \tag{4.11}$$

the real part of the exponent in Eq. (4.10) can be written as

$$p_1(M) = -\frac{M\omega\tau}{2Q} + \frac{3\sqrt{3}}{4}G^{2/3}M^{1/3} - \sqrt{3}\Delta_{\rm rms}^2 M^{5/3}G^{-2/3}. \tag{4.12}$$

This exponent reaches a maximum value†

$$\frac{\bar{p}_1}{\bar{p}} = \left(\frac{2 + 3\lambda}{5}\right)\left(\frac{1 + \lambda}{2}\right)^{-3/2}, \tag{4.13}$$

at

$$\frac{\bar{M}_1}{\bar{M}} = \left(\frac{1 + \lambda}{2}\right)^{-3/2}, \tag{4.14}$$

where $\bar{p}$ and $\bar{M}$ are given in Eqs. (3.3) and (3.4), and where

$$\lambda^2 = 1 + 20\varepsilon^2, \qquad \varepsilon^2 = Q^2\Delta_{\rm rms}^2/\omega^2\tau^2. \tag{4.15}$$

Equations (4.12) and (4.13) have been derived under the assumption of small $\Delta_{\rm rms}^2$. Nevertheless, they can be expected to be approximately valid for $\varepsilon \gtrsim 1$, since one expects a rapid decrease of the transient growth if the relative frequency spread is greater than Q^{-1}. In fact, if one assumes $\varepsilon\sqrt{20} \gg 1$, one

† As in Eq. (3.4a), the coefficient $M^{-5/6}$ causes the maximum to shift to

$$M_1^{\max} \cong \bar{M}_1 - (5/2)(Q/\omega\tau)/(1 + 20\varepsilon^2)^{1/2}.$$

finds, from Eqs. (3.3) and (3.4), a maximum exponent

$$\bar{p}_1 \simeq \frac{0.8}{\varepsilon^{1/2}} \bar{p} \simeq 0.65 N r^{1/2} \Delta_{\rm rms}^{-1/2}, \tag{4.16}$$

independent of Q. The factor Q^{-1} in Eqs. (3.3) and (3.4) has been replaced by the relative frequency spread $\Delta_{\rm rms}/\omega\tau$, as might have been expected. This result is similar to that obtained by Yokoya[8] using a uniform distribution of transverse deflecting-mode frequency to modify an integral representation.

The envelope associated with the single-pulse transient in the presence of frequency fluctuation is therefore given by

$$\frac{\xi_{\rm env}}{\xi_0} \cong \frac{G^{1/3}}{M^{5/6}\sqrt{6\pi}} e^{p_1(M)}, \tag{4.17}$$

where $p_1(M)$ is given in Eq. (4.10). (The subscript 1 here refers to the case of frequency fluctuation.) Comparisons with numerical simulations have been carried out with the same parameters as in Fig. 1.[9] The validity of the above analysis was confirmed by plotting

$$\begin{aligned} w_1(M) &= \ln\left(\frac{\xi_{\rm env}}{\xi_0}\right) + M\frac{\omega\tau}{2Q} - \frac{3\sqrt{3}}{4} G^{2/3} M^{1/3} + \tfrac{5}{6}\ln M \\ &= \frac{\ln G}{3} - \frac{\ln(6\pi)}{2} - \frac{\sqrt{3}\,\Delta_{\rm rms}^2 M^{5/3}}{G^{2/3}} \end{aligned} \tag{4.18}$$

against $M^{5/3}$ for different values of $\varepsilon = Q\Delta_{\rm rms}/\omega\tau$, where $\xi_{\rm env}$ is taken from the simulations. The second form in Eq. (4.18) depends on the validity of Eq. (4.15). Figures 3 and 4 contain such plots for two different random-number sets for

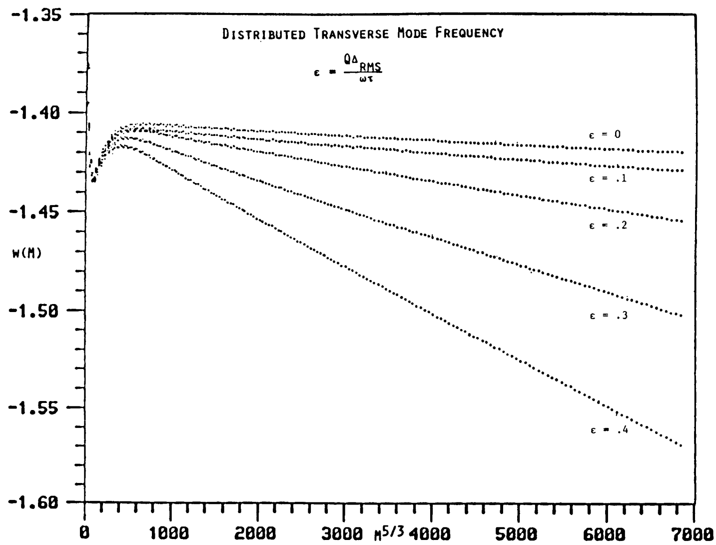

FIGURE 3 Plot of $w(M)$ in Eq. (4.18) vs $M^{5/3}$ for frequency distribution (random seed #1).

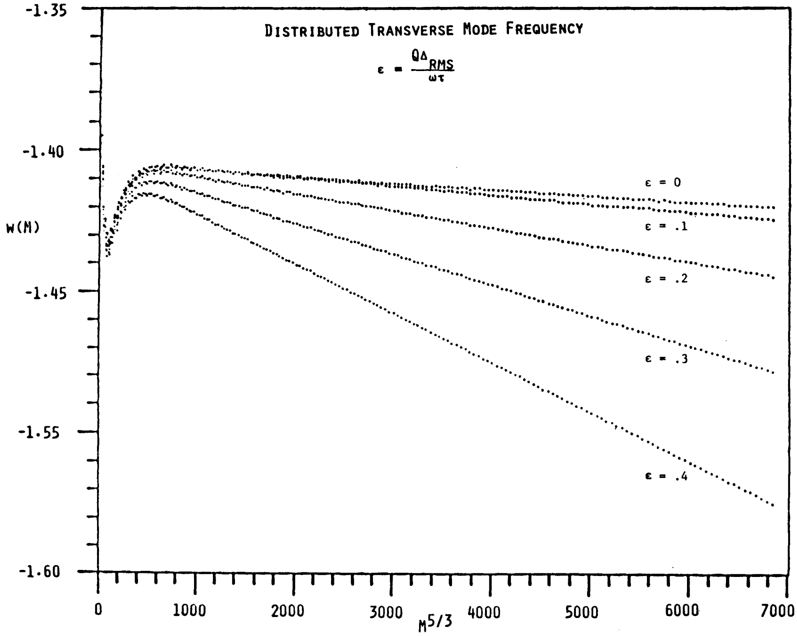

FIGURE 4 Plot of $w(M)$ in Eq. (4.18) vs $M^{5/3}$ for frequency distribution (random seed #2).

values of ε from 0 to 0.4. As predicted by Eq. (4.18), the plots are straight lines in a wide range around the maximum of ξ_{env}, and the slopes and intercepts are in good agreement with the predicted values. A more precise test is shown in Fig. 5, where

$$w_1(M) - w_1(M)\big|_{\varepsilon=0}$$

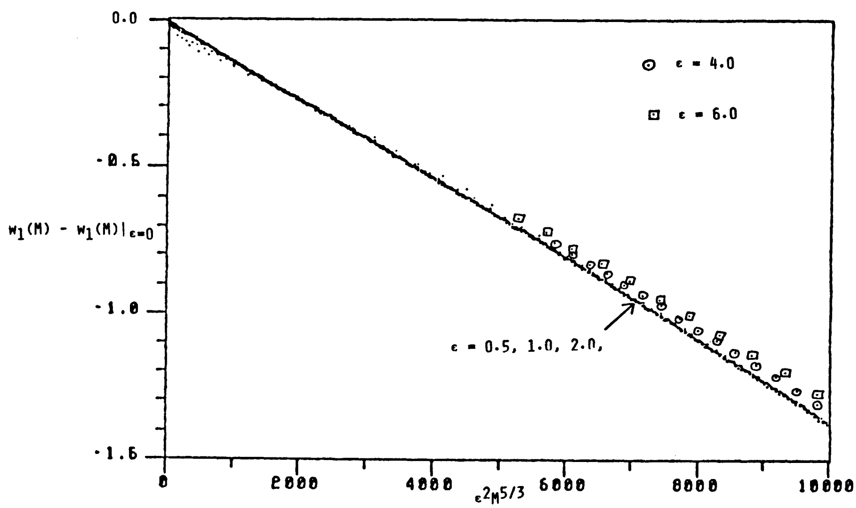

FIGURE 5 Plot of $w_1(M) - w_1(M)\big|_{\varepsilon=0}$ vs $\varepsilon^2 M^{5/3}$ (random seed #1).

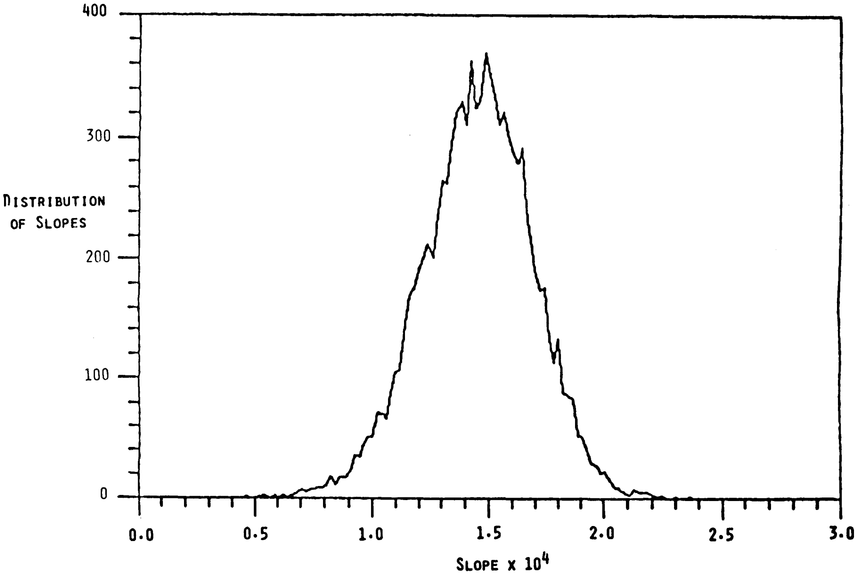

FIGURE 6 Distribution of slopes for 10,000 samples like the one in Fig. 5.

has been plotted against $\varepsilon^2 M^{5/3}$ for $\varepsilon = 0.5$, 1.0, 1.5, 2.0, 4.0, and 6.0 for the same random-number seed as in Fig. 3. The slope as $\varepsilon^2 M^{5/3} \to 0$ is obtained for 10,000 random sets, and the distribution is plotted in Fig. 6. The mean slope obtained is 1.45×10^{-4} while the slope predicted by Eq. (4.18) is 1.69×10^{-4}. If one includes the modifying factor $1 - (j/2N)$ in Eq. (4.8), the predicted slope is 1.52×10^{-4}, in good agreement with that obtained from the simulation.

It therefore appears that Eq. (4.10) is an accurate representation of the effect of frequency fluctuation, even for values of $\varepsilon = Q\Delta_{\text{rms}}/\omega\tau$ above 1. This is quite surprising since, for $\varepsilon > 1$, only a few cavities will become excited.

In a companion paper[10] we derive results for smooth variation of parameters. The result for smooth variation of deflecting-mode frequency is close to that in Eq. (4.10), but the value of E has been generalized to

$$E = e^{-i\pi/6} M^{1/3} \left[\int_0^N \sqrt{r(n)}\, dn \right]^{2/3},$$
$$r(n) = L_n R_n / \gamma_n$$

takes into account smooth variation of L, R, and γ with cavity number. This value of E can therefore be used in Eqs. (4.10) to (4.12) when r is dependent on n. In this case, however, a weighting factor $\sqrt{r(n)}$ must be used to calculate the rms frequency spread.[10]

5. FLUCTUATION OF CHARGE PER BUNCH

If the charge per bunch fluctuates, the parameter R in Eq. (1.2) must be written as

$$R \to R(1 + \delta_m),$$

where δ_m is the fractional change in the charge of the mth bunch. It is clear from the origin of Eq. (2.3) that the field stimulation due to the lth bunch is connected with the l in s_{M-l}. For this reason the summand in Eq. (2.7) must be modified by the factor

$$(1+\delta_\alpha)(1+\delta_\beta)\cdots(1+\delta_\delta)(1+\delta_0). \tag{5.1}$$

The sums over $\alpha, \beta, \ldots, \delta$ will still lead to the dominance of the term involving the sum of phases, but the modification due to the factors $(1+\delta_\alpha)(1+\delta_\beta)\cdots(1+\delta_\delta)$ in Eq. (5.1) will lead to corrections of order $(1+\delta_{\text{rms}})$. These corrections will not lead to any significant change for charge fluctuations of the order of 10% or less.

The above comments do not include the factor $(1+\delta_0)$ in Eq. (5.1), since we do not sum over the index 0. However, the conclusion for the single-pulse transient is the same: the factor $(1+\delta_0)$ will only contribute a small correction.

The situation is, however, different for a beam with a systematic initial displacement, such as might occur in the presence of steering errors. In this case the transient peak will be governed by the sum

$$\xi(N, M) = \text{Re} \sum_{M'=0}^{M-1} e^{i\omega\tau(M-M')}(1+\delta_{M'})\xi(0, M')h(M-M'), \tag{5.2}$$

where $h(M-M')$ is the single-pulse transient given in Eq. (2.15) and has the form

$$h(m) = \frac{e^{-m\omega\tau/2Q}}{m\sqrt{6\pi}}\sqrt{E}\, e^{(3/2)E}, \tag{5.3}$$

with

$$E = N^{2/3}r^{1/3}m^{1/3}e^{-i(\pi/6)}. \tag{5.4}$$

For constant $\xi(0, M') = \xi_0$, the term not involving $\delta_{M'}$ describes the usual approach to equilibrium, where the amplitude of the single-pulse transient is modified by the factor $|K|$ in Eq. (2.16), which is of order 1 away from the resonance $\omega\tau = 2n\pi$. On the other hand, the term involving $\delta_{M'}$ will accumulate in an rms sense to a value that is identical to that in Eqs. (3.1) and (3.2) for

$$\langle \xi_0^2 \rangle = \xi_0^2 \langle \delta^2 \rangle = \xi_0^2 \delta_{\text{rms}}^2. \tag{5.5}$$

If we assume a factor $2^{-1/2}$ for oscillation of the displacement and a factor $2^{-1/2}$ for the oscillation of the envelope, we find that Eq. (5.5) corresponds to a modification of the peak of the single-pulse transient by the factor

$$\delta_{\text{rms}}\bar{M}^{1/2}\left(\frac{12\pi}{\bar{p}}\right)^{1/4}. \tag{5.6}$$

The two effects together will then lead to a correction factor to the maximum of the single-pulse transient given by $\zeta(\delta)$, where

$$\zeta^2(\delta) = |K|^2 + \langle \delta^2 \rangle (12\pi/\bar{p})^{1/2}. \tag{5.7}$$

It is clear that, for large $\bar{M}$, the term in the charge fluctuation can dominate, even for small δ_{rms}.

Simulations have been run for several random samples of a Gaussian

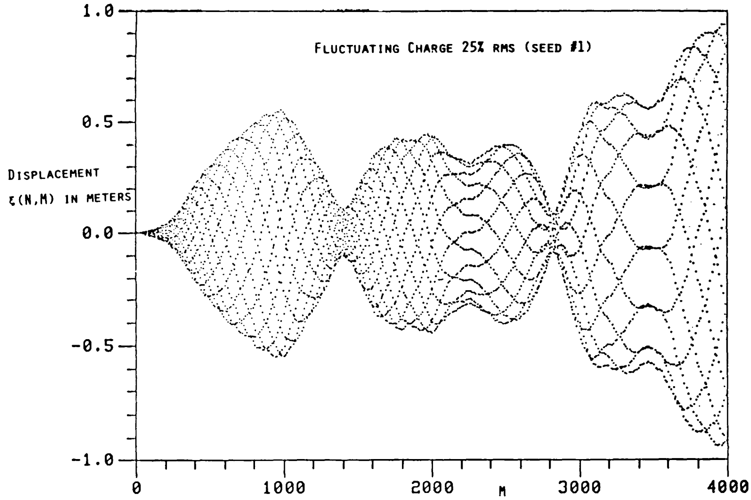

FIGURE 7 Displacement vs M for $\omega\tau/2\pi = 24/13$, $r = 2.88 \times 10^{-3}$, $Q = 1000$, $N = 30$, $\xi_0 = 1$ mm, 25% fluctuating charge.

distribution in the charge fluctuation. Two of these for $\delta_{\mathrm{rms}} = 0.25$ and 1.0 (using the same random-number seed) are shown in Figs. 7 and 8. From such simulations one obtains $\zeta(\delta)$ for the first major peak. A plot of $[\zeta^2(\delta) - \zeta^2(0)]^{1/2}$ vs δ_{rms} is shown in Fig. 9. It clearly shows the validity of the linear prediction even for $\delta_{\mathrm{rms}} \sim 1$ (which corresponds to a charge fluctuation that can change the sign of the bunch charge). It is clear that, even for $\delta_{\mathrm{rms}} = 0.25$, the second term in Eq. (5.7) dominates. And the comparison of Fig. 5 with Fig. 1 confirms the strong

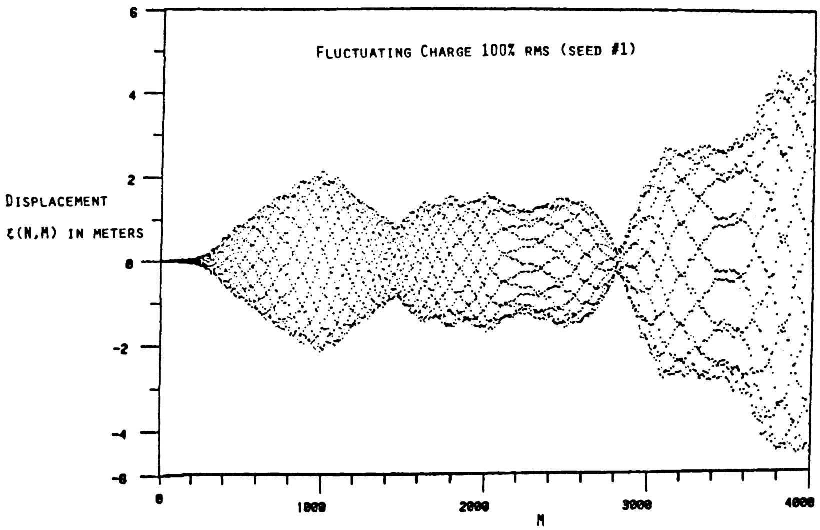

FIGURE 8 Displacement vs M for $\omega\tau/2\pi = 24/13$, $r = 2.88 \times 10^{-3}$, $Q = 1000$, $N = 30$, $\xi_0 = 1$ mm, 100% fluctuating charge.

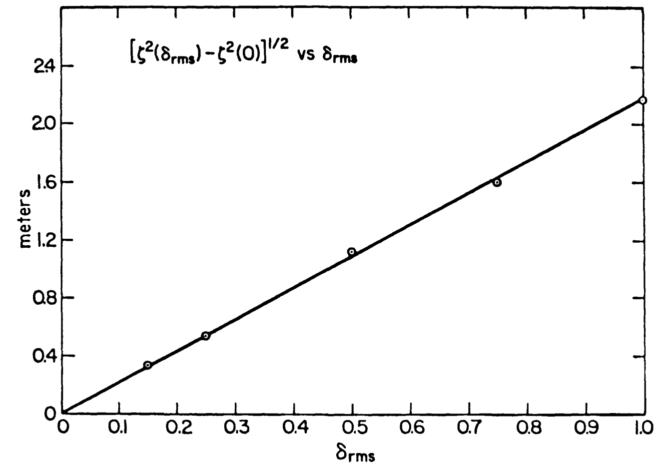

FIGURE 9 Maximum displacement vs rms charge fluctuation (random seed #1) for $\xi_0 = 1$ mm.

similarity of the results for random initial displacement and charge fluctuation. In particular, the curves differ essentially by a factor 0.25, as predicted by Eq. (5.5).

The above result implies that beam breakup will be enhanced by charge fluctuation if there are steering errors (systematic displacement of the beam injected into the accelerator). This enhancement can be estimated from Eq. (5.6) or (5.7).

DISCUSSION

We have provided an alternate derivation of the transient in cumulative beam breakup, yielding the same result as the more rigorous derivation used earlier. This alternate derivation provides a convenient vehicle for analyzing random fluctuation of various parameters. An earlier result for random initial displacement is included for completeness.

The analysis for random fluctuation of the deflecting-mode frequency is carried out for a small frequency spread, but the final result, presented in Eqs. (4.10) and (4.15), appears to be valid for values of $\varepsilon = Q\Delta_{rms}/\omega\tau$ at least as large as 6. The result is particularly simple and not unexpected for $\varepsilon > 1$: the value of Q^{-1} in the constant-frequency result is replaced by $\Delta_{rms}/\omega\tau$, the equivalent Q^{-1} due to the frequency spread.

This result has particular significance where one has a high value of Q for each cavity, such as for a superconducting rf linac. A standard feature in such a design[11] is to couple the most troublesome deflecting modes (the ones with the highest value of R) to a normally conducting load, thus reducing the Q by several orders of magnitude. The effective Q is also reduced by the expected fabrication tolerances, which can lead to a frequency spread in the deflecting modes of a few parts in 10,000, even if the accelerating modes are tuned more precisely to one another.

The effect of fabrication tolerances may also be relevant to SLAC. In their study of the scaling laws for current, accelerator length, and pulse length, the effect of deflecting-mode frequency spread was not included in reaching the conclusion that beam breakup corresponded to an exponent between 15 and 20. According to Eqs. (4.11) and (4.13), this exponent would have been between 12 and 16 for $\varepsilon = 2$ and between 8 and 10 for $\varepsilon = 3$. Clearly the implied growth of the initial noise stimulus would be significantly reduced from the original estimate of 10^7 to 10^8 in these cases.

The analysis of the effect of fluctuation in the charge per bunch is carried out both for a single offset bunch and for all bunches offset. The effect is not serious in the former case, but when all bunches are offset the charge fluctuation corresponds to an equivalent fluctuation in initial displacement described in Eqs. (5.5) to (5.7) that can amplify the single-bunch transient by a large factor (of order of the square root of the "width" of the single-bunch transient corresponding to Fig. 1). Clearly this effect will be important in the presence of steering errors in the beam.

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