Particle Accelerators, 1987, Vol. 20, pp. 159–170
0031-2460/87/2004–0159/$20.00/0

CHROMATICITY OPTIMIZATION BY TUNING IN LARGE COLLIDER RINGS

DAVID NEUFFER†

Physics Department, Texas A & M University, College Station, TX 77843

(Received February 25, 1986; in final form April 28, 1986)

In large colliders such as the Superconducting Super Collider (SSC) the nonlinear chromaticity can be dominated by the interaction region (IR) focusing quads. Their contributions to the chromaticity are minimized by placing the superperiod tune near a quarter integer. The reasons for this minimum are discussed and extensions to more complicated lattices with multiple IRs per superperiod are developed. Applications to the SSC and to LEP are described.

1. CHROMATICITY IN THE SSC

It has recently been noted that second-order chromaticity, the second-order dependence of tune upon momentum in large Superconducting Super Collider (SSC)[1] lattices, may be minimized by placing superperiod tunes near a quarter integer; that is,

$$\nu \cong N.25 \quad \text{or} \quad N.75,$$

where N, the integer part of the tune ν is arbitrary.[2] In this paper we explore the reasons for this minimum and obtain guidelines for minimized chromaticities in more complicated lattices.

In the limit where the change in betatron tune is small, it may be written as

$$\Delta\nu = \frac{1}{4\pi}\int \beta(s)\,\Delta k(s)\,ds, \tag{1}$$

where $\beta(s)$ is the Courant–Snyder betatron function[3] in the unperturbed lattice (the lattice at $\Delta p/p = 0$), and $\Delta k(s)$ is the momentum-dependent perturbation of the focusing strength. In a lattice with only quadrupole and sextupole magnets as focusing elements, this is

$$\Delta k(s) = Q(-\Delta - \Delta^2) + S\eta\Delta(1-\Delta)$$

to second order in $\Delta = \Delta p/p$, where $Q = B'/B\rho$ is the quadrupole focusing strength and $S = B''/B\rho$ is the sextupole strength, $B\rho$ is the magnetic rigidity of the central energy, and η is the off-momentum orbit dispersion function.

For the second-order evaluation of the chromaticity, we must expand β and η in Δ:

$$\beta(s) = \beta_0(s) + \beta'(s)\Delta,$$
$$\eta(s) = \eta_0(s) + \eta'(s)\Delta.$$

† On leave from AT-6, Los Alamos National Laboratory, Los Alamos, NM 87545.

From betatron perturbation theory,[3] we can obtain expressions for these derivatives:

$$\beta'(s) = \frac{-\beta(s)}{2\sin(2\pi\nu)} \int ds_1 \beta(s_1)(Q - S\eta)_1 \cos\{2[\pi\nu - |\mu(s) - \mu(s_1)|]\}, \qquad (2a)$$

$$\eta'(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi\nu)} \int ds_1 \sqrt{\beta(s_1)}\,(Q - S\eta)_1 \cos[\pi\nu - |\mu(s) - \mu(s_1)|]. \qquad (2b)$$

In each case, the integrals are evaluated over the superperiod length, and the tune ν is the superperiod tune.

Expanding Eq. (1) to second order in Δ, we obtain

$$\begin{aligned}\Delta\nu = &-\frac{\Delta}{4\pi}\int \beta_0(s)(Q - S\eta)\,ds \\ &-\frac{\Delta^2}{4\pi}\int \beta_0(s)(Q + S\eta - S\eta')\,ds \\ &-\frac{\Delta^2}{4\pi}\int \beta'(s)(Q - S\eta)\,ds, \end{aligned} \qquad (3)$$

which produces the coefficients in the expansion

$$\Delta\nu = \xi_1\Delta + \xi_2\Delta^2 + \cdots.$$

Note that the expression above is valid for both transverse degrees of freedom (x and y). $\Delta\nu_x$ and $\Delta\nu_y$ are obtained from β_x and β_y, respectively, using the appropriate focusing functions ($Q_x = -Q_y$, $S_x = -S_y$). We will apply Eqs. (2) and (3) to SSC lattices to determine optimum chromaticity conditions.

2. SAMPLE SSC LATTICES

The SSC has 6 interaction regions (IRs), so the simplest lattice is a periodicity-6 or hexagon lattice. A superperiod, shown in Fig. 1, starts at an IR center and includes a strong focusing triplet, a matching section, a $\pi/3$ bending arc (about 25 km long with 3-T bending magnets), a matching section, and a triplet focusing to a second IR center. Lattice parameters are shown in Table I. The important components are the long arcs and the IR triplets. The arcs are composed of about 220 identical FODO cells in this example, and sample cell parameters are shown in Table II. A FODO cell is shown in Fig. 2. Focusing is provided by quadrupoles ($Q \cong 0.0017$, $L_Q \cong 5$ m). All SSC lattices contain similar unit cells. The IR region consists of a low-beta focus point, a free space of ± 20 m about the IR focus, followed by a strong-focusing triplet with $Q \cong \pm 0.003$ and a total length of 50 m. A typical IR region is displayed in Fig. 3. Sample IR region parameters with lattice functions passing through that region are displayed in Table III. Note that $\beta(s)$ at the IR triplet is 2000 to 4000 m, approximately 10 times greater that in the arcs. Also, the tune advance from IR center to triplet is $\delta\nu \cong 0.25$.

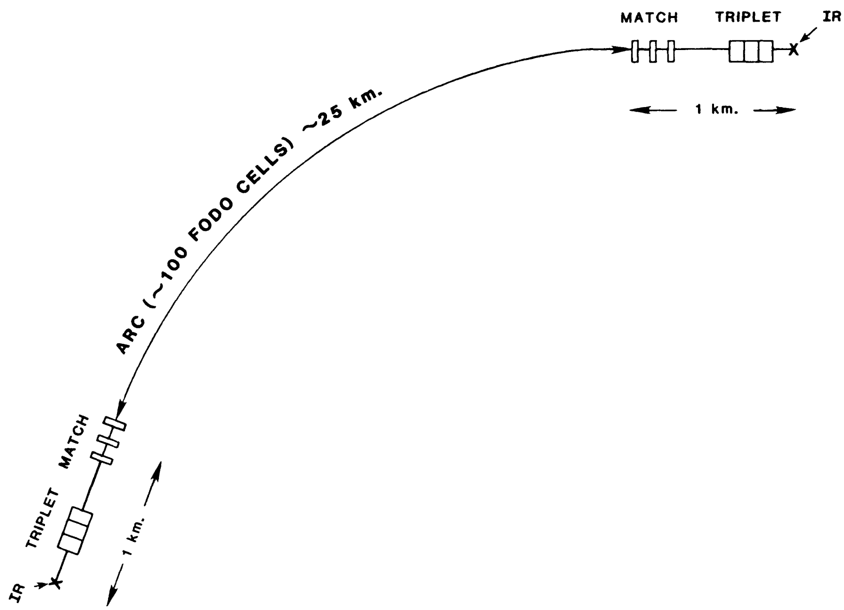

FIGURE 1 Layout of one superperiod of a hexagon (periodicity-six) lattice, showing the IR triplet, matching elements, and the 60° arc.

The examples displayed here all use 3-T bending fields; however, the conclusions of this paper are completely independent of bending field. The results apply equally well to ~6-T SSC lattices.

3. APPROXIMATE EVALUATION OF CHROMATICITY FUNCTIONS

The chromaticity functions can be numerically evaluated for a particular lattice; however, approximate evaluations may be obtained by simple integrations within

TABLE I
SSC lattice parameters

Parameter	Periodicity-6 lattice	Periodicity-2 lattice
Circumference	162.8 km	165.6 km
Tunes	118.56	122.55
Superperiodicity	6	2
Superperiod tune	19.76	61.27
FODO cell half-length	115 m	115 m
FODO cells/superperiod	112	328
β^* (IR)	1.0 m	1.0 m
β_{max} (IR quads)	4200 m	4200 m
IR triplet length	43.75 m	43.75 m
IR quad gradient	200 T/m	200 T/m

TABLE II
FODO cell parameters

Parameter	Value
Cell half-length	115 m
Total dipole length	105 m
Dipole field, bend	3.0 T, 0.27° bend
Quadrupole length and strength	5.0 m, 118 T/m
Sextupole/spool piece slot length	5.0 m
Phase advance/cell	60°
β_{max}	398 m
β_{min}	133 m
η_{max}	2.68 m
η_{min}	1.63 m

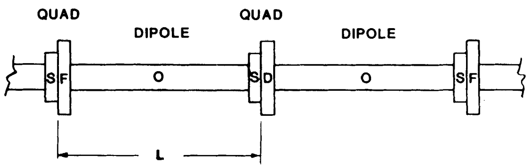

FIGURE 2 A FODO cell of an accelerator arc, showing focusing and defocusing quads (F and D), sextupole/spool pieces (S), and dipoles (O).

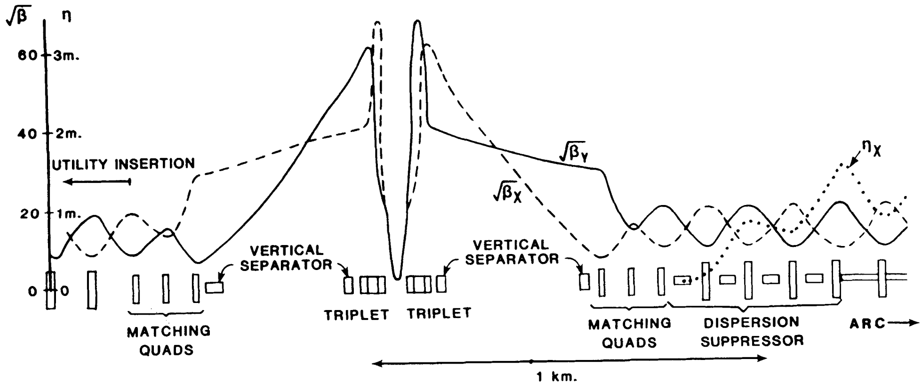

FIGURE 3 Layout and lattice functions near an IR region. Note the antisymmetric design which naturally places $\nu_x \cong \nu_y$. Both focusing triplets and matching optics into the arcs are shown.

TABLE III
Lattice functions in an IR region

Element	Type	Length, m	Cumulative length, m	β_x, m	β_y, m	Tune advances from IR center $\delta\nu_x$	$\delta\nu_y$
0	IR center	—	0	1.0	1.0	0	0
F0	Drift	20	20	401	401	0.24205	0.24205
QF1	Quad	15.75	35.75	723	2090	0.24615	0.24514
QD1	Half-quad	12.50	48.25	906	4230	0.24881	0.24576
QD1	Half-quad	12.50	60.75	2430	2960	0.25028	0.24627
QF1	Quad	13.0	73.75	3810	1820	0.25092	0.24724
F1	Drift	431.75	505.5	50	848	0.47255	0.30378
Q1	Quad	5.0	510.5	52	817	0.48821	0.30473
F2	Drift	75.0	585.5	220	218	0.61209	0.33314
Q2	Quad	5.0	590.5	233	199	0.61559	0.33697
F2	Drift	75.0	665.5	364	80	0.65700	0.43791
Q3	Quad	7.0	672.5	357	82	0.66006	0.45177
	Arc						

simplified lattice segments. As noted above, the critical components are the IR quads in the triplets and the arc FODO cells.

The first term in Eq. (3) contains an expression of the form

$$\frac{-1}{4\pi}\int \beta_0(s)Q\,ds = \left.\frac{d\nu}{d\Delta}\right|_{S=0} = \xi_0,$$

where ξ_0 is identified as the natural chromaticity. This can be evaluated in a lattice cell, obtaining

$$\xi_{\text{cell}} \cong -\frac{1}{4\pi}(\beta_{\max} - \beta_{\min})QL_Q.$$

Using simplified lattice functions, this becomes

$$\xi_{\text{cell}} = -\frac{1}{4\pi}\left[\frac{2L}{\cos(\varphi/2)}\right]\left[\frac{2\sin(\varphi/2)}{L}\right] \cong -\frac{\varphi}{2\pi},$$

where φ is the phase advance per cell. Summing over all cells in the arc, we obtain

$$\xi_0(\text{cells}) \cong -\nu_{\text{arc}} \cong -20$$

from a typical superperiod of an SSC hexagon lattice.

The other important contribution to the natural chromaticity is that of the IR quads. This may be estimated using the sample IR of the previous section. An approximate evaluation obtains

$$\xi_{\text{IR}} = -\int_{\text{IR}} \beta_0(s)Q(s)\,ds \cong -10 \text{ per triplet}$$

or about −20 per hexagon superperiod, roughly equal to the contribution of the arc. Note that the IR contributes very little to the tune v, only about 0.5 per IR.

The sextupoles in the arcs have strengths chosen such that the total linear chromaticity of the superperiod ξ_1 is zero. Therefore,

$$\frac{1}{4\pi}\int \beta(s)\eta(s)S(s)\,ds = \frac{1}{4\pi}\int \beta(s)Q(s)\,ds \cong 40$$

per superperiod.

The last two terms in Eq. (3) give the nonlinear chromaticity ξ_2. The first of these can be evaluated using the linear chromaticity calculation above, to give

$$\xi_2 = 2\frac{d\xi}{d\Delta} \cong -\frac{1}{4\pi}\int \beta_0(s)[Q(s) + S(s)\eta(s)]\,ds \cong -80 \tag{4}$$

per hexagon superperiod. In Eq. (4) we have removed the term in Eq. (3) that is proportional to $S\eta'$, which is relatively small for SSC lattices. Equation (4) is labeled as providing the "natural" second-order chromaticity. It is quickly estimated as twice the natural linear chromaticity, since in second order the sextupole correction term changes sign, interfering constructively with the quadrupole-dependent linear chromaticity that it cancels in first order.

SSC lattices often have much larger second-order chromaticity. The dominant contribution is due to the last term in Eq. (3), from which we obtain

$$\xi_2 = -\frac{1}{4\pi}\int \beta'(s)(Q - S\eta)\,ds, \tag{5}$$

with

$$\beta'(s) = \frac{-\beta(s)}{2\sin(2\pi v)}\int ds_1\beta(s_1)(Q - S\eta)_1 \cos\{2[\pi v - |\mu(s) - \mu(s_1)|]\}. \tag{2a}$$

We remind the reader that all derivatives here are taken with respect to $\Delta = \Delta p/p$. Within the arcs, $\beta'(s)$ is of the order of $\beta(s)$ and oscillates in sign following the phase function. The contribution to ξ' is of the same order as that of $\eta'(s)$ discussed above and is small.

However, at the IR quads, $\beta'(s)$ is magnified. Assuming the phase factors in Eq. (2a) are of order unity, we may estimate this as

$$\beta'(s) \cong -\tfrac{1}{2}\overline{\beta_{\mathrm{IR}}}^2 Q_{\mathrm{IR}} L_{Q_{\mathrm{IR}}} \cong -4\times 10^5\ \mathrm{m}, \tag{6a}$$

using typical IR values. The contribution to ξ' from the IR triplets can be estimated by placing this value into Eq. (5) and integrating over the IR region, obtaining

$$\xi_{2_{\mathrm{IR}}} \cong -\frac{1}{4\pi}\beta'_{\mathrm{IR}} Q_{\mathrm{IR}} L_Q \cong 4\times 10^3 \text{ per superperiod}, \tag{6b}$$

more than tenfold larger than the natural second-order chromaticity.

4. MINIMIZATION OF ξ_2

ξ_2 can be substantially reduced by reducing β' at the IR triplets, and this can be done by choosing the tune ν such that the phase factors in Eq. (2a) are small. We rewrite Eq. (2a) as

$$\beta'(s) = -\frac{\beta(s)}{2}\cot(2\pi\nu)\int \beta(s_1)(Q - S\eta)_1 \cos[2\,|\mu(s) - \mu(s_1)|]\,ds_1$$
$$-\frac{\beta(s)}{2}\int \beta(s_1)(Q - S\eta)_1 \sin[2\,|\mu(s) - \mu(s_1)|]\,ds_1. \tag{7}$$

β' becomes relatively small at the IR quads as $\cot(2\pi\nu)\to 0$; that is, $\nu \cong N.25$ or $N.75$. Then the first term of Eq. (6) vanishes and $\beta'(s)$ is dominated by the second term. However, as $\nu \to N.25$ or $N.75$, the second term also becomes small at the IR quads. The phase advance from IR triplet to IR triplet within the superperiod is $\Delta\mu \cong 2\pi(\nu - 0.5)$, so $\sin(2\Delta\mu)\to 0$. The phase factor is small within an IR triplet itself ($\Delta\mu \cong 0$). $\beta'(s)$ thus becomes small at the IRs as ν approaches a quarter integer, and, from Eq. (5), ξ_2 also becomes relatively small.

Approaching a quarter integer, superperiod tune will reduce second-order chromaticity until it approaches the magnitude of the "natural" second-order chromaticity contributed by the lattice outside the IRs. From our estimates above, this occurs if $\cot(2\pi\nu) < 0.1$, or $|\nu - (N.25 \text{ or } N.75)| \leq 0.02$. In this region, fine tuning may obtain an even smaller second-order chromaticity. However, chromaticity in this region may be dominated by higher orders, and tuning space for avoiding resonances must be allowed.

Figure 4 and Table IV show the dependence of chromaticity upon fractional tune for the hexagon lattice of Table I. Zero second-order chromaticity is obtained at $\nu \cong N.757$ per superperiod, with a range about this ($\Delta\nu$) of ± 0.02, in which chromaticity is reasonably well-optimized.

5. EXTENSION TO MORE COMPLICATED LATTICES

While the hexagon is the simplest SSC lattice, practical considerations may indicate a preference for lattices with clustered IRs, such as the "racetrack" lattice shown in Fig. 5 which has three IRs per superperiod. These lattices can have much worse chromaticity than higher-periodicity lattices if the chromatic effects of IRs interfere constructively. It is therefore essential to extend the chromaticity minimization principles discussed above to more complex lattices and to ensure adequate chromatic behavior.

The relevant equations are still Eqs. (2) and (3), with the dominant contributions coming from the IR quads through $\beta'(s)$ in Eq. (6). All integrals are taken over a complete superperiod. As before, the first term in Eq. (6) approaches zero as $\cot(2\pi\nu)\to 0$, or

$$\nu \to N.25 \quad \text{or} \quad N.75, \tag{8}$$

where ν is the superperiod tune.

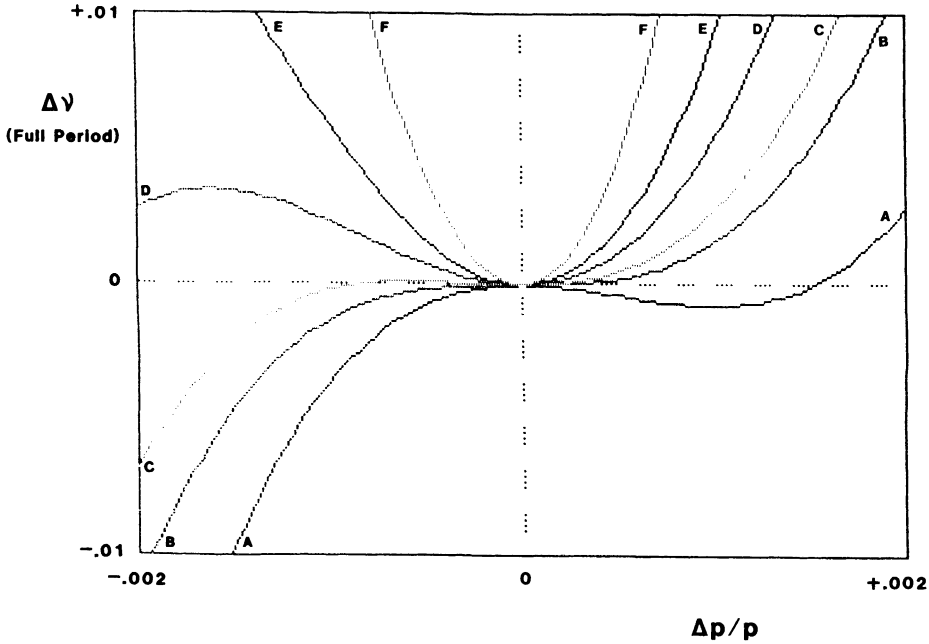

FIGURE 4 Chromaticity ($\nu - \nu_0$ as a function of $\Delta p/p$) for six SSC lattices with superperiodicity 6. The superperiod tune $\nu_6 = \nu_0/6$ is varied across the quarter integer $\nu_6 = 19.75$. For lattice A, $\nu_6 = 19.7115$; B, $\nu_6 = 19.7583$; C, $\nu_6 = 19.7817$; D, $\nu_6 = 19.8283$; E, $\nu_6 = 19.8749$; F, $\nu_6 = 19.9330$.

In a superperiod with only one IR, the second term of Eq. (6) is naturally small at the IR quads when ν is near a quarter integer. However, with multiple IRs per superperiod, $\sin(2|\mu - \mu'|)$ between IR triplets need not always be small. Figure 3 shows a superperiod of a racetrack SSC lattice with three IRs per superperiod. To keep $\sin(2|\mu - \mu'|)$ near zero between each set of IR quads, it is necessary (and sufficient) that the phase advance between IR centers be any multiple of a

TABLE IV

Nonlinear chromaticity in superperiod-6 SSC lattices ($d\beta^*/d(\Delta p/p)$ is evaluated at the IR center)

Superperiod tune (ν_6)	$\cot(2\pi\nu_6)$	$\frac{d\beta^*}{d(\Delta p/p)}$	ξ_2 (superperiod)
19.7115	0.2471	−27.2 m	−297
19.7467	0.0207	−1.38	−60.5
19.7583	−0.0524	6.97	15.7
19.7817	−0.2018	24.0	170
19.8283	−0.536	62.1	514
19.8749	−0.999	117	985
19.9330	−2.234	258	+2230

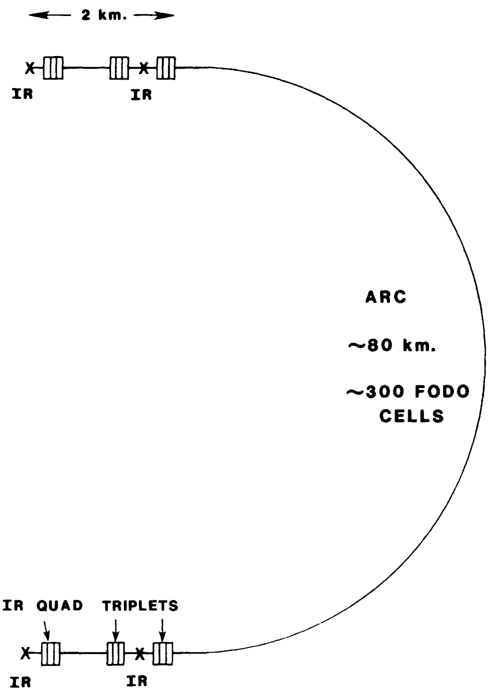

FIGURE 5 Layout of one superperiod of a racetrack (2-sided) lattice, showing three IRs per superperiod.

quarter integer times 2π; that is,

$$\delta\mu_{\mathrm{IR}\to\mathrm{IR}} = 2\pi\, \delta\nu_{\mathrm{IR}\to\mathrm{IR}} = 2\pi \frac{M}{4}, \tag{9}$$

where M is any integer. To obtain this conclusion, we have used the fact that the phase advance between IR centers and adjacent triplets is always nearly exactly $(0.25) \times 2\pi$. Equations (8) and (9) set the conditions for optimum second-order chromaticity.

Figure 6 shows chromaticity plots for a number of SSC racetrack lattices. Chromaticity is modified first by altering the superperiod tune to near a quarter integer and then by tuning the IR→IR phase advances near a quarter integer in $\delta\nu$. Chromaticity is substantially improved by these tuning steps. (Figure 6 displays results for lattices with 3 IRs per superperiod; similar results are also obtained with 2 IRs per superperiod.)

Peggs and Talman[4] have proposed a modification of this chromaticity cancellation scheme[1] in which IRs are clustered in identical pairs with an odd quarter-integer tune advance between the paired IRs, or

$$\delta\nu_{\mathrm{IR}\to\mathrm{IR}} = \frac{(2N+1)}{4}.$$

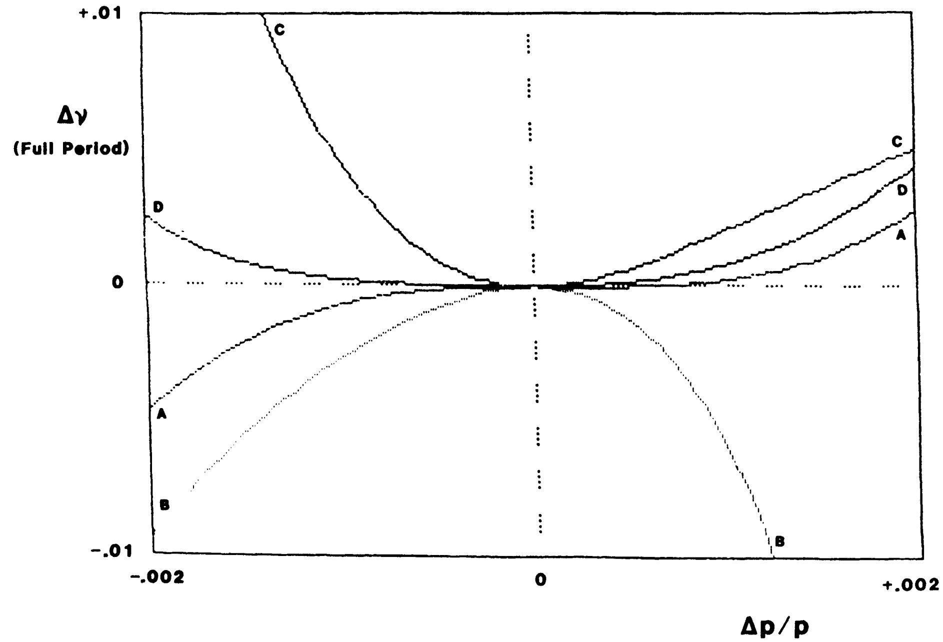

FIGURE 6 Chromaticity for sample SSC lattices, showing the dependence of chromaticity on superperiod tune ν_2 and $\delta\nu_{\mathrm{IR}\to\mathrm{IR}}$ in superperiodicity-2 lattices. For lattice A (superperiodicity-6), $\nu_6 = 19.783$; B (superperiodicity-2), $\nu_2 = 60.15$, $\delta\nu_{\mathrm{IR}\to\mathrm{IR}} = 2.33$; C (superperiodicity-2), $\nu_2 = 60.77$, $\delta\nu_{\mathrm{IR}\to\mathrm{IR}} = 2.33$; D (superperiodicity-2), $\nu_2 = 61.27$, $\delta\nu_{\mathrm{IR}\to\mathrm{IR}} = 2.77$.

With this pairing, the betatron function perturbation $\beta'(s)$ due to the IR triplets is zero outside the IR pair. This is so, since, for points outside the paired IRs, the phase factors $2|\mu - \mu'|$ in Eq. (2a) for the two IR regions differ by π and therefore their contributions to $\beta'(s)$ cancel. Note that this cancellation occurs only outside the paired-IR region and that this cancellation is independent of the superperiod tune ν.

This pairing does reduce betatron function distortion in the arcs but does not reduce it within the paired-IR region. It does not greatly affect the second-order chromaticity, since that is dominated by the effects of betatron function distortion at the IR triplets. Minimization of chromaticity requires a superperiod tune near a quarter integer, as well as quarter-integer IR→IR tune advances.

The paired cancellation of $\beta'(s)$ does not occur unless the paired IRs are identical. However, the chromaticity reduction procedure [Eqs. (8) and (9)] is valid even if the different IRs in the superperiod have different optical properties, since it relies on setting phase factors near zero, not on a cancellation of equal terms.

The IR-pairing scheme, however, does provide a good reason for choosing quarter-integer tune advances between IRs in a superperiod (rather than half or full integers). Betatron distortion reduction is very desirable and its reduction should reduce third- and higher-order chromaticities.

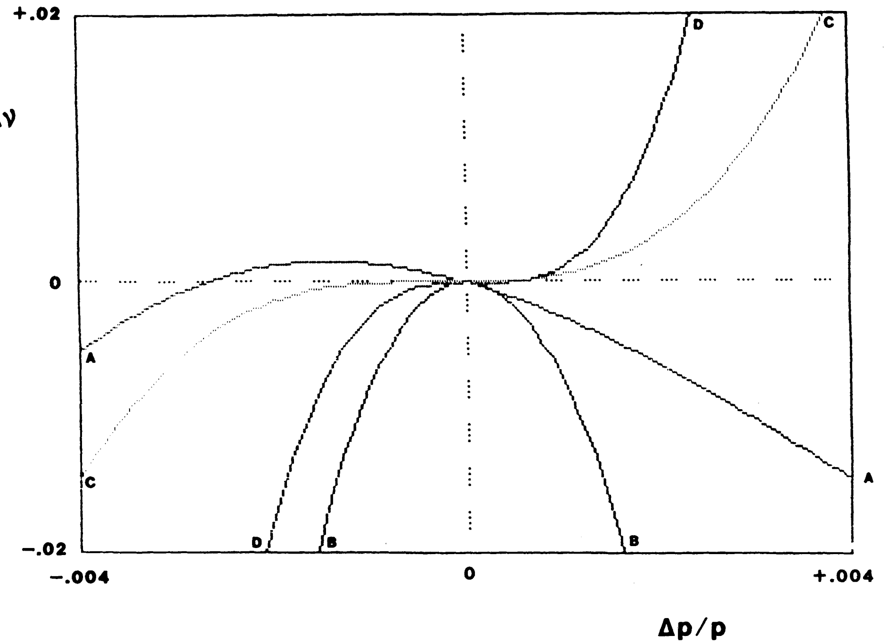

FIGURE 7 Chromaticity for various LEP lattices, showing the dependence on superperiod tune. All lattices have superperiodicity four. For lattice A, $\nu_4 = 16.55$, 3-family chromaticity correction; B, $\nu_4 = 16.55$, 1-family linear chromaticity correction; C, $\nu_4 = 16.70$, 1-family linear correction only; D, $\nu_4 = 16.74$, 1-family linear correction only.

6. APPLICATION TO THE LEP LATTICE

The optimization technique discussed above may be applied to other lattices; however, it does require that the chromaticity be dominated by the IR quads. Most earlier lattices are not dominated by this effect. As an example, we consider the LEP e^+e^- collider lattice, as presented by Iselin.[5] Lattice parameters are displayed in Table V. Some large differences from the SSC lattices are apparent. The beam is not round at the IR ($\beta_x^* = 1.6\,\text{m}$, $\beta_y^* = 0.1\,\text{m}$), IR focusing is provided by a doublet rather than a triplet, and the length and strength of the quads are somewhat less than for the SSC, with smaller high β values.

The vertical chromaticity is, in fact, larger than the horizontal. It can be estimated as discussed above for the SSC. If we insert LEP IR quad parameters into the equations for β' and ξ', we find

$$\beta'_{\text{IR}} \cong 4 \times 10^3$$

and

$$|\xi_{2_{\text{IR}}}| \cong 160 \text{ per superperiod.}$$

The second-order chromaticity due to the IR is only about 3 times as large as that due to the remainder of the lattice, unlike the SSC where the comparison factor is greater than 10.

TABLE V
LEP lattice parameters

Parameter	Value
Circumference	26,759 m
Tunes	$\nu_x = 58.35$, $\nu_y = 66.20$
Superperiodicity	4
Superperiod tunes	$\nu_x = 14.5875$, $\nu_y = 16.55$
Cell half-length	39.5 m
Cell β_{max}	135 m
Cell phase advance	$\phi_x = 60°$, $\phi_y = 60°$
IR low-beta values	$\beta_x^* = 1.6$, $\beta_y^* = 0.1$ m
IR quad lengths	$L_1 = 5$, $L_2 = 3$ m
IR quad strengths ($B'/B\rho$)	$Q_1 = -0.0528$, $Q_2 = +0.045$ m^{-2}
IR quad high-beta values	500 m

However, the present procedure can significantly improve chromaticity, as is demonstrated in the chromaticity plots of Fig. 5. Lattice A is the CERN design lattice, with superperiod tune $\nu_y = 16.55$, which has three independent families of sextupoles (six sets) adjusted to minimize total chromaticity. Lattice B is the same lattice, but with only one family of sextupoles correcting only linear chromaticity. In lattice C, the tune is moved near the quarter integer per superperiod ($\nu_y = 16.70$), and only linear chromaticity is corrected. The chromaticity contributions of the IR and the arc nearly cancel. In lattice D, the tune is closer to the quarter integer, but the contributions do not cancel as well. These examples show that by modifying the superperiod tune toward the quarter integer, improved chromaticity can be obtained with only linear chromaticity correction. Chromaticity can be as flat as in the CERN design, which has three-family correction.

ACKNOWLEDGMENTS

I gratefully acknowledge helpful discussions with S. Peggs, A. Chao, and R. Talman.

REFERENCES

1. *Report of the Reference Designs Study Group on the Superconducting Super Collider,* Department of Energy (1984).
2. D. Neuffer, "Comment on the Fractional Part of the Superperiod Tune," in *SSC Aperture Workshop Proceedings,* SSC-TR-2001 (1984), p. 116.
3. E. D. Courant and H. L. Snyder, *Ann. Phys.* (*NY*) **3,** 1 (1958).
4. S. Peggs and R. Talman, unpublished communications, Clustered IR Study Group (1985).
5. F. C. Iselin, *The MAD Program,* CERN-LEP-TH/85-15 (1985).