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BEAM BEHAVIOR STUDIES IN ACCELERATORS USING PERTURBATION THEORY†

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Using canonical perturbation theory with perturbed tunes (instead of the linear tunes used in ordinary perturbation theory), we obtain results similar to those deduced from superconvergent perturbation theory (of the same order) and illustrate the amplitude dependence of tune due to nonlinear elements. We note that when the perturbed tunes are on resonance (e.g., $2\nu_x - 2\nu_z = 0$ for the AGS Booster), ordinary perturbation theory would not show the resonance conditions, whereas superconvergent perturbation theory would. Our results indicate that $\nu_x - \nu_z > 0.01$ and that a chromaticity range of -5.5 to -2.5 is needed for the operation of the booster, which seem to agree rather well with the tracking results.

1. INTRODUCTION

In circular accelerators the presence of nonlinearities can greatly affect the beam behavior. In many cases, this behavior can be examined using perturbation theory. Using second-order perturbation theory, we have studied nonlinear effects in accelerators.[1–4] We illustrate some of our analyses for the AGS Booster at Brookhaven National Laboratory, with emphasis on the $2\nu_x - 2\nu_z = 0$ resonance, since the $2\nu_x - 2\nu_z$ coupling becomes large when the chromaticity of the machine is corrected to zero. We have studied the behavior of the beam with respect to chromaticity and have calculated the perturbation to betatron tune, resonance strengths, stop bandwidth,[5] fixed points, island width, and Chirikov[6] criteria for various chromaticities. Our results seem to indicate a chromaticity range of -5.5 to -2.5 for the operation of the booster with $\nu_x - \nu_z > 0.01$ (see Figs 1–5 below), a range that agrees well with results obtained from tracking.[7] Using canonical perturbation theory via our algorithm NONLIN with the perturbed tunes (instead of the linear tune used in ordinary perturbation theory), we obtain results similar to those deduced from the superconvergent perturbation theory (of the same order, e.g., to second order). We note that when the perturbed tunes are on resonance (e.g., $2\nu_x - 2\nu_z = 0$ for the AGS Booster), ordinary perturbation theory would not show the resonance conditions, whereas superconvergent perturbation theory would. This effect illustrates the amplitude dependence of the tune due to nonlinear elements (e.g., sextupoles in the Booster). Superconvergent perturbation theory[8] is a canonical perturbation theory where one uses successive canonical transformations to reduce the perturbing potential. Thus, in each successive transformation the perturbed tunes

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are used rather than the linear tunes, as with ordinary canonical perturbation theory, which can lead to different results if either the linear or perturbed tunes are on resonance. (There are other differences as well.)

In Section 2, we present a theoretical discussion, and in Section 3, we tabulate some of our results for the AGS Booster lattice with eddy currents and chromaticity sextupoles. Since the Booster is designed to accelerate protons as well as heavy ions, we also have studied the Booster lattice with the chromaticity sextupole, including the iron-saturation effect of the magnets.

2. THEORY

The Hamiltonian of a dynamical system can be expressed by

$$H = \frac{J_x}{\beta_x(s)} + \frac{J_z}{\beta_z(s)} + V(J_x, J_z, \phi_x, \phi_z, s), \tag{1}$$

where (J_x, ϕ_x) and (J_z, ϕ_z) are the action-angle variables. We shall use ν_x^0 and ν_z^0 for the linear tunes, C for the circumference of the machine, and V for the perturbing potential (periodic in ϕ_x, ϕ_z, and s).

To determine the perturbation on the tune and the variation of emittance due to nonlinearities (e.g., sextupole, octupole, etc.), we make a canonical transformation of the Hamiltonian [Eq. (1)] with a generating function of the following form, in order to make the new Hamiltonian cyclic in the new action variables K_x and K_z up to a given order in K [e.g., $O(K_{x,z}^{5/2})$]:

$$G(K_x, K_z, \phi_x, \phi_z, s) = K_x\phi_x + K_z\phi_z + \sum_k \frac{g_k(K_x, K_z s)}{\sin \pi(n_{x_k}\nu_x + n_{z_k}\nu_z)} \cos(n_{x_k}\phi_x + n_{z_k}\phi_z + \theta_k), \tag{2}$$

where $g_k(K_x, K_z, s)$ are the generating function resonance strengths whose magnitudes show to what extent J_x and J_z deviate from the invariants of motion. The terms n_{x_k} and n_{z_k} are integers defining a given resonance, and θ_k is the phase, with K_x, K_z, Ψ_x, and Ψ_z as the new action and angle variables, respectively (also see the Appendix),

$$\psi_x = \phi_x + \sum_k \left[\frac{\partial}{\partial K_x} g_k(K_x, K_z, s) \cos(n_{x_k}\phi_x + n_{z_k}\phi_z + \theta_k) - g_k(K_x, K_z, s) \frac{\partial}{\partial K_x} \theta_k(K_x, K_z, s) \sin(n_{x_k}\phi_x + n_{z_k}\phi_z + \theta_k) \right] \times \frac{1}{\sin \pi(n_{x_k}\nu_x + n_{z_k}\nu_z)}, \tag{3}$$

and

$$\psi_z = \phi_z + \sum_k \left[\frac{\partial}{\partial K_z} g_k(K_x, K_z, s) \cos(n_{x_k}\phi_x + n_{z_k}\phi_z + \theta_k) - g_k(K_x, K_z, s) \frac{\partial}{\partial K_z} \theta_k(K_x, K_z, s) \sin(n_{x_k}\phi_x + n_{z_k}\phi_z + \theta_k) \right] \times \frac{1}{\sin \pi(n_{x_k}\nu_x + n_{z_k}\nu_z)}. \quad (4)$$

Thus the perturbation on the tune due to the nonlinear elements (e.g., sextupoles and octupoles) can be found as

$$\frac{d}{ds}\psi_x(s) = \frac{\partial H}{\partial K_x}, \quad (5)$$

$$\frac{d}{ds}\psi_x = \frac{1}{\beta_x(s)} + 2a(s)K_x + b(s)K_z,$$

and

$$\nu_x \equiv \frac{1}{2\pi} \int_0^C \left[\frac{d}{ds}\psi_x(s) \right] ds, \quad (6)$$

or

$$\nu_x = \nu_x^0 + 2\alpha_{xx}K_x + 2\alpha_{xz}K_z, \quad (7)$$

where

$$\alpha_{xx} = 1/\pi \int_0^C a(t)\,dt, \qquad \alpha_{xz} = \frac{1}{2\pi}\int_0^C b(t)\,dt.$$

Similarly,

$$\frac{d}{ds}\psi_z = \frac{\partial H}{\partial K_z},$$

$$\frac{d}{ds}\psi_z = \frac{1}{\beta_z(s)} + 2c(s)K_z + b(s)K_x,$$

$$\nu_z \equiv \frac{1}{2\pi} \int_0^C \left[\frac{d}{ds}\psi_z(s) \right] ds,$$

or

$$\nu_z = \nu_z^0 + 2\alpha_{zz}K_z + 2\alpha_{zz}K_x, \quad (8)$$

with

$$\alpha_{zz} = 1/\pi \int_0^C c(t)\,dt,$$

where the coefficients $a(s)$, $b(s)$, and $c(s)$ are given in Ref. 1, and the machine tunes ν_x and ν_z depend on the beam emittance. To use Eqs. (7) and (8), we assume that K_x and K_z are approximately invariants (i.e., $\approx$ constants) of the motion. The ν_x^0 and ν_z^0 terms are the unperturbed tunes, and the new actions $2K_x, 2K_z$ were taken to be equal to the average beam emittance divided by π, $\langle E_x \rangle$ and $\langle E_z \rangle$. Thus, Eqs. (7) and (8) lead to

$$\begin{aligned}\Delta\nu &= \Delta\nu^0 + \alpha_{xx}\langle E_x\rangle + \alpha_{xz}\langle E_z\rangle - \alpha_{zz}\langle E_z\rangle - \alpha_{xz}\langle E_x\rangle \\ &\equiv \Delta\nu^0 + \Delta\nu'\end{aligned} \tag{9}$$

where $\Delta\nu = \nu_x - \nu_z$ and $\Delta\nu^0 = \nu_{x^0} - \nu_{z^0}$ (unperturbed), and $\Delta\nu'$ is the measure of the nonlinear (perturbed) tune split. Additionally, the maximum the beam emittance may grow to can be obtained from the generating function[1] (see Appendix) as

$$E_x \leqq 2\pi\left[K_x + \sum_x n_{x_k}\left|\frac{g_k(K_x, K_z, s)}{\sin\pi(n_{x_k}\nu_x + n_{z_k}\nu_z)}\right|\right], \tag{10}$$

$$E_z \leqq 2\pi\left[K_z + \sum_k n_{z_k}\left|\frac{g_k(K_x, K_z, s)}{\sin\pi(n_{x_k}\nu_x + n_{z_k}\nu_z)}\right|\right]. \tag{11}$$

Further, we can deduce the contribution of a single resonance to the emittance growth:

$$E_x = 2\pi\left[K_x + n_x\frac{a(K_x, K_z)}{\delta}\cos\left(n_x\phi_x + n_z\phi_z - \frac{2\pi}{C}ps + \theta\right)\right], \tag{12}$$

$$E_z = 2\pi\left[K_z + n_z\frac{a(K_x, K_z)}{\delta}\cos\left(n_x\phi_x + n_z\phi_z - \frac{2\pi}{C}ps + \theta\right)\right], \tag{13}$$

with $n_x > 0$ when $n_z < 0$ for difference resonances. The emittance oscillates about its average value (with oscillation amplitude proportional to $g(J_x, J_z) = a(J_x, J_z)/\delta$), where δ is the bandwidth (e.g., $\delta \cong 0$ near resonance), defined as

$$\delta \equiv n_x\nu_x + n_z\nu_z - p. \tag{14}$$

Equation (14) determines how far the tunes ν_x and ν_z are from the resonance (defined by integers n_x, n_z, and p). The measure of the extent to which the emittance deviates from an invariant of the motion (called "smear") is defined as

$$\text{Smear} \equiv \left[\frac{\langle E_x\rangle + \langle E_z\rangle}{\langle E_x^{1/2}\rangle^2 + \langle E_z^{1/2}\rangle^2} - 1\right]^{1/2}, \tag{15}$$

which is the measure of the oscillation amplitude. (For a detailed discussion of smear, see Refs. 3 and 4.) As a simpler alternative, smear can be defined as:

$$\text{Smear}_x \cong \frac{1}{\sqrt{3}}\frac{E_x(\max) - E_x(\min)}{\langle E_x\rangle} = \frac{1}{\sqrt{3}}|n_x g(J_x, J_z)|, \tag{16}$$

$$\text{Smear}_z \cong \frac{1}{\sqrt{3}}\frac{E_z(\max) - E_z(\min)}{\langle E_z\rangle} = \frac{1}{\sqrt{3}}|n_z g(J_x, J_z)|. \tag{17}$$

Equations (16) and (17) are useful for obtaining the resonance coupling strength $g(J_x, J_z)$ (e.g., from the tracking results).

Additionally, one can calculate the resonance strengths, fixed points, Chirikov criteria, island width, stop bandwidths, etc. Finally, in our analysis of beam behavior versus chromaticity from Eq. (1), using the methods given in Refs. 2 and 5,

$$\xi_{x(\text{or } z)} = \frac{1}{4\pi} \int_0^C \left[-k_{x(\text{or } z)}^{(s)} \pm S(s)D(s) \right] \beta_{x(\text{or } z)}\, ds, \tag{18}$$

with the quadrupole strength

$$k(s) = -\frac{1}{B\rho} \frac{dB_z}{dx} \bigg|_{x,z=0},$$

and the sextupole strength (including the chromaticity-correcting sextupoles and eddy-current sextupoles in one case, and sextupoles due to saturation in the second case, respectively):

$$S(s) = \frac{1}{B\rho} \frac{d^2 B_z}{dx^2} \bigg|_{x,z=0},$$

where $D(s)$ is the horizontal dispersion and $\beta(s)$ is the betatron function.

Thus, we first calculate the sextupole strengths in order to obtain the desired chromaticity. Then, using second-order perturbation theory, we study the effect of the sextupoles (e.g., due to eddy currents, chromatic correction, and saturation) on the beam. In addition, we compare the ordinary perturbation theory with the superconvergent perturbation theory,[8] illustrating the amplitude dependence of the tune due to nonlinear elements in an accelerator.

3. AGS BOOSTER

We have investigated the nonlinear effects, including the systematic resonances (e.g., $2\nu_x - 2\nu_z = 0$), for the AGS Booster, an intermediate synchrotron injector for the AGS[9] at Brookhaven National Laboratory. The Booster is designed to increase the intensity of the protons and polarized protons by factors of 4 and 20–30, respectively, as well as to allow the acceleration of all species of heavy ions in the AGS. We have studied the behavior of the beam with respect to the chromaticity of the machine, because the $2\nu_x - 2\nu_z$ coupling becomes large when the chromaticity of the machine is corrected to zero. We illustrate some of our results in Table I for the Booster lattice with eddy currents and chromaticity sextupoles. The inclusion of the iron-saturation effect of the magnet is important, since the Booster is designed to accelerate heavy ions as well as protons. Thus, our results for the Booster lattice with the saturation and chromaticity sextupoles is given in Ref. 10. Next we present our analytic results for the AGS Booster in Table I and Figs. 1–5.

Table I includes the bandwidth with perturbed tunes, as well as the maximum

TABLE I

Analytical Results for The AGS Booster, Brookhaven National Laboratory

Chromaticity	Bandwidth ($2\nu_x - 2\nu_z$)	Maximum Emittance (π mm-mrad)	Generating-Function Resonance Strength	Hamiltonian Resonance Strength
−6.0000000	−0.0156880	108.9210000	0.0370876	0.0215430
−5.5000000	−0.0172100	108.2140000	0.0370590	0.0212390
−5.0000000	−0.0186480	108.1590000	0.0383278	0.0221530
−2.0000000	−0.0254620	111.8600000	0.0984171	0.0762360
0.0	−0.0282800	118.8650000	0.1999800	0.1658900

Chromaticity	Stop Bandwidth	Fixed Points	Island Width	Chirikov Criteria
−6.0000000	0.0017235	0.0010516	6.6580000	0.0004030
−5.5000000	0.0016991	0.0010728	8.0884000	0.0003271
−5.0000000	0.0017723	0.0010686	10.4350000	0.0002644
−2.0000000	0.0060989	0.0004842	15.4580000	0.0006143
0.0	0.0132710	0.0003068	11.7630000	0.0017565

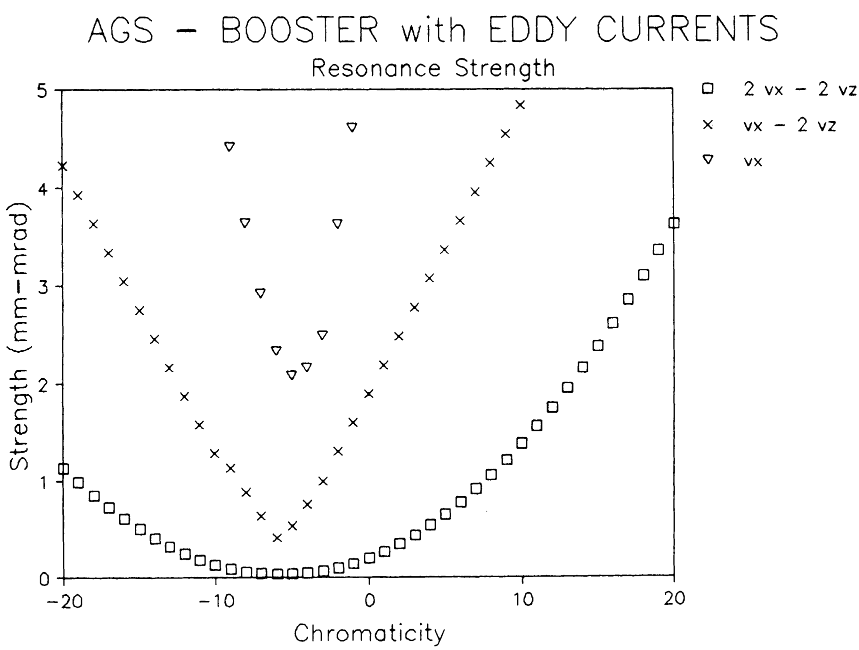

FIGURE 1 Strength of $2\nu_x - 2\nu_z$, $\nu_x - 2\nu_z$, and ν_x resonances (generating-function resonance strengths) as functions of machine chromaticity for the Booster lattice with eddy currents and chromaticity sextupoles.

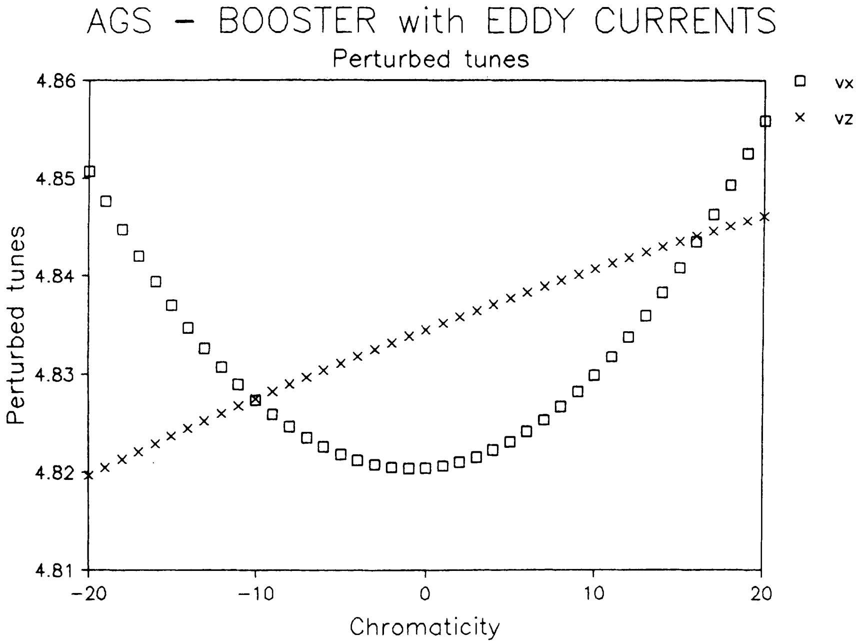

FIGURE 2 Perturbation to tune versus machine chromaticity for operating tunes of 4.82 and 4.83 and average beam emittance of 50 π mm-mrad. This figure illustrates the amplitude dependence of the tune due to sextupoles in the Booster.

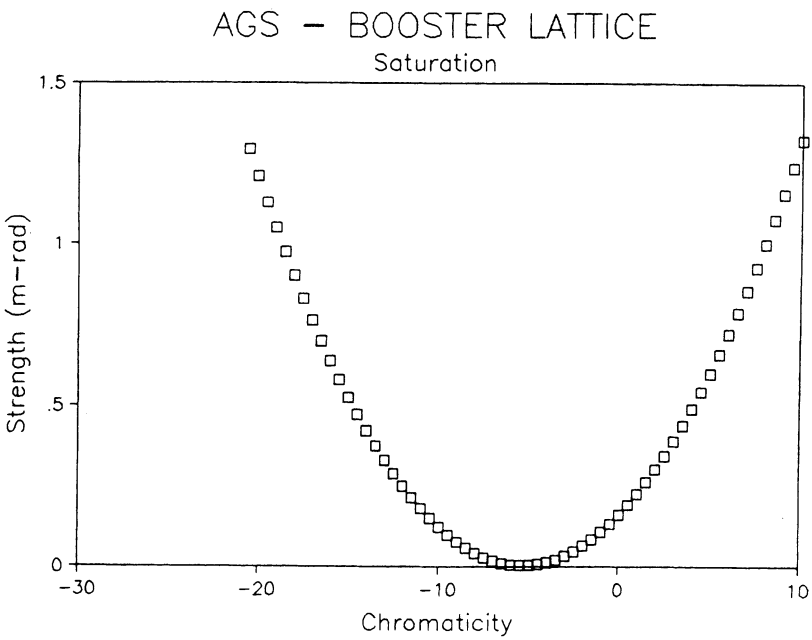

FIGURE 3 Strength of the $2\nu_x - 2\nu_z$ resonance versus machine chromaticity for the Booster lattice with saturation and chromaticity sextupoles.

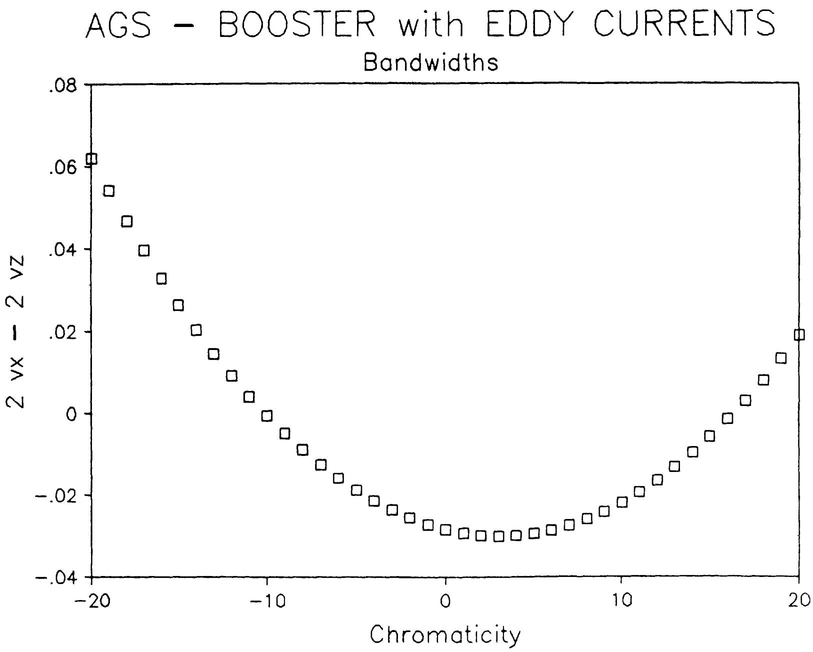

FIGURE 4 Bandwidth versus machine chromaticity for the $2\nu_x - 2\nu_z$ resonance for the Booster lattice with eddy currents and chromaticity sextupoles.

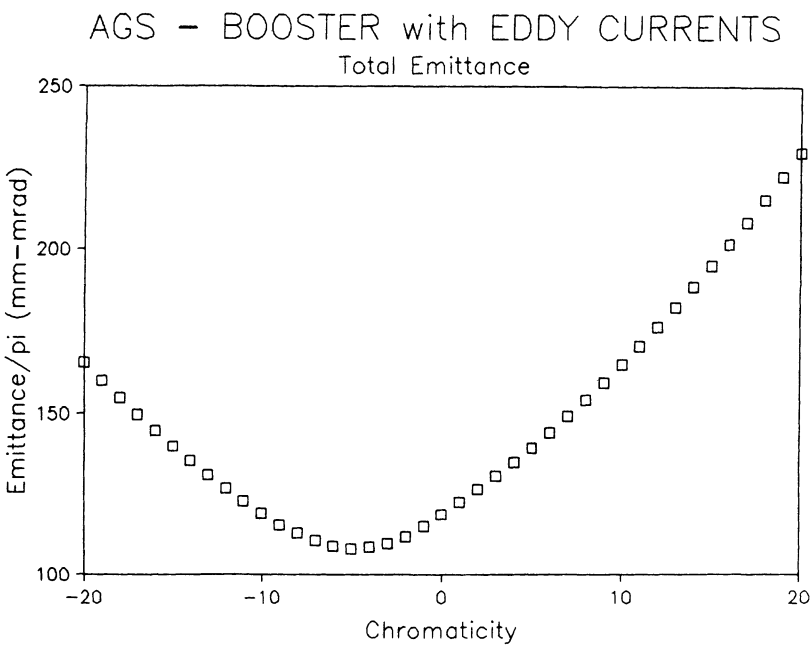

FIGURE 5 Maximum total emittance for the Booster lattice with eddy currents and chromaticity sextupoles versus machine chromaticity.

total value of the emittance, generating-function resonance strengths, Hamiltonian resonance strengths, stop bandwidth, fixed points, island width, and Chirikov criteria for the Booster as functions of different chromaticities. (See Ref. 1 for detailed analytic formulations.) Since the $2\nu_x - 2\nu_z$, $\nu_x - 2\nu_z$, and ν_x resonance contributions are of interest in Fig. 1, we show the strength of these resonances (generating-function resonance strengths) as functions of the chromaticity of the machine for the Booster lattice with eddy currents and chromaticity sextupoles.

Figure 2 shows perturbation to tune versus machine chromaticity for operating tunes of 4.82 and 4.83 and an average beam emittance of 50 π mm-mrad. Figure 2 nicely illustrates the amplitude dependence of the tune due to sextupoles in the Booster. If, instead of the perturbed tune (used in our calculation), we had used the linear tune (used in ordinary perturbation theory), we would not have seen the resonance conditions when on resonance. Thus, with the perturbed tunes and canonical perturbation theory, we obtain results similar to those deduced from the superconvergent perturbation theory of the same order. Note that $\nu_x - \nu_z > 0$ for chromaticities < -10 or $> +16$ and $\nu_x - \nu_z < 0$ for chromaticities between -10 and 16. Furthermore, at chromaticities of about -10 and 16, the two tunes are equal, which excites the $2\nu_x - 2\nu_z$ resonance. Since this is a difference resonance, the sum of the emittances remains conserved (coupling) and does not lead to emittance growth, which is confirmed by tracking results obtained from the programs ORBIT and PATRICIA.[7]

Figure 3 shows the $2\nu_x - 2\nu_z$ resonance strength versus the machine chromaticity for the Booster lattice with saturation and chromaticity sextupoles. Figure 4 shows bandwidth versus the machine chromaticity for the $2\nu_x - 2\nu_z$ resonance for the Booster lattice with eddy currents and chromaticity sextupoles. Figure 5 shows maximum total emittance for the Booster lattice with eddy currents and chromaticity sextupoles versus the machine chromaticity, which agrees quite well with the results obtained from tracking (as can be seen in Ref. 3).

These results depend on the initial conditions of the particles (e.g., we assumed an initial condition such that the initial beam emittances equal the averaged beam emittances, which must be considered in comparing our results with tracking.

4. CONCLUSION

We have illustrated the amplitude dependence of the tune due to nonlinear elements (e.g., sextupoles in the Booster) and have shown that using the perturbed tune with ordinary perturbation theory can produce results similar to those obtained from superconvergent theory of the same order. Furthermore, in order to find the optimum operating conditions (e.g., operating tune, chromaticity, etc.) for the AGS Booster, we have investigated the systematic resonances, with emphasis on the $2\nu_x - 2\nu_z$ resonance as well as the $\nu_x - 2\nu_z$ and ν_x resonances. As seen from Table I, the $2\nu_x - 2\nu_z$ coupling becomes large when the chromaticity of the machine is corrected to zero. Our results indicate that, for this difference resonance, the sum of the emittances remains conserved (coupling) and does not lead to emittance growth, as confirmed by tracking results obtained from the programs ORBIT and PATRICIA.

We have also studied the behavior of the beam with respect to chromaticity. Our results indicate a chromaticity range of −5.5 to −2.5 for the operation of the booster (and for a tune split >0.01), which agrees well with the results obtained from tracking (that is, no coupling when chromaticity ranges from −5 to −2.5).

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APPENDIX

Second-Order Perturbation Theory

In this appendix we develop the second-order perturbation theory needed to obtain the emittance growth and generating-function resonance strengths in accelerators.

Consider the following Hamiltonian:

$$H = \mathbf{a}(s)\cdot\mathbf{J} + \varepsilon\sum_{\mathbf{m}} v_{\mathbf{m}}(\mathbf{J}, s)e^{i\mathbf{m}\cdot\boldsymbol{\phi}},$$

$$\mathbf{a}(s)\cdot\mathbf{J} \equiv \frac{J_x}{\beta_x(s)} + \frac{J_z}{\beta_z(s)}, \tag{A-1}$$

where $(\mathbf{J}, \boldsymbol{\phi})$ are the action-angle variables, ε is the perturbation strength used to keep track of the order of the perturbation, and $v_{\mathbf{m}}(\mathbf{J}, s)$ are the Fourier components of the perturbing potential. We choose the generating function

$$F = \mathbf{K}\cdot\boldsymbol{\phi} + \sum_{\substack{\mathbf{l}\\ |\mathbf{l}|\neq 0}} [\varepsilon g_{\mathbf{l}}(\mathbf{K}, s) + \varepsilon^2 h_{\mathbf{l}}(\mathbf{K}, s)]e^{i\mathbf{l}\cdot\boldsymbol{\phi}}, \tag{A-2}$$

with $\mathbf{K}$ and ψ the new action-angle variables. The vector $\mathbf{l}$ is a vector with integer components ($|\mathbf{l}|$ is the sum of the absolute value of $\mathbf{l}$'s components), and $g_{\mathbf{l}}(\mathbf{K}, s)$

and $h_{\mathbf{l}}(\mathbf{K}, s)$ are the first- and second-order functions chosen to cancel the first- and second-order terms in the new Hamiltonian $\bar{H}$.

We can find the action-angle variables from the generating function F, Eq. (A-2), as

$$\mathbf{J} = \nabla_{\phi}\mathbf{F} = \mathbf{K} + \sum_{\substack{\mathbf{l} \\ |\mathbf{l}|\neq 0}} i\mathbf{l}[\varepsilon g_{\mathbf{l}}(\mathbf{K}, s) + \varepsilon^2 h_{\mathbf{l}}(\mathbf{K}, s)]e^{i\mathbf{l}\cdot\phi} \quad \text{(A-3)}$$

and

$$\psi = \nabla_{\mathbf{K}}F = \phi + \sum_{\substack{\mathbf{l} \\ |\mathbf{l}|\neq 0}} [\varepsilon\nabla_{\mathbf{K}}f_l(\mathbf{K}, s) + \varepsilon^2\nabla_{\mathbf{K}}h_{\mathbf{l}}(\mathbf{K}, s)]e^{i\mathbf{l}\cdot\phi}, \quad \text{(A-4)}$$

and the new Hamiltonian

$$\bar{H} = H + \frac{\partial}{\partial s}F, \quad \text{(A-5)}$$

or

$$\bar{H} = \mathbf{a}(s)\cdot\mathbf{K} + \sum_{\mathbf{m}} i\mathbf{m}\cdot\mathbf{a}(s)[\varepsilon g_{\mathbf{m}}(\mathbf{K}, s) + \varepsilon^2 h_{\mathbf{m}}(\mathbf{K}, s)]e^{i\mathbf{m}\cdot\phi}$$
$$+ \varepsilon\sum_{\mathbf{m}} v_{\mathbf{m}}\Big(\mathbf{K} + \sum_{\substack{\mathbf{l} \\ |\mathbf{l}|\neq 0}} i\mathbf{l}[\varepsilon g_l(\mathbf{K}, s) + \varepsilon^2 h_{\mathbf{l}}(\mathbf{K}, s)]e^{il\cdot\phi}, s\Big)e^{i\mathbf{m}\cdot\phi}$$
$$+ \sum_{\mathbf{m}}\left[\varepsilon\frac{\partial}{\partial s}g_{\mathbf{m}}(\mathbf{K}, s) + \varepsilon^2\frac{\partial}{\partial s}h_{\mathbf{m}}(\mathbf{K}, s)\right]e^{i\mathbf{m}\cdot\phi}. \quad \text{(A-6)}$$

Expanding $v_{\mathbf{m}}$ in a Taylor series about $\varepsilon = 0$ leads to

$$\bar{H} = \mathbf{K}\cdot\mathbf{a}(s) + \sum_{\mathbf{m}}\left\{i\mathbf{m}\cdot\mathbf{a}(K)[\varepsilon g_{\mathbf{m}}(\mathbf{K}, s) + \varepsilon^2 h_{\mathbf{m}}(\mathbf{K}, s)] + \varepsilon\frac{\partial}{\partial s}g_{\mathbf{m}} + \varepsilon^2\frac{\partial}{\partial s}h_{\mathbf{m}}\right\}e^{i\mathbf{m}\cdot\phi}$$
$$+ \varepsilon\sum_{\mathbf{m}}\left[v_{\mathbf{m}}(\mathbf{K}, s) + \sum_{\substack{\mathbf{l} \\ |\mathbf{l}|\neq 0}} i\varepsilon g_{\mathbf{l}}(\mathbf{K}, s)e^{il\cdot\phi} * \mathbf{l}\cdot\nabla_{\mathbf{K}}v_{\mathbf{m}}(\mathbf{K}, s)\right]e^{i\mathbf{m}\cdot\phi} + O(\varepsilon^3). \quad \text{(A-7)}$$

Collecting the ε, ε^2, etc., terms gives

$$\bar{H} = \mathbf{a}(s)\cdot\mathbf{K}\sum_{\mathbf{m}}\left\{\varepsilon\left[\mathbf{m}\cdot\mathbf{a}(s)g_{\mathbf{m}}(\mathbf{K}, s) + \frac{\partial}{\partial s}g_{\mathbf{m}}(\mathbf{K}, s) + v_{\mathbf{m}}(\mathbf{K}, s)\right]\right.$$
$$\left. + \varepsilon^2\left[\mathbf{a}(s)\cdot\mathbf{m} * h_{\mathbf{m}}(\mathbf{K}, s) + \frac{\partial}{\partial s}h_{\mathbf{m}}(\mathbf{K}, s) + \sum_{\substack{\mathbf{l} \\ |\mathbf{l}|\neq \mathbf{m}}} ig_{\mathbf{m}-\mathbf{l}}(\mathbf{K}, s) * (\mathbf{m}-\mathbf{l})\cdot\nabla_{\mathbf{K}}v_{\mathbf{l}}(\mathbf{K}, s)\right]\right\}$$
$$\times e^{i\mathbf{m}\cdot\phi} + O(\varepsilon^3). \quad \text{(A-8)}$$

To eliminate the phase variables, we consider the first-order perturbation theory by setting the coefficient of the ε term to zero for $\mathbf{m}\neq 0$:

$$\mathbf{m}\cdot\mathbf{a}(s)g_{\mathbf{m}}(\mathbf{K}, s) + \frac{\partial}{\partial s}g_{\mathbf{m}}(\mathbf{K}, s) + v_{\mathbf{m}}(\mathbf{K}, s) = 0. \quad \text{(A-9)}$$

Then solving for $g_{\mathbf{m}}(\mathbf{K}, s)$ results in

$$g_{\mathbf{m}}(\mathbf{K}, s) = \frac{i}{2 \sin \pi \mathbf{m} \cdot \mathbf{v}} \int_s^{s+C} v_{\mathbf{m}}(\mathbf{K}, t) \times \exp \{i\mathbf{m} \cdot [\boldsymbol{\xi}(t) - \boldsymbol{\xi}(s) - \pi \mathbf{v}]\}\, dt, \tag{A-10}$$

with $\mathbf{m} = (n_{x_k}, n_{z_k} \cdots)$.

Further, to eliminate the phase (angle) variables to second order in the perturbing potential, we set the coefficients of the ε^2 term equal to zero in Eq. (A-8):

$$\mathbf{m} \cdot \mathbf{a}(s) h_{\mathbf{m}}(\mathbf{K}, s) + \frac{\partial}{\partial s} h_{\mathbf{m}}(\mathbf{K}, s) + \sum_{\substack{\mathbf{l} \\ |\mathbf{l}| \neq 0}} i g_{\mathbf{m}-\mathbf{l}}(\mathbf{K}, s)(\mathbf{m} - \mathbf{l}) \cdot \nabla_{\mathbf{K}} v_{\mathbf{l}}(\mathbf{K}, s) = 0 \tag{A-11}$$

for $|\mathbf{m}| \neq 0$. Solving for $h_m(\mathbf{K}, s)$ we obtain

$$h_{\mathbf{m}}(\mathbf{K}, s) = \frac{i}{2 \sin \pi \mathbf{m} \cdot \mathbf{v}} \int_s^{s+C} \left[\sum_{\mathbf{l}} i g_{\mathbf{m}-\mathbf{l}}(\mathbf{K}, t)(\mathbf{m} - \mathbf{l}) \cdot \nabla_{\mathbf{K}} v_{\mathbf{l}}(\mathbf{K}, t) \right] \times \exp \{i\mathbf{m}[\boldsymbol{\xi}(t) - \boldsymbol{\xi}(s) - \pi \mathbf{v}]\}\, dt, \tag{A-12}$$

where

$$\boldsymbol{\xi}(s) \equiv \int_0^s \mathbf{a}(t)\, dt, \tag{A-13}$$

since $a(s)$ is periodic in s, the unperturbed time v^0 defined as

$$\mathbf{v}^0 \equiv \frac{\boldsymbol{\xi}(s + C) - \boldsymbol{\xi}(s)}{2\pi} \tag{A-14}$$

is independent of s.

Thus the new Hamiltonian $\bar{H}$ can be expressed as

$$\bar{H} = \mathbf{a}(s) \cdot \mathbf{K} + \varepsilon v_{\mathbf{l}}(\mathbf{K}, s)\,|_{\mathbf{l}=0} - \varepsilon^2 \sum_{\substack{\mathbf{l} \\ |\mathbf{l}| \neq 0}} i g_{-\mathbf{l}}(\mathbf{K}, s)\mathbf{l} \cdot \nabla_{\mathbf{K}} v_{\mathbf{l}}(\mathbf{K}, s) + O(\varepsilon^3), \tag{A-15}$$

and the emittance becomes:

$$\mathbf{E} = 2\pi \mathbf{J} = 2\pi \left[\mathbf{K} + \sum_{\substack{\mathbf{m} \\ |\mathbf{m}| \neq 0}} i\mathbf{m}[\varepsilon g_{\mathbf{m}}(\mathbf{K}, s) + \varepsilon^2 h_{\mathbf{m}}(\mathbf{K}, s)] e^{i\mathbf{m} \cdot \boldsymbol{\Phi}} \right], \tag{A-16}$$

where

$$|\varepsilon g_{\mathbf{m}}(\mathbf{K}, s) + \varepsilon^2 h_{\mathbf{m}}(\mathbf{K}, s)\,|\, \sin \pi \mathbf{m} \cdot \mathbf{v}$$

is the generating-function resonance strength, corresponding to $g_k(K_x, K_z, s)$ in Eqs. (10) and (11).